ON SLIGHTLY $\alpha$-CONTINUOUS FUNCTIONS

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Abstract. In [11] the feeble continuity is introduced and then the weak and strong forms of feeble (or, equivalently $\alpha$-continuity) continuity are studied. In this note, we introduce a type of function called a slightly $\alpha$-continuous function and study several properties of it.

1. Introduction

Since the concept of feeble continuity is introduced in [9], the weak and strong forms of it are defined and studied here and there. For example, after a year Mashhour, Hasanem and El-Deeb have defined $\alpha$-continuity in [13] and the notion of almost feeble continuity is, in [12], defined and studied its properties and relations. Among them feeble continuity and $\alpha$-continuity are equivalent because it is proved in [6] that feebly open sets coincide with $\alpha$-open sets.

We denote topological spaces by $X$, $Y$ and $Z$ on which no separation axioms are assumed, and the closure and the interior of a subset $S$ of $X$ by $\text{Cl}_X(S)$ and $\text{Int}_X(S)$ (simply, $\text{Cl}(S)$ and $\text{Int}(S)$), respectively. $S$ is said to be semi-open [7] if there exists an open set $O$ such that $O \subset S \subset \text{Cl}(O)$ and its complement is called semi-closed. The intersection of all semi-closed sets containing $S$ is called the semi-closure of $S$ and denoted by $s\text{Cl}(S)$. $S \subset X$ is said to be $\alpha$-open.
if \( S \subseteq \text{Int} \left( \text{Cl} \left( \text{Int}(S) \right) \right) \) and its complement is called \( \alpha \)-closed. The intersection of all \( \alpha \)-closed sets containing \( S \) is called the \( \alpha \)-closure of \( S \) and denoted by \( \alpha \text{Cl}(S) \). It is known in [9] that a feebly open set, which coincides with an \( \alpha \)-open set, is defined as a set if there is an open set \( U \) such that \( U \subseteq S \subseteq \text{sCl}(U) \).

Throughout this paper, we also denote the family of all \( \alpha \)-open (resp. semi-open, open and clopen) sets of \( X \) by \( \alpha \text{O}(X) \) (resp. \( \tau(X) \) and \( \text{C}(X) \)), and denote the family of \( \alpha \)-open (resp. semi-open, open and clopen) sets of \( X \) containing \( x \) by \( \alpha \text{O}(X,x) \) (resp. \( \text{SO}(X,x) \), \( \tau(X,x) \) and \( \text{C}(X,x) \)).

**Definition 1.1.** A function \( f : X \rightarrow Y \) is called semi-continuous (s.C.) \([8]\) (resp. almost semi-continuous (a.s.C.) \([1]\), semi \( \theta \)-continuous (s.\( \theta \).C.) \([1]\) and weakly semi-continuous (w.s.C.) \([1]\)) if for each \( x \in X \) and each \( V \in \tau(Y,f(x)) \), there exists \( U \in \text{SO}(X,x) \) such that \( f(U) \subseteq V \) (resp. \( f(U) \subseteq \text{Int} \left( \text{Cl}(V) \right) \), \( f(\text{Cl}(U)) \subseteq \text{Cl}(V) \) and \( f(U) \subseteq \text{Cl}(V) \)).

**Definition 1.2.** A function \( f : X \rightarrow Y \) is called slightly semi-continuous (sl.s.C.) \([15]\) (resp. slightly continuous (sl.C.) \([4]\)) if for each \( x \in X \) and each \( V \in \text{CO}(Y,f(x)) \), there exists \( U \in \text{SO}(X,x) \) (resp. \( U \in \tau(X,x) \)) such that \( f(U) \subseteq V \).

**Definition 1.3** A function \( f : X \rightarrow Y \) is called almost continuous (a.C.) \([17]\) (resp. \( \theta \)-continuous (\( \theta \).C.) \([3]\) and weakly continuous (w.C.) \([7]\)) if for each \( x \in X \) and each \( V \in \tau(Y,f(x)) \), there is \( U \in \tau(X,x) \) such that \( f(U) \subseteq \text{Int} \left( \text{Cl}(V) \right) \) (resp. \( f(\text{Cl}(U)) \subseteq \text{Cl}(V) \) and \( f(U) \subseteq \text{Cl}(V) \)).

2. Slightly \( \alpha \)-continuous functions

**Definition 2.1.** A function \( f : X \rightarrow Y \) is called slightly \( \alpha \)-continuous (sl.\( \alpha \).C.) if for each \( x \in X \) and each \( V \in \text{CO}(Y,f(x)) \), there exists \( U \in \alpha \text{O}(X,x) \) such that \( f(U) \subseteq V \).

