LIE SEMIGROUPS IN $O(2,2)$

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Abstract. We study Ol' shanskiı semigroups admitting triple decompositions in the Lie group $O(2,2)$.

1. Introduction

An impotent class of subsemigroups of Lie groups is Ol'shanskiı semigroups that play the role of noncommutative analogue of tube domains in the harmonic analysis of hermitian semisimple Lie groups. In [4] the author gave some conditions for the existence of the Ol' shanskiı type semigroup (a semigroup variant of the Cartan decomposition) in a Lie group, and in [5] the authors investigate some conditions for the existence of a triple decomposition (a semigroup variant of the Harish-Chandra decomposition) from an Ol' shanskiı semigroup. The class of semigroups for which the triple decomposition obtains contains symplectic semigroups, or more generally the conformal compression semigroup of a symmetric cone in an euclidean Jordan algebra (see [5] and [7]).

The Lie algebra $so(2,2)$ of the Lie group $O_o(2,2)$, the connected component of the identity in the group of linear transformations of $\mathbb{R}^4$ preserving a metric of signature $(2,2)$, is a symmetric algebra of Cayley type, $q^+ + h + q^-$, with $\dim q^\pm = 1$. In this paper we show that this symmetric algebra induces an Ol' shanskiı semigroup in $O(2,2)$ admitting a triple decomposition.
2. Lie semigroups with triple decompositions

Let $G$ be a Lie group with Lie algebra $\mathcal{L}(G)$ and $S$ be a closed subsemigroup of $G$ with identity. The tangent wedge of $S$ is defined by

$$\mathcal{L}(S) = \{ X \in \mathcal{L}(G) : \exp(tX) \in S \text{ for all } t \geq 0 \}.$$ 

Then it is a closed convex cone containing zero and is a Lie wedge, i.e.,

$$e^{adX} \mathcal{L}(S) = \mathcal{L}(S), \forall X \in \mathcal{L}(S) \cap -\mathcal{L}(S).$$

The systematic groundwork for a Lie theory of semigroups was worked out by K. H. Hofmann, J. Hilgert and J. D. Lawson (cf. [1]).

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\tau : G \to G$ be a differentiable involution of $G$. The pair $(G, \tau)$ is called an involutive group. Then the derivative of $\tau$ at the identity $e$, $d\tau(e) : \mathfrak{g} \to \mathfrak{g}$, is a Lie algebra involution and leads to a decomposition of $\mathfrak{g}$ into the $+1$-eigenspace $\mathfrak{h}$ and $-1$-eigenspace $\mathfrak{q}$, $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, which satisfies

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, [\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{q}, [\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}.$$ 

Lie algebras with a given decomposition with these properties are called symmetric algebras. Let $H$ be a subgroup of $G_r := \{ g \in G : \tau(g) = g \}$ containing the identity component of $G_r$. If $C$ is an $\text{Ad}(H)$-invariant cone in $\mathfrak{q}$ and if $S = H(\exp C)$ is a subsemigroup of $G$ for which the mapping

$$(h, X) \mapsto h(\exp C) : H \times C \to S$$

is a homeomorphism, then $S$ is called an Ol'shanski semigroup, and the factorization $s = h(\exp X)$ for $s \in S$ is called the Ol'shanski polar factorization.

The following appears at the Theorem 3.1 in [4].

Theorem 2.1 Let $(G, \tau)$ be a finite dimensional involutive Lie group, and let $H \subset G_r$ be a closed subgroup containing the identity component of $G_r$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the corresponding symmetric decomposition of the Lie algebra of $G$, and let $\mathfrak{z}$ denote the center of $\mathfrak{g}$. Let $C$ be a closed convex cone in $\mathfrak{q}$ which is invariant under the adjoint action of $H$, and for which $\text{ad}(X)$ has real spectrum for each $X \in C$. Then the following conditions are equivalent.
(1) \((h,X) \mapsto h(\exp X) \cdot H \times C \to H(\exp C)\) is a diffeomorphism onto a closed subset of \(G\).

(2) The mapping \(\exp : \mathfrak{h} \to G/H\) defined by \(\exp(X) = H(\exp X)\) restricted to \(C\) is a diffeomorphism onto a closed subset of \(G/H\).

