ON MAXIMAL PRERADICAL RATIONAL EXTENSIONS

Yong Uk Cho

Abstract. The concepts of $t$-rational extensions and $t$-essential extensions of modules, where $t$ a preradical for $R$-Mod, are introduced. The structures of such extensions are determined. Relations between maximal $t$-rational extensions and other concepts of modules are studied.

1. Introduction

The notion of rational extensions of modules was introduced by Utumi [7] and Findlay-Lambek [2]. For a preradical $t$, as torsion theoretically, we will define $t$-rational extensions of modules. This is a dual concept of $t$-corational extensions [5].

In general, rational extensions do not preserve direct sums. However, $t$-rational extensions preserve direct sums. In this paper, first, we determine the form of $t$-rational extension of $t$-torsionfree module.

Next, we show that every $t$-torsion free module has the maximal $t$-rational extension uniquely up to isomorphism. Moreover, we characterize $t$-rationally complete module.

Throughout this paper, $R$ denotes a ring with identity and all modules are unitary left $R$-modules. We denote the category of all modules by $R$-Mod and the injective hull of a module $A$ by $E(A)$.

For a preradical $t$ of $R$-Mod, a module $A$ is said to be $t$-torsion (resp. $t$-torsionfree) if $t(A) = A$ (resp. $t(A) = 0$). The $t$-torsion class (resp. $t$-torsionfree class) of modules or the class of $t$-torsion (resp.

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t-torsionfree) modules will be denoted by $T(t)$ (resp. $F(t)$). Also, for two preradicals $t$ and $s$ of $R$-$\text{Mod}$, we shall say that $s$ is less than $t$ (or $t$ is larger than $s$) if $s(A) \subseteq t(A)$ for every module $A$.

We denote the left linear topology corresponding to a left exact preradical $t$ by $L(t)$, that is,

$$L(t) = \{ R \leq R | R/I \in T(t) \}$$

Also, for each module $M$, $Z(M)$ denotes the singular submodule of $M$, that is,

$$Z(M) = \{ x \in M | Ix = 0, \text{ for some essential left ideal } I \text{ of } R \}$$

We refer the reader to Stenström [6] for additional terminology and properties of preradicals and torsion theories.

2. t-rational extensions

Let $Q$ be a module. We define a preradical $k_Q$ by

$$k_Q(M) = \cap \{ \text{Ker} f | f \in \text{Hom}_R(M,Q) \}$$

for all modules $M$. As is well known, $k_Q$ is the largest preradical $t$ such that $t(Q) = 0$. In fact, $k_Q$ is a radical. All preradicals in this paper are over $R$-$\text{Mod}$ for a fixed ring $R$. Let $t$ be a preradical. We call an exact sequence $0 \rightarrow A \rightarrow B$ is a t-rational extension of $A$ if $B$ is in $F(t)$ and any submodule of $\text{Coker}(\alpha)$ is in $T(t)$. If $\alpha$ is an inclusion map, then we say that $A$ is a t-rational submodule of $B$.

It is immediate that an exact sequence $0 \rightarrow A \rightarrow B$ is a rational extension of $A$ if and only if it is a $k_B$-rational extension of $A$. Also, if an exact sequence $0 \rightarrow A \rightarrow B$ is a t-rational extension of $A$, then it is a rational extension of $A$

The converse of the above remark is not true in general, as follows:

**Example 2.1.** Let $R$ be the ring $\mathbb{Z}$ of integers. We put $A = 12\mathbb{Z}$, $B = \mathbb{Z}$ and $t = \text{Soc}$. Since $Z(B) = 0$ and $A$ is an essential submodule of $B$, $0 \rightarrow A \rightarrow B$ is a rational extension of $A$, where $i$ an inclusion map. However, $\text{Soc}(B/A) = 2\mathbb{Z}/12\mathbb{Z} \neq B/A$. Thus $0 \rightarrow A \rightarrow B$ is not a t-rational extension of $A$. 