**Theorem 2.1.** For a function \( f : X \rightarrow Y \), the following are equivalent:
(a) \( f \) is \( \alpha \)-C.,
(b) \( f^{-1}(V) \in \alpha O(X) \) for each \( V \in CO(Y) \),
(c) \( [X - f^{-1}(V)] \in \alpha O(X) \) for each \( V \in CO(Y) \).

**Proof** (a) \( \Rightarrow \) (b) \( \cdot \) Let \( V \in CO(Y) \) and let \( x \in f^{-1}(V) \). Then \( f(x) \in V \) and there is \( U_x \in \alpha O(X, x) \) such that \( f(U_x) \subset V \) since \( f \) is \( \alpha \)-C. Thus we have \( f^{-1}(V) = \bigcup \{ U_x : x \in f^{-1}(V) \} \) and so \( f^{-1}(V) \) is the union of \( \alpha \)-open sets. Hence \( f^{-1}(V) \in \alpha O(X) \) because \( \alpha O(X) \) is a topology on \( X \). The remainders of proof are easy and are thus omitted \( \square \).

The following are obtained easily since \( \alpha O(X) \subset SO(X) \) in any space \( X \).

**Theorem 2.2** Slight continuity implies slight \( \alpha \)-continuity

**Theorem 2.3** Slight \( \alpha \)-continuity implies slight semi-continuity.

**Example 2.1.** Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{b, c\}, X\} \). Define a function \( f : (X, \tau) \to (X, \sigma) \) by \( f(a) = f(c) = a \) and \( f(b) = b \). Then it is easy to prove \( f \) is \( \alpha \)-C. However \( f \) is not \( \alpha \)-C because \( f^{-1}(\{a\}) = \{a, c\} \) is not \( \alpha \)-open in \( (X, \tau) \).

**Theorem 2.4** If \( f : X \to Y \) is \( \alpha \)-C. and \( A \in \alpha O(X) \), then the restriction \( f|A \) is \( \alpha \)-C.

**Proof** Let \( V \in CO(Y) \). Then \( (f|A)^{-1}(V) = A \cap f^{-1}(V) \in \alpha O(X) \) since \( \alpha O(X) \) is a topology on \( X \). Therefore, \( f|A \) is \( \alpha \)-C. \( \square \)

**Definition 2.2** A function \( f : X \to Y \) is said to be \( \alpha \)-irresolute [10] if for each \( V \in \alpha O(Y) \), \( f^{-1}(V) \in \alpha O(X) \), and to be pre-feeably-open [6] if for each \( U \in \alpha O(X) \), \( f(U) \in \alpha O(Y) \)

**Theorem 2.5.** If \( f : X \to Y \) is \( \alpha \)-irresolute and \( g : Y \to Z \) is \( \alpha \)-C., then \( g \circ f \) is \( \alpha \)-C.

**Proof** Let \( V \in CO(Z) \). Then \( g^{-1}(V) \in \alpha O(Y) \). Since \( f \) is \( \alpha \)-irresolute, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \in \alpha O(X) \). Thus \( g \circ f \) is \( \alpha \)-C. \( \square \)
Theorem 2.6. Let \( f : X \to Y \) be \( \alpha \)-irresolute and pre-feebly-open surjection, and let \( g : Y \to Z \) be a function. Then \( g \circ f \) is sl.\( \alpha \).C. if and only if \( g \) is sl.\( \alpha \).C.

Proof. Let \( g \circ f \) be sl.\( \alpha \).C. and \( V \in CO(Z) \). Then \((g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \in \alpha O(X) \). Since \( f \) is pre-feebly-open, \( f(f^{-1}(g^{-1}(V))) \in \alpha O(Y) \). Hence \( g^{-1}(V) \in \alpha O(Y) \). Thus \( g \) is sl.\( \alpha \).C. We have its opposite from Theorem 2.5. \( \square \)

The following diagram is obtained from the above and the references:

\[
\begin{array}{cccccc}
C. & \Rightarrow & a.C. & \Rightarrow & \theta.C. & \Rightarrow & w.C. & \Rightarrow & sl.C. & \Rightarrow & sl.\alpha.C. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \checkmark \\
s.C. & \Rightarrow & a.s.C. & \Rightarrow & \theta.s.C. & \Rightarrow & w.s.C. & \Rightarrow & sl.s.C. \\
\end{array}
\]