(3) The mapping \(\exp\) restricted to \(C\) is a diffeomorphism onto a closed subset of \(G\).

(4) If \(Z \in \mathfrak{z} \cap (C - C)\) satisfies \(\exp Z = e\), then \(Z = 0\). For each non-zero \(X \in C \cap \mathfrak{z}\), the closure of \(\exp(\mathbb{R}X)\) is not compact.

If these conditions hold, then \(S := H(\exp C)\) is a closed semigroup with the tangent wedge \(\mathcal{L}(S) = \mathfrak{h} + C\).

Let \(\mathfrak{g}\) be a symmetric algebra, \(\mathfrak{g} = \mathfrak{h} + \mathfrak{q}\). An element \(X \in \mathfrak{g}\) is called hyperbolic if the spectrum of \(\text{ad}(X)\) is real and \(\text{ad}(X)\) is semisimple (i.e., diagonalizable) as a linear operator. If a closed convex cone \(C\) in \(\mathfrak{q}\) has dense interior in the vector space \(C - C\), and if \(\text{ad}(X)\) is hyperbolic for each \(X\) in the interior of \(C\), then the cone is said to be hyperbolic.

The symmetric Lie algebra \(\mathfrak{g}\) is called a symmetric algebra of Cayley type if there exist abelian subalgebras \(\mathfrak{q}_-\) and \(\mathfrak{q}_+\) of \(\mathfrak{g}\) contained in \(\mathfrak{q}\) such that \(\mathfrak{q} = \mathfrak{q}_- \oplus \mathfrak{q}_+\). Note that the triple \((\mathfrak{q}_-, \mathfrak{h}, \mathfrak{q}_+)\) is a \((-1,0,1)\)-graded Lie algebra. Let \(H\) be a closed subgroup of \(G\) with Lie algebra \(\mathfrak{h}\). We define a smooth mapping by

\[
\phi : \mathfrak{q}_- \times H \times \mathfrak{q}_+ \to G, \quad \phi(X,h,Y) = (\exp X)h(\exp Y)
\]

The following is the principal theorem of [5].

**Theorem 2.2.** Let \((G,\tau)\) be a finite dimensional involutive Lie group such that the Lie algebra \(\mathfrak{g} = \mathfrak{q}_- + \mathfrak{h} + \mathfrak{q}_+\) is a symmetric algebra of Cayley type. Let \(H\) be a closed subgroup of \(G,\tau\) with Lie algebra \(\mathfrak{h}\). Suppose that \(C^-\) is a cone in \(\mathfrak{q}_-\) and \(C^+\) is a cone in \(\mathfrak{q}_+\) such that \(C := C^+ + C^-\) is a hyperbolic \(\text{Ad}(H)\)-invariant cone. Set \(S := (\exp C^-)h(\exp C^+)\) if any of the conditions (1)-(4) of Theorem 2.1 is satisfied, then the mapping

\[
\phi : C^- \times H \times C^+ \to S, \quad \phi(X,h,Y) = (\exp X)h(\exp Y)
\]

is diffeomorphism. If further the set \(S\) is closed, then \(S\) is semigroup and equal to the Ol’shanski\'i semigroup \(H(\exp C)\). The set \(S\) is closed and the conclusions follow in the case that \(C\) is pointed.
3. Groups with Lie algebra type $D_r$

Throughout we fix $n \in \mathbb{N}$. Let $\mathbb{R}^n$ be the Euclidean $n$-space with the usual inner product $\langle \cdot , \cdot \rangle$. We define a matrix $2n$ by $2n$ matrix $J$ by

$$J = \begin{bmatrix} O & I \\ I & O \end{bmatrix},$$

where $I = I_n$ denotes the $n \times n$ real identity matrix.