On the other hand, for any preradical \( t \), an exact sequence \( 0 \to A \xrightarrow{\alpha} B \) is called a \( t \)-essential extension of \( A \) if it is an essential extension of \( A \) and every submodule of \( \operatorname{Coker}(\alpha) \) is in \( T(t) \). If \( \alpha \) is an inclusion map, we say that \( A \) is a \( t \)-essential submodule of \( B \).

Obviously, we see that if \( 0 \to A \xrightarrow{\alpha} B \) is a \( t \)-rational extension of \( A \), then it is a \( t \)-essential extension of \( A \).

The converse of this remark is not true in general, as follows:

**Example 2.2.** Let \( R \) be the ring \( \mathbb{Z} \) of rational integers. We put \( A = 4\mathbb{Z}/8\mathbb{Z} \), \( B = \mathbb{Z}/8\mathbb{Z} \) and \( t = \mathbb{Z} \) be the singular torsion functor. Then \( Z(B) = B \), and so \( Z(B/A) = B/A \). Also since \( A \) is an essential submodule of \( B \), \( 0 \to A \xrightarrow{i} B \) is a \( t \)-essential extension of \( A \), where \( i \) is an inclusion map. However, \( 0 \to A \xrightarrow{i} B \) is not a \( t \)-rational extension of \( A \).

We note that for any left exact preradical \( t \), an exact sequence \( 0 \to A \xrightarrow{\alpha} B \) is a \( t \)-rational extension of \( A \) if and only if \( B \) is in \( F(t) \) and \( \operatorname{Coker}(\alpha) \) is in \( T(t) \).

Let \( t \) be a preradical. We say that a left \( R \)-module \( A \) is \( t \)-injective if the functor \( \operatorname{Hom}_R(-, A) \) preserves exactness for all exact sequences

\[
0 \to C' \to C \to C'' \to 0
\]

with \( C'' \in T(t) \).

**Proposition 2.3.** Let \( t \) be a left exact radical, and \( 0 \to A \xrightarrow{\alpha} B \) and \( 0 \to B \xrightarrow{\beta} C \) be two exact sequences. Then \( 0 \to A \xrightarrow{\alpha} B \) is a \( t \)-rational extension of \( A \) and \( 0 \to B \xrightarrow{\beta} C \) is a \( t \)-rational extension of \( B \) if and only if \( 0 \to A \xrightarrow{\beta \alpha} C \) is a \( t \)-rational extension of \( A \).

**Proof.** Assume that both \( 0 \to A \xrightarrow{\alpha} B \) and \( 0 \to B \xrightarrow{\beta} C \) are \( t \)-rational extensions. Then since

\[
0 \to \beta(B)/\beta\alpha(A) \to C/\beta\alpha(A) \to C/\beta(B) \to 0
\]

is exact,

\[
(C/\beta\alpha(A))/(\beta(B)/\beta\alpha(A)) \cong C/\beta(B)
\]
On the other hand, since $\beta(B)/\beta\alpha(A) \cong B/\alpha(A)$, $\beta(B)/\beta\alpha(A)$ and $C/\beta(B)$ are in $T(t)$, we see that $C/\beta\alpha(A)$ is in $T(t)$. Thus $0 \to A^{\beta\alpha}C$ is a $t$-rational extension of $A$.

Conversely, suppose that $0 \to A^{\beta\alpha}C$ is a $t$-rational extension of $A$. We must show that $B/\alpha(A)$ and $C/\beta(B)$ are in $T(t)$. Since

$$B/\alpha(A) \cong \beta(B)/\beta\alpha(A) \subseteq C/\beta\alpha(A),$$

$B/\alpha(A)$ is in $T(t)$. On the other hand, since

$$C/\beta(B) \cong (C/\beta\alpha(A))/(\beta(B)/\beta\alpha(A))$$

which is contained in $T(t)$, it follows that $C/\beta(B)$ is in $T(t)$. This completes the proof. □

**Proposition 2.4.** Let $t$ be a left exact preradical and

$\{0 \to A_\lambda^{\alpha_\lambda}B_\lambda\}_{\lambda \in \Lambda}$ be a family of $t$-rational extensions. Then

$$0 \to \oplus_{\lambda \in \Lambda} A_\lambda^{\alpha_\lambda}B_\lambda$$

is a $t$-rational extension.