3. More Characterizations and Comparisons

It is well known that a filterbase \( B \) in \( X \) is said to be residually in \( U \subset X \) if there is \( B \in B \) such that \( B \subset U \), and a net \( \{s\lambda : \lambda \in D\} \) in \( X \) is said to be residually in \( U \subset X \) if there is a \( \lambda_0 \in D \) such that \( \lambda_0 \leq \lambda \) implies \( \{s\lambda\} \in U \). We say that a filterbase \( B \) in \( X \) converges to \( x \in X \) if \( B \) is residually in every \( U \in \tau(X, x) \) and a net \( \{s\lambda\}_{\lambda \in D} \) in \( X \) converges to \( x \in X \) if \( \{s\lambda\}_{\lambda \in D} \) is residually in every \( U \in \tau(X, x) \).

In [11] a filterbase having a concept, which is weaker than one of convergent filterbase, is defined to study more properties of \( \alpha \)-irresolute functions. We defined the following to obtain more characterizations of sl.\( \alpha \).C. function.

Definition 3.1. A filterbase \( B \) in \( X \) is said to be \( \alpha \)-converge (resp. \( c \)-converge) to \( x \in X \) [11] if \( B \) is residually in \( U \) for each \( U \in \alpha O(X, x) \) (resp. \( U \in CO(X, x) \)).

Definition 3.2. Let \( (D, \preceq) \) be a directed set. A net \( \{s\lambda : \lambda \in D\} \) in \( X \) is said to \( \alpha \)-converge (resp. \( c \)-converge) to \( x \in X \) if \( \{s\lambda\}_{\lambda \in D} \) is residually in \( U \) for each \( U \in \alpha O(X, x) \) (resp. \( U \in CO(X, x) \)).
THEOREM 3.1. For a function \( f : X \to Y \), the following are equivalent:

(a) \( f \) is \( \text{sl.}\alpha.C. \) at \( x \).

(b) If a filterbase \( \mathcal{B} \) in \( X \) is residually in each \( U \in \alpha O(X, x) \), then \( f(\mathcal{B}) \) in \( Y \) is residually in every \( V \in CO(Y, f(x)) \).

(c) If a net \( \{ s_\lambda \}_{\lambda \in \Delta} \) in \( X \) is residually in each \( U \in \alpha O(X, x) \), then \( \{ f(s_\lambda) \}_{\lambda \in \Delta} \) is residually in every \( V \in CO(Y, f(x)) \).

Proof. (a) \( \Rightarrow \) (b) Let (a) be true and \( V \in CO(Y, f(x)) \) and \( U \in \alpha O(X, x) \) such that \( f(U) \subset V \). Assume a filterbase \( \mathcal{B} \) in \( X \) is residually in each \( U \in \alpha O(X, x) \). Then there is \( E \in \mathcal{B} \) such that \( E \subset U \). So we have \( f(E) \subset f(U) \subset V \), which proves (b).

(b) \( \Rightarrow \) (c): Let (b) be true and \( V \in CO(Y, f(x)) \). Assume a net \( \{ s_\lambda \}_{\lambda \in \Delta} \) is residually in each \( U \in \alpha O(X, x) \). Thus there is \( \lambda_0 \in \Delta \) such that \( \lambda_0 \leq \lambda \) implies \( s_\lambda \in U \). To show (c) let \( E_k = \{ s_\lambda : k \leq \lambda \} \) and \( B = \{ E_k \} \). Then \( B \) is also residually in the \( U \) since it is a filterbase in \( X \) which is generated by \( \{ s_\lambda \}_{\lambda \in \Delta} \). Thus from (b), \( f(B) = \{ f(E_k) \} \) is residually in \( V \in CO(Y, f(x)) \), that is, there is an \( f(E_{k_0}) \in f(\mathcal{B}) \) such that \( f(E_{k_0}) \subset V \) and there is thus a \( k_0 \in \Delta \) such that \( f(s_{k_0}) \in V \) and \( k_0 \leq \lambda \) implies \( f(s_\lambda) \in V \) because \( E_{k_0} = \{ s_\lambda : k_0 \leq \lambda \} \). Hence \( \{ f(s_\lambda) \}_{\lambda \in \Delta} \) is is residually in \( V \). So (c) holds.