Note that $J^2 = I_{2n}$ and $J^{-1} = J = J^t$. We define the symmetric bilinear form on $\mathbb{R}^{2n}$ by

$$\langle x | y \rangle = \langle Jx | y \rangle, \quad x, y \in \mathbb{R}^{2n}.$$ 

Let

$$G := \{ M \in \text{GL}(2n, \mathbb{R}) \cdot (Mx | My) = \langle x | y \rangle \text{ for all } x, y \in \mathbb{R}^{2n} \}.$$ 

Note that for $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$, the inverse of $M$ is given by

$$M^{-1} = \begin{bmatrix} D^t & B^t \\ C^t & A^t \end{bmatrix}.$$ 

**Proposition 3.1.** Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{GL}(2n, \mathbb{R})$. Then the following are equivalent:

1. $M \in G$, i.e., $M$ preserves $\langle \cdot , \cdot \rangle$.
2. $M^tJMJ = J$.
3. $A^tC$, $B^tD$ are skew-symmetric and $A^tD + C^tB = I$.
4. $DC^t$, $BA^t$ are skew-symmetric and $DA^t + CB^t = I$.

**Proof.** Straightforward. \(\square\)

**Remark 3.2.** Let $O(n, n)$ be the group of all pseudo-orthogonal real matrices of signature $(n, n)$. Then this group can be expressed as

$$O(n, n) = \{ M \in \text{GL}(2n, \mathbb{R}) : M^t I_{n,n} M = I_{n,n} \},$$

where $I_{n,n} := \begin{bmatrix} I & O \\ O & -I \end{bmatrix}$ and $I$ is the identity matrix of size $n \times n$.

The group $O(n, n)$ has four connected components and the identity component of $O(n, n)$ is equal to the identity component of $\text{SO}(n, n) = \{ M \in \text{GL}(2n, \mathbb{R}) : M^t M = I_{2n} \}$. 
Observe that $R I_{n,n} R^{-1} = J$, where $R$ is the real Cayley transform, i.e.,

$$R = \begin{bmatrix} I & -I \\ I & I \end{bmatrix}$$

Thus we have $G = RO(n, n) R^{-1}$.

We define an involution $\tau : G \to G$ by

$$\tau \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix}.$$  

Then $(G, \tau)$ is an involutive Lie group with the fixed group

$$G_\tau = \left\{ \begin{bmatrix} A & O \\ O & (A^t)^{-1} \end{bmatrix} \cdot A \in GL(n, \mathbb{R}) \right\}.$$

Let $\mathfrak{g}$ be the Lie algebra of $G$. Then obviously, we have

$$\mathfrak{g} = \{ M \in M_{2n}(\mathbb{R}) \cdot JM^t J = -M \}$$

$$= \left\{ \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \cdot B^t = -B, C^t = -C, A \in M_{n}(\mathbb{R}) \right\}$$

$$\cong \mathfrak{so}(n, n)$$

and hence $\mathfrak{g}$ is the classical Lie algebra type $D_n$ with its dimension $2n^2 - n$. If $n \geq 3$, then $D_n$ is simple.

Furthermore, the differential $d\tau(e) \cdot \mathfrak{g} \to \mathfrak{g}$ at the identity $e$ is given by

$$d\tau(e) \left( \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \right) = \begin{bmatrix} A & -B \\ -C & -A^t \end{bmatrix}$$

and it defines a symmetric algebra

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q}.$$

Furthermore, by setting

$$\mathfrak{q}^+ = \left\{ \begin{bmatrix} O & X \\ O & O \end{bmatrix} \cdot X^t = -X \right\},$$

$$\mathfrak{q}^- = \left\{ \begin{bmatrix} O & O \\ X & O \end{bmatrix} \cdot X^t = -X \right\},$$
$g = q^+ + h + q^-$ becomes a symmetric algebra of Cayley type. Let

$$Q^+ = \left\{ \begin{bmatrix} I & X \\ O & I \end{bmatrix} : X^t = -X \right\} = \exp q^+$$

$$Q^- = \left\{ \begin{bmatrix} I & O \\ X & I \end{bmatrix} : X^t = -X \right\} = \exp q^-$$

$$H = G_\tau = \left\{ \begin{bmatrix} D & O \\ O & (D^{-1})^t \end{bmatrix} : D \in \text{GL}(n, \mathbb{R}) \right\}.$$  

Then $Q^\pm$ is a Lie subgroup of $G$ with its Lie algebra $q^\pm$, respectively.