**Proof.** Since $B_\lambda$ is in $F(t)$ for all $\lambda \in \Lambda$, $\oplus_{\lambda \in \Lambda} B_\lambda$ is in $F(t)$. Also,

$$f : \oplus_{\lambda \in \Lambda} B_\lambda / \oplus_{\lambda \in \Lambda} \alpha_\lambda(A_\lambda) \to \oplus_{\lambda \in \Lambda} (B_\lambda/\alpha_\lambda(A_\lambda)),$$

which is defined by

$$f((b_\lambda)_{\lambda \in \Lambda} + \oplus_{\lambda \in \Lambda} \alpha_\lambda(A_\lambda)) = (b_\lambda + \alpha_\lambda(A_\lambda))_{\lambda \in \Lambda}$$

is a monomorphism. Since $\oplus_{\lambda \in \Lambda} (B_\lambda/\alpha_\lambda(A_\lambda))$ is in $T(t)$,

$$\oplus_{\lambda \in \Lambda} B_\lambda / \oplus_{\lambda \in \Lambda} \alpha_\lambda(A_\lambda)$$

is in $T(t)$. Thus $0 \to \oplus_{\lambda \in \Lambda} A_\lambda^{\alpha_\lambda}B_\lambda$ is a $t$-rational extension. □

Let $t$ be a preradical. We call a module $A$ $t$-uniform (resp. strongly $t$-uniform) if every nonzero submodule of $A$ is $t$-essential (resp. $t$-rational) submodule. Clearly, every strongly $t$-uniform module is $t$-uniform. Also, every nonzero submodule of a strongly $t$-uniform module is strongly $t$-uniform. If $R$ is $t$-uniform (resp. strongly $t$-uniform), we say that the ring $R$ is left $t$-uniform (resp. left strongly $t$-uniform).

If $A$ is in $F(t)$ and $t$-uniform, then it is strongly $t$-uniform.
Proposition 2.5. Let \( t \) be a left exact radical and \( M \) be a submodule of a module \( N \). If \( M \) is both a strongly \( t \)-uniform and a \( t \)-rational submodule of \( N \), then \( N \) is strongly \( t \)-uniform.

Proof. Let \( N' \) be a nonzero submodule of \( N \). Then \( 0 \neq N' \cap M \) is a \( t \)-rational submodule of \( M \). Also, since \( N' \cap M \subseteq M \subseteq N \) and \( M \) is a \( t \)-rational submodule of \( N \), \( N' \cap M \) is a \( t \)-rational submodule of \( N \), by Proposition 2.3. Thus \( N' \) is a \( t \)-rational submodule of \( N \). Consequently, \( N \) is strongly \( t \)-uniform.

A ring \( R \) is called a domain if \( R \) has no nonzero divisors of zero and a left Ore domain if, in addition, \( Ra \cap Rb \neq 0 \), for all \( a \neq 0 \) and \( b \neq 0 \) in \( R \).

Lemma 2.6. A ring \( R \) is left Ore domain if and only if \( Z(R) = 0 \) and \( R \) is left uniform.

Proof. Suppose that \( R \) is left Ore domain. Then clearly, \( R \) is left uniform and so that \( L(Z) \), the family of essential left ideals of \( R \), actually consists of all nonzero left ideals. If \( a \in Z(R) \), \( Ia = 0 \), for some \( I \in L(Z) \). Thus \( a = 0 \).

Conversely, assume that \( ab = 0 \) and \( a \neq 0 \) in \( R \). Then \( Rab = 0 \) and \( Ra \in L(Z) \), that is, \( b \in Z(R) = 0 \). Consequently, \( R \) is a domain.