(c) \( \Rightarrow \) (a): Suppose that \( f \) is not \( \text{sl.}\alpha.C. \) at \( x \in X \). Then there exists a \( V \in CO(Y, f(x)) \) such that \( f(U) \notin V \) for each \( U \in \alpha O(X, x) \). Thus \( U \notin f^{-1}(V) \). For each \( U \in \alpha O(X, p) \), we have \( U \subset Y - f^{-1}(V) = f^{-1}(Y - V) \). So \( U \cap f^{-1}(Y - V) \neq \emptyset \). In order to find a net not \( \alpha \)-converging to \( f(x) \), we may partially order \( \alpha O(X, x) \) by set-inclusion and also direct it by \( \leq \) as defined by \( A \leq B \) iff \( B \subset A \) for each \( A, B \in \alpha O(X, x) \). Let \( s : \alpha O(X, x) \to X \) be a selection function defined by \( s(U) \equiv s_U \subset U \cap f^{-1}(Y - V) \) for each \( U \in \alpha O(X, x) \). Then \( \{ s_U \}_{U \in \alpha O(X, x)} \) is a net in \( X \) \( \alpha \)-converging to \( x \). Since \( s_U \in U \cap f^{-1}(Y - V) \) and \( f(s_U) \in f(U \cap f^{-1}(Y - V)) \subset f(U) - V \), we have \( f(s_U) \notin V \) for each \( U \in \alpha O(X, x) \). Thus \( \{ f(s_U) \}_{U \in \alpha O(X, x)} \) is not residually in \( V \in CO(Y, f(x)) \). It contradicts. Thus \( f \) is \( \text{sl.}\alpha.C. \). \( \square \)
Corollary 3.1. For a function \( f : X \to Y \), the following are equivalent:

(a) \( f \) is \( \text{sI.a.C.} \).

(b) For each \( x \in X \) and each filterbase \( \mathcal{B} \) in \( X \) \( \alpha \)-converging to \( x \), \( f(\mathcal{B}) \) \( c \)-converges to \( f(x) \).

(c) For each \( x \in X \) and each net \( \{s_\lambda\}_{\lambda \in D} \) in \( X \) \( \alpha \)-converging to \( x \), \( \{f(s_\lambda)\}_{\lambda \in D} \) \( c \)-converges to \( f(x) \).

Definition 3.3. A space \( X \) is called:

(a) \( \alpha \)-Hausdorff [11] (resp. ultra Hausdorff (written as UT\( \alpha \)) [18]) if every two distinct points of \( X \) can be separated by disjoint \( \alpha \)-open (resp. clopen) sets,

(b) ultra normal [18] if each pair of nonempty disjoint closed sets can be separated by disjoint clopen sets,

(c) mildly compact [18] if every clopen cover of \( X \) has a finite subcover,

(d) quasi H-closed (written as QHC) [16] if every open cover of \( X \) has a finite proximate subcover,

(d) \( F \)-closed (written as FC) [2] if every \( \alpha \)-open cover of \( X \) has a finite proximate subcover.

Definition 3.4. A space \( X \) is called \( \alpha \)-normal if each pair of nonempty disjoint closed sets can be separated by disjoint \( \alpha \)-open sets.

Theorem 3.2. If \( f : X \to Y \) is an \( \text{sI.a.C.} \) injection and \( Y \) is UT\( \alpha \), then \( X \) is \( \alpha \)-Hausdorff.

Proof. Let \( x_1, x_2 \in X \) and \( x_1 \neq x_2 \). Then there are \( V_1, V_2 \in \text{CO}(Y) \) such that \( f(x_1) \in V_1, f(x_2) \in V_2 \) and \( V_1 \cap V_2 = \emptyset \) because \( Y \) is UT\( \alpha \). By Theorem 2.1, \( x_i \in f^{-1}(V_i) \in \alpha O(X) \) for \( i = 1, 2 \). Since \( f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \), \( X \) is \( \alpha \)-Hausdorff. \( \square \)

Theorem 3.3. If \( f : X \to Y \) is an \( \text{sI.a.C.} \) and closed injection and \( Y \) is ultra normal, then \( X \) is \( \alpha \)-normal.
**Proof.** Let $F_1$ and $F_2$ be any disjoint closed subsets of $X$. Since $Y$ is ultra normal, two disjoint closed subsets of $Y$, $f(F_1)$ and $f(F_2)$, are separated by disjoint clopen sets $V_1$ and $V_2$, respectively. So by Theorem 2.1, $F_i \subset f^{-1}(V_i), f^{-1}(V_i) \in \alpha O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus $X$ is $\alpha$-normal. □