**Proposition 3.3.** Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$ Then the following are equivalent:

1. $g \in Q^+HQ^-$
2. $D$ is invertible

**Proof.** If $g \in Q^+HQ^-$, then the element $g$ is of the form

$$\begin{bmatrix} * & (F^{-1})^t \\ * & (F^{-1})^t \end{bmatrix}$$

for some $F \in \text{GL}(n, \mathbb{R})$ and hence $D$ is invertible. Conversely, we note that if $D$ is invertible, then

$$g = \begin{bmatrix} I & BD^{-1} \\ O & I \end{bmatrix} \left( D^{-1} \right)^t \begin{bmatrix} O & A \\ D & D^{-1} \end{bmatrix} \begin{bmatrix} I & O \\ D^{-1}C & I \end{bmatrix}$$

By Proposition 3.1, $B^t = -D^tBD^{-1}$ and $C^t = -D^{-1}CD^t$. It follows that

$$(BD^{-1})^t = (D^{-1})^tB^t = -(D^{-1})^tD^tBD^{-1} = -BD^{-1}$$

and

$$(D^{-1}C)^t = C^t(D^{-1})^t = -D^{-1}CD^t(D^{-1})^t = -D^{-1}C$$

Thus $g \in Q^+HQ^-$. \hfill \Box

**Note:** The factorization in the Proposition 3.3 is uniquely determined (see Theorem 5.2 in [5]).
Remark 3.4 The set \( P := HQ^- \) is a closed subgroup of \( G \). Let \( \mathcal{M} := G/P \). Then there is a canonical imbedding

\[
(3) \quad \text{Skew}(n, \mathbb{R}) \hookrightarrow q^+ \hookrightarrow \mathcal{M}, \quad X \mapsto \begin{bmatrix} O & X \\ O & O \end{bmatrix} \mapsto \begin{bmatrix} I & X \\ O & I \end{bmatrix} P
\]

If \( n = 2 \), then \( \text{Skew}(2, \mathbb{R}) \) is one-dimensional and it contains the positive cone \( \Omega := \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \cdot x > 0 \right\} \). In the last section of this paper, we will devote to show that the compression semigroup of \( \Omega \) defined by \( \Gamma_\Omega = \{ g \in G \mid g \cdot \Omega \subset \Omega \} \) has an Ol'shanskiï and triple factorizations

4. Factorizations of \( \Gamma_\Omega \)

Throughout this section we denote by \( G \) the group of 4 by 4 real matrices \( M \) satisfying the condition

\[ M^tJM = J, \quad \text{where} \quad J = \begin{bmatrix} O & I \\ I & O \end{bmatrix}. \]

Note that \( G \) is the group \( RO(2, 2)R^{-1} \).

Let \( \mathfrak{g} \) be the Lie algebra of \( G \). Then \( \mathfrak{g} \) is the classical Lie algebra type \( D_2 \). It is well-known that

\[ D_2 \cong \mathfrak{so}(2, 2) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \]

Lemma 4.1. Let \( A \in \text{GL}(2, \mathbb{R}) \) and let \( X \in \text{Skew}(2, \mathbb{R}) \). Then \( AXA^t = A^tXA = \det(A)X \in \text{Skew}(2, \mathbb{R}) \).

Proof For \( A \in \text{GL}(2, \mathbb{R}) \) and \( X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \text{Skew}(2, \mathbb{R}) \), we have

\[
AXA^t = A^tXA = \begin{bmatrix} 0 & \det(A)x \\ -\det(A)x & 0 \end{bmatrix} = \det(A)X \in \text{Skew}(2, \mathbb{R}).
\]

\[ \square \]
For convenience, we let
\[ \text{Skew}^+(2, \mathbb{R}) := \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x \geq 0 \right\}, \]
\[ \text{Skew}^-(2, \mathbb{R}) := \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x \leq 0 \right\}, \]
and let
\[ C^+ := \left\{ \begin{bmatrix} O & X \\ O & O \end{bmatrix} : X \in \text{Skew}^+(2, \mathbb{R}) \right\}, \]
\[ C^- := \left\{ \begin{bmatrix} O & Y \\ Y & O \end{bmatrix} : Y \in \text{Skew}^-(2, \mathbb{R}) \right\}, \]
\[ C := C^+ + C^- . \]

Then \( C^- \) is a cone in \( q_- \), \( C^+ \) is a cone in \( q_+ \), and we can easily show that \( C \) is a pointed closed convex cone in the Lie algebra \( \mathfrak{g} \) of \( G \).