Corollary 2.7. Let \( t \) be a preradical. If \( R \) is left strongly \( t \)-uniform, then \( R \) is a left Ore domain. The converse is not true in general.

Proof. Assume that \( R \) is strongly \( t \)-uniform. Then clearly, \( R \) is left uniform and \( Z = k_{EC(R)} \) which is indicated previously, that is, \( Z(R) = 0 \). Therefore, by Lemma 2.6, \( R \) is left Ore domain.

Next, we put \( R = \mathbb{Z}, t = \text{Soc} \) and \( I = 8\mathbb{Z} \). Then \( R \) is left Ore domain. But

\[ \text{Soc}(R/I) = 4\mathbb{Z}/8\mathbb{Z} \neq R/I \]

This means \( I \) is not even a \( t \)-rational submodule of \( R \).

Theorem 2.8. Let \( t \) be a preradical and \( 0 \rightarrow A \rightarrow B \) is a \( t \)-rational extension of \( A \) and \( 0 \rightarrow A \rightarrow B' \) an exact sequence. If there exist homomorphisms \( f : B \rightarrow B' \) and \( g : B' \rightarrow B \) such that \( f\alpha = \alpha' \) and \( g\alpha' = \alpha \), then \( gf = 1_B \).
Proof. By the assumption, $gf\alpha = \alpha$ and so $(gf - 1_B)\alpha = 0$. Thus we have
\[ \alpha(A) \subseteq \text{Ker}(gf - 1_B) \]
and
\[ B/\text{Ker}(gf - 1_B) \cong \text{Im}(gf - 1_B) \subseteq B \]
Since $B/\text{Ker}(gf - 1_B)$ is $t$-torsionfree and $B/\alpha(A)$ is $t$-torsion, we see that $B/\text{Ker}(gf - 1_B) = 0$, that is, $\text{Ker}(gf - 1_B) = B$. Hence $gf - 1_B = 0$, that is, $gf = 1_B$. \hfill \Box

Corollary 2.9. Let $t$ be a preradical, and both $0 \rightarrow A \overset{\alpha}{\rightarrow} B$ and $0 \rightarrow A \overset{\alpha'}{\rightarrow} B'$ $t$-rational extensions of $A$. If there exist homomorphisms $f : B \rightarrow B'$ and $g : B' \rightarrow B$ such that $f\alpha = \alpha'$ and $g\alpha' = \alpha$, then $B \cong B'$.

3. Maximal $t$-rational extensions

Let $t$ be a preradical. We call an exact sequence $0 \rightarrow A \overset{\alpha}{\rightarrow} B$ a maximal $t$-rational extension of $A$ if

(i) $0 \rightarrow A \overset{\alpha}{\rightarrow} B$ is a $t$-rational extension of $A$,

(ii) For any $t$-rational extension $0 \rightarrow A \overset{\alpha'}{\rightarrow} B'$ of $A$, there exists a homomorphism $f : B' \rightarrow B$ such that $f\alpha' = \alpha$.

By Corollary 2.9, we obtain the following important statement

Theorem 3.1. Let $t$ be a left exact radical and $A$ a $t$-torsionfree module. Then there exists a maximal $t$-rational extension of $A$, uniquely up to isomorphism.

Proof. We put that $t(E(A)/A) = B/A$, where $A \subseteq B \subseteq E(A)$. Then as is easily seen, $0 \rightarrow A \overset{i}{\rightarrow} B$ is a $t$-rational extension of $A$ where $i$ is an inclusion map. Let $0 \rightarrow A \overset{\alpha'}{\rightarrow} B'$ be any $t$-rational extension of $A$. Then we have the following commutative diagram.

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow & & \downarrow \\
\text{E}(A) & \rightarrow & B'
\end{array}
\]
where \( t' \) is an inclusion map. Thus \( f\alpha' = t' \) and \( f \) is monomorphism, we show that \( f(B') \subseteq B \), since \( f\alpha'(A) = A \) and \( \alpha(A) \subseteq B' \), \( A \subseteq Imf \).