**Lemma 3.1.** QHC spaces coincide with FC spaces.

**Proof.** Let $X$ be FC. Then $X$ is also QHC because $\tau(X) \subset \alpha O(X)$. Conversely, let $X$ be QHC and let $\mathcal{G} = \{U_i \mid U_i \in \alpha O(X), i \in \nabla\}$ such that $X \subset \bigcup_{i \in \nabla} U_i$. Then for each $i \in \nabla$, $U_i \subset \text{IntClInt}(U_i)$ because $U_i \in \alpha O(X)$. Thus $X \subset \bigcup_{i \in \nabla} \text{IntClInt}(U_i)$. Since $X$ is QHC and $\mathcal{G}^* = \{\text{IntClInt}(U_i) \mid i \in \nabla\}$ is an open cover of $X$, there exists a finite subset $\nabla_0 = \{i_1, i_2, \ldots, i_n\}$ of $\nabla$ such that $X \subset \bigcup_{k=1}^{k=m} \text{Cl}(\text{IntClInt}(U_{i_k}))$ since $\text{IntClInt}(U_{i_k}) \subset \text{Cl}(U_{i_k})$ for $k = 1, 2, \ldots, n$, we have $X \subset \bigcup_{k=1}^{k=m} \text{Cl}(U_{i_k})$. Hence $X$ is FC. □

**Theorem 3.4.** If $f : X \to Y$ is an sl.$\alpha$.C surjection and $X$ is quasi $H$-closed, then $Y$ is mildly compact.

**Proof.** Let $\{V_\lambda \mid V_\lambda \in CO(Y), \lambda \in \nabla\}$ be a cover of $Y$. Since $f$ is sl.$\alpha$.C., $f^{-1}(V_\lambda) \in \alpha O(X)$ for each $\lambda \in \nabla$. Thus $\{f^{-1}(V_\lambda) \mid \lambda \in \nabla\}$ is an $\alpha$-open cover of $X$. Since $X$ is quasi $H$-closed and is thus FC from Lemma 3.1, there is a finite subclass $\nabla_0$ of $\nabla$ such that $X \subset \bigcup_{\alpha \in \nabla_0} \text{Cl}(f^{-1}(V_\alpha))$. Since $f^{-1}(V_\alpha) \in \alpha O(X), f^{-1}(V_\alpha) \subset \text{IntClInt}(f^{-1}(V_\alpha))$ and so $\text{Cl}(f^{-1}(V_\alpha)) \subset \text{ClIntClInt}(f^{-1}(V_\alpha)) \subset \text{ClIntCl}(f^{-1}(V_\alpha))$. Moreover, by Theorem 2.1 $f^{-1}(V_\alpha)$ is $\alpha$-closed and $\text{ClIntCl}(f^{-1}(V_\alpha)) \subset f^{-1}(V_\alpha)$ Consequently, we obtain $X = \bigcup_{\alpha \in \nabla_0} \text{Cl}(f^{-1}(V_\alpha)) \subset \bigcup_{\alpha \in \nabla_0} f^{-1}(V_\alpha)$. Therefore, $Y = \bigcup_{\alpha \in \nabla} V_\alpha$. Hence $Y$ is mildly compact. □

**Example 3.1.** Let $(R, T)$ and $(R, U)$ be the indiscrete and the usual space of set of real numbers, respectively. Then the identity $I : (R, T) \to (R, U)$ is sl.$\alpha$.C., but not a.C.

**Theorem 3.5.** If $f : X \to Y$ is sl.$\alpha$.C. and $Y$ is extremally disconnected, then $f$ is a.C.
Proof. Let \( x \in X \) and \( V \in \tau(Y, f(x)) \). Since \( Y \) is extremally disconnected, \( Cl(V) \in \mathcal{O}(Y) \) and by Theorem 2.1 \( f^{-1}(Cl(V)) \) is \( \alpha \)-open and \( \alpha \)-closed in \( X \). Therefore, we have \( x \in f^{-1}(V) \subset f^{-1}(Cl(V)) \subset \text{IntCl} \cap f^{-1}(Cl(V)) \subset Cl(\text{IntCl}(f^{-1}(Cl(V)))) \subset f^{-1}(Cl(V)) \). Putting \( U = \text{IntCl}(f^{-1}(Cl(V))) \), \( U \) is an open set of \( X \), \( x \in U \) and \( f(U) \subset Cl(V) = \text{IntCl}(V) \). This shows that \( f \) is a C. □

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