**Theorem 4.2.** We have \( H_o(\exp C) \) is an Ol'shanskiĭ semigroup with the following triple decomposition,
\[ H_o(\exp C) = (\exp C^+)H_o(\exp C^-), \]
where \( H_o \) is the identity component of \( H \).

**Proof.** In order to prove this theorem, we will show that our setting satisfies the conditions given in Theorem 2.2.

**Step 1:** The cone \( C \) is invariant under the adjoint action of the identity component \( H_o \) of \( H = G_r \).

We note that the identity component of \( H \) is equal to
\[ H_o = \left\{ \begin{bmatrix} A & O \\ O & (A^{-1})^t \end{bmatrix} \in H \ | \ \det(A) > 0 \right\}. \]

Let \( K = \begin{bmatrix} A & O \\ O & (A^{-1})^t \end{bmatrix} \in H_o \) and let \( C = \begin{bmatrix} O & X \\ Y & O \end{bmatrix} \in C \). Then
\[ KCK^{-1} = \begin{bmatrix} O & AXA^t \\ (A^{-1})^tYA^{-1} & O \end{bmatrix} \]
and by Lemma 4.1,
\[ AXA^t = \det(A)X \in \text{Skew}^+(2, \mathbb{R}), \]
\[ (A^{-1})^tYA^{-1} = \frac{1}{\det(A)}Y \in \text{Skew}^-(2, \mathbb{R}). \]

Thus we have the cone \( C \) is \( \text{Ad}(H_o) \)-invariant.

**Step 2:** For any \( C \in C \), \( \text{ad}(C) \) has a real spectrum.

Let
\[ C = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 \\ 0 & -y & 0 & 0 \\ y & 0 & 0 & 0 \end{bmatrix} \in C. \]

Then the characteristic polynomial of \( C \) is
\[ p(\lambda) = \det(\lambda I - C) = (\lambda^2 - xy)^2 \]
and hence \( C \) has a real spectrum. By Lemma 4.1 in [4], \( \text{ad}(C) \) has a real spectrum.

**Step 3** The cone \( C \) is hyperbolic.

We note that \( C \) is a closed convex cone in \( \mathfrak{q} \) and \( C \) has dense interior in the vector space \( C - C \). By step 2, the spectrum of \( \text{ad}(C) \) is real for each \( C \in C \). To show that \( \text{ad}(C) \) is semisimple for each \( C \) in the interior of \( C \), it is sufficient to show that \( C \) is semisimple ([3]). For \( x, y > 0 \), let
\[ C = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 \\ 0 & -y & 0 & 0 \\ y & 0 & 0 & 0 \end{bmatrix} \in C. \]

Then we can easily show that the matrix
\[ P = \begin{bmatrix} x & 0 & -x & 0 \\ 0 & -x & 0 & x \\ 0 & \sqrt{xy} & 0 & \sqrt{xy} \\ \sqrt{xy} & 0 & \sqrt{xy} & 0 \end{bmatrix} \]
diagonalizes \( C \), i.e.,
\[ P^{-1}CP = \text{diag}(\sqrt{xy}, \sqrt{xy}, -\sqrt{xy}, -\sqrt{xy}). \]
Thus $C$ is a hyperbolic cone

Obviously, the Lie algebra $\mathfrak{g}$ is semisimple and hence the center $\mathfrak{z}$ of $\mathfrak{g}$ is trivial. By Theorem 2.1 and Theorem 2.2, the proof is now complete. □

Set

$$\Omega := \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x > 0 \right\} = \text{Skew}^+(2, \mathbb{R}) \setminus \{O\},$$

$$\overline{\Omega} := \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x \geq 0 \right\} = \text{Skew}^+(2, \mathbb{R}).$$

By (3), the group $G$ naturally acts on $\mathcal{M} = G/P$. We define a compression semigroup with respect to $\overline{\Omega}$

$$\Gamma_\Omega := \{ g \in G \ : \ g \cdot \overline{\Omega} \subset \overline{\Omega} \}.$$

Theorem 4.3. We have $\Gamma_\Omega = \Gamma^+ H_o \Gamma^- = H_o (\exp C)$, where $\Gamma^+ = \exp C^+$ and $\Gamma^- = \exp C^-$.