Also since

\[
(Imf + B)/B \cong Imf/(Imf \cap B) \cong (Imf/A)/((Imf \cap B)/A)
\]

and

\[
Imf/A \cong B'/\alpha'(A)
\]

\((Imf + B)/B\) is in \( T(t) \). Since

\[
0 \longrightarrow B/A \longrightarrow (Imf + B)/A \longrightarrow (Imf + B)/B \longrightarrow 0
\]

is exact, \( B/A \) and \( (Imf + B)/B \) are in \( T(t) \). Thus \( (Imf + B)/A \) is in \( T(t) \).

Moreover,

\[
B/A = t(E(A)/A) \supseteq t((Imf + B)/A) = (Imf + B)/A
\]

and

\[
t(B/A) = B/A.
\]

Thus \( B = Imf + B \), that is, \( Imf \subseteq B \). Hence \( 0 \longrightarrow A \longrightarrow B \) is a maximal \( t \)-rational extension of \( A \).

Finally, let two exact sequences \( 0 \longrightarrow A \longrightarrow B \) and \( 0 \longrightarrow A \longrightarrow B \)

be maximal \( t \)-rational extensions of \( A \). Then by Corollary 2.9, we have \( B_1 \cong B_2 \) \( \square \)

**Lemma 3.2** ([3], Proposition 3.3). Let \( t \) be a preradical and \( 0 \longrightarrow A \longrightarrow B \) an exact sequence. If \( B \) is \( t \)-injective and \( \text{Coker}(\alpha) \) is in \( F(t) \), then \( A \) is \( t \)-injective.

**Proof.** Since \( \alpha(A) \cong A \), we will show that \( \alpha(A) \) is \( t \)-injective. Let

\[
0 \longrightarrow X \longrightarrow Y \longrightarrow Y/X \longrightarrow 0
\]

be an exact sequence with \( Y/X \in T(t) \). Then there exists a homomorphism \( g : Y \longrightarrow B \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
0 & \longrightarrow & X \\
\downarrow f & & \downarrow \alpha(A) \\
0 & \longrightarrow & Y \\
\downarrow i & & \downarrow i \\
Y/X & \longrightarrow & B
\end{array}
\]
Thus $g$ induces a homomorphism $g': Y/X \to B/\alpha(A)$, since $Y/X$ is in $T(t)$ and $B/\alpha(A)$ is in $F(t)$, $g' = 0$. Hence $g(Y) \subseteq \alpha(A)$, that is, $\alpha(A)$ is $t$-injective.\hfill $\Box$

**Theorem 3.3.** Let $t$ be a left exact radical. If $0 \to A \to B$ is the maximal $t$-rational extension of $A$, then $B$ is $t$-injective.

**Proof** From Lemma 3.2 and the method of construction of maximal $t$-rational extensions of $A$ in Theorem 3.1, $B$ is $t$-injective.\hfill $\Box$

Let $t$ be a preradical. We call a module $A$ $t$-rationally complete if $0 \to A \to B$ a $t$-rational extension of $A$ implies $A$ is isomorphic to $B$.

**Theorem 3.4.** Let $t$ be a left exact radical and $A$ a $t$-torsion free module. Then $A$ is $t$-rationally complete if and only if $A$ is $t$-injective.

**Proof** Assume that $A$ is $t$-rationally complete. By Theorem 3.1, there exists a maximal $t$-rational extension $0 \to A \to B$ of $A$. From Theorem 3.3, $B$ is $t$-injective.

Conversely, suppose that $A$ is $t$-injective. If $0 \to A \to B$ is a $t$-rational extension of $A$, then the fact that $B/\alpha(A)$ is in $T(t)$ implies $\alpha(A)$ is a direct summand of $B$. Also, since $\alpha(A)$ is an essential submodule of $B$, $B = \alpha(A)$, that is, $A \cong B$. Thus $A$ is $t$-rationally complete.\hfill $\Box$

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Department of Mathematics
Silla University
Pusan 617-736, Korea
E-mail: yuchon@silla.ac.kr