Proof. Obviously, the sets $\Gamma^+$ and $H_o$ are contained in $\Gamma_\Omega$. To show that $\Gamma^+ H_o \Gamma^- \subset \Gamma_\Omega$, it is sufficient to show that $\Gamma^- \subset \Gamma_\Omega$. Let

$$g = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \in \Gamma^-, \text{ where } Y = \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \text{ for some } y \leq 0$$

Note that $I + Y X$ is invertible for all $X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \overline{\Omega}$. By Proposition 3.3, we have

$$\begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & X \\ Y & I + Y X \end{bmatrix} \in \begin{bmatrix} I & X(I + Y X)^{-1} \\ 0 & I \end{bmatrix} P.$$

Since

$$X(I + Y X)^{-1} = \begin{bmatrix} 0 & \frac{x}{1-xy} \\ -\frac{x}{1-xy} & \frac{1}{1-xy} \end{bmatrix}$$
and \( \frac{x}{1 - xy} \geq 0 \) for \( x \geq 0, y \leq 0 \), we have
\[
g \cdot X = X(I + YX)^{-1} \in \overline{\Omega}.
\]

Now suppose that
\[
g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma \Omega.
\]
Then since \( g \cdot 0 \in \overline{\Omega} \), \( g \in Q^+P \). By Proposition 3.3, \( D \) is invertible and we have the following factorization
\[
g = \begin{bmatrix} I & BD^{-1} \\ O & I \end{bmatrix} \begin{bmatrix} (D^{-1})^t & O \\ O & D \end{bmatrix} \begin{bmatrix} I & O \\ D^{-1}C & I \end{bmatrix}
\]
with \( BD^{-1} = g \cdot 0 \in \overline{\Omega} \). Thus the first term in the righthand of (4) belongs to \( \Gamma^+ \), i.e.,
\[
\begin{bmatrix} I & BD^{-1} \\ O & I \end{bmatrix} \in \Gamma^+.
\]
For convenience, we let
\[
Y = D^{-1}C = \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix}.
\]
Since \( g \cdot \overline{\Omega} \subset \overline{\Omega} \),
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & X \\ O & I \end{bmatrix} = \begin{bmatrix} * & * \\ * & CX + D \end{bmatrix} \in Q^+P
\]
for all \( X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \overline{\Omega} \). By Proposition 3.3, \( CX + D \) is invertible and hence
\[
I + YX = I + D^{-1}CX = D^{-1}(D + CX)
\]
is invertible. We note that
\[
I + YX \text{ is invertible for all } X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \overline{\Omega}
\]
\[
\iff \begin{bmatrix} 1 - xy & 0 \\ 0 & 1 - xy \end{bmatrix} \text{ is invertible for all } x \geq 0
\]
\[
\iff y \leq 0
\]
Thus we have that the last term in the righthand of (4) belongs to $\Gamma^-$, i.e.,

$$
\begin{bmatrix}
1 \\
D^{-1}C \\
O \\
I
\end{bmatrix} \in \Gamma^-.
$$

Finally, to show that the middle term in the righthand of (4) is contained in $H_o$, we have to prove that $\det(D) > 0$.

By (4.1), $g \cdot \overline{\Omega} \subseteq \overline{\Omega}$ implies that

$$
g \cdot X = BD^{-1} + (D^t)^{-1}X(I + YX)^{-1}D^{-1} \in \overline{\Omega}
$$

for all $X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \overline{\Omega}$. Since $BD^{-1} \in \text{Skew}^+(2, \mathbb{R})$, we can write

$$
BD^{-1} = \begin{bmatrix} 0 & z \\ -z & 0 \end{bmatrix} \text{ with } z \geq 0.
$$

Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the matrix $BD^{-1} + (D^t)^{-1}X(I + YX)^{-1}D^{-1}$ is of the form

$$
\begin{bmatrix}
0 & z + \frac{x(ad - bc)}{1 - xy} \\
-z - \frac{x(ad - bc)}{1 - xy} & 0
\end{bmatrix}.
$$

Thus we have

$$
z + \frac{x(ad - bc)}{1 - xy} \geq 0 \text{ for all } x \geq 0.
$$

It follows that $\det(D) = ad - bc > 0$. We have

$$
(D^{-1})^t \begin{bmatrix} O \\ D \end{bmatrix} \in H_o.
$$

By (4), (5), (6), and (7), we have $g \in \Gamma^+H_o\Gamma^-$. \hfill \Box

**Theorem 4.4.** We have $\Gamma_\Omega = \{g \in G : g \cdot \Omega \subseteq \Omega\}$.

**Proof.** Let $S = \{g \in G : g \cdot \Omega \subseteq \Omega\}$ and let $g \in \Gamma_\Omega$. Since $g \in \Gamma_\Omega = \Gamma^+H_o\Gamma^-$, the element $g$ is of the form

$$
\begin{bmatrix}
I & A \\
O & I
\end{bmatrix} \begin{bmatrix} (D^{-1})^t & O \\ O & D \end{bmatrix} \begin{bmatrix} I & O \\ O & I \end{bmatrix}.
$$
By Theorem 3.3, we have
\[ g \left[ \begin{array}{cc} I & X \\ O & I \end{array} \right] \in \left[ \begin{array}{cc} I & A + (D^t)^{-1} X (I + BX)^{-1} D^{-1} \\ O & I \end{array} \right] P \]
for all
\[ X = \left[ \begin{array}{cc} 0 & x \\ -x & 0 \end{array} \right] \in \Omega \]
Let
\[ A = \left[ \begin{array}{cc} 0 & a \\ -a & 0 \end{array} \right] \quad \text{and} \quad B = \left[ \begin{array}{cc} 0 & b \\ -b & 0 \end{array} \right] \]
for some \( a \geq 0, b \leq 0 \). Then \( g \cdot X = A + (D^t)^{-1} X (I + BX)^{-1} D^{-1} \) is of the form
\[ \left[ \begin{array}{cc} 0 & a + x \frac{\det(D)}{1 - xb} \\ -(a + x \frac{\det(D)}{1 - xb}) & 0 \end{array} \right] . \]
Since \( \det(D) > 0 \) and since \( x > 0 \),
\[ a + x \frac{\det(D)}{1 - xb} > 0 \]
Therefore \( g \cdot X \in \Omega \) and hence \( \Gamma_\Omega \subset S \).

Conversely, suppose that \( g \in S \). Set for each \( n \in \mathbb{N} \),
\[ h_n = \left[ \begin{array}{cc} I & X_n \\ O & I \end{array} \right] , \quad \text{where} \quad X_n = \left[ \begin{array}{cc} 0 & \frac{1}{n} \\ -\frac{1}{n} & 0 \end{array} \right] \]
Then \( h_n \in \Gamma_\Omega \) and \( h_n \cdot \overline{\Omega} \subset \Omega \). It follows that
\[ g h_n \overline{\Omega} \subset g \cdot \Omega \subset \Omega \subset \overline{\Omega} \]
Thus we have \( g h_n \in \Gamma_\Omega \) for all \( n \in \mathbb{N} \). Since \( h_n \) converges to the identity of \( G \) and since \( \Gamma_\Omega \) is closed, we have
\[ g = \lim_{n \to \infty} g h_n \in \Gamma_\Omega \]
This completes the proof. \( \square \)

Let
\[ \text{SL}(2, \mathbb{R})^+ := \left\{ \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in \text{SL}(2, \mathbb{R}) : a, b, c, d \geq 0 \right\} . \]
Then $\text{SL}(2, \mathbb{R})^+$ is a closed subsemigroup of $\text{SL}(2, \mathbb{R})$ with the following factorization.

**Lemma 4.5.** We have $\text{SL}(2, \mathbb{R})^+ = UDL$, where

$$U = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \geq 0 \right\}, L = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} : x \geq 0 \right\}$$

and

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a > 0 \right\}$$

**Proof.** Proposition 3.2 in [6]. \hfill \Box

Now we let

$$H' = \left\{ \begin{bmatrix} \lambda I & O \\ O & \frac{1}{\lambda} I \end{bmatrix} : \lambda > 0 \right\}$$

$$H'' = \left\{ \begin{bmatrix} D & O \\ O & (D^{-1})^t \end{bmatrix} : D \in \text{SL}(2, \mathbb{R}) \right\}.$$ Then $H'$ and $H''$ are subgroups of $H_o$ with $H_o = H''H'$.

**Theorem 4.6.** We have

(1) $H' \exp C = \Gamma^+ H' \Gamma^-$ and is a subsemigroup of $H_o \exp C$.

(2) $\Gamma^+ H' \Gamma^-$ is isomorphic to $\text{SL}(2, \mathbb{R})^+$.

**Proof.** (1) Let $g, h \in \Gamma^+ H' \Gamma^-$. Then

$$g = \begin{bmatrix} I & X \\ O & I \end{bmatrix} \begin{bmatrix} \lambda I & O \\ O & \frac{1}{\lambda} I \end{bmatrix} \begin{bmatrix} I & O \\ Y & I \end{bmatrix}$$

and

$$h = \begin{bmatrix} I & X' \\ O & I \end{bmatrix} \begin{bmatrix} \lambda' I & O \\ O & \frac{1}{\lambda'} I \end{bmatrix} \begin{bmatrix} I & O \\ Y' & I \end{bmatrix}$$

for some $\lambda, \lambda' > 0$, $x, x' \geq 0$ and $y, y' \leq 0$ with

$$X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}, \quad X' = \begin{bmatrix} 0 & x' \\ -x' & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix}, \quad Y' = \begin{bmatrix} 0 & y' \\ -y' & 0 \end{bmatrix}.$$
We have

\[(9) \quad gh = \begin{bmatrix} I & A \\ O & I \end{bmatrix} \begin{bmatrix} (D^{-1})^t & O \\ O & D \end{bmatrix} \begin{bmatrix} I & O \\ B & O \end{bmatrix},\]

where

\[A = \begin{bmatrix} 0 & \frac{\lambda^2 x' - xyx' + x}{1 - yx'} \\ -\frac{\lambda^2 x' - xyx' + x}{1 - yx'} & 0 \end{bmatrix} \in \text{skew}(2, \mathbb{R})^+\]

\[B = \begin{bmatrix} 0 & \frac{\lambda^2 y - yx'y' + y'}{1 - yx'} \\ -\frac{\lambda^2 y - yx'y' + y'}{1 - yx'} & 0 \end{bmatrix} \in \text{skew}(2, \mathbb{R})^-\]

\[D = \frac{1 - yx'}{\lambda x}, \quad \frac{1 - yx'}{\lambda x} > 0\]

Thus we have \(gh \in \Gamma^+ H' \Gamma^-\) Therefore, \(\Gamma^+ H' \Gamma^-\) is a subsemigroup of \(\Gamma_\Omega\). Now since \(H'\) is closed, \(\Gamma^+ H' \Gamma^-\) is a closed subsemigroup of \(G\). By considering the tangent wedge of this semigroup, we conclude that \(H' \exp C \subset \Gamma^+ H' \Gamma^-\). Conversely, suppose that \(g \in \Gamma^+ H' \Gamma^-\). Then since \(g \in \Gamma_\Omega = H_o \exp C, g = h \exp C\) for some \(h \in H_o, C \in \mathbb{C}\). We show that \(h \in H'\). Let

\[C = \begin{bmatrix} O & X \\ Y & O \end{bmatrix}\]

with \(X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -y \\ y & 0 \end{bmatrix}, x, y \geq 0\).

Then by a direct matrix computation, the \(H_o\)-part of \(\exp C\) is

\[h_o := \begin{bmatrix} \text{sech}(xy)^{\frac{1}{2}} I & 0 \\ 0 & \cosh(xy)^{\frac{1}{2}} I \end{bmatrix} \]

Therefore \(hh_o \in H'\) which implies that \(h \in H'\).

(2) Define a mapping \(f : \Gamma^+ H' \Gamma^- \rightarrow \text{SL}(2, \mathbb{R})^+\) by

\[f(g) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix}\]

for

\[g = \begin{bmatrix} I & X \\ O & I \end{bmatrix} \begin{bmatrix} \lambda I & O \\ O & \frac{1}{\lambda} I \end{bmatrix} \begin{bmatrix} I & O \\ Y & I \end{bmatrix} \in \Gamma^+ H' \Gamma^-\].
\[ X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \]

Since the factorization of $\Gamma^+ \Gamma^-$ is unique and since the factorization of $\text{SL}(2, \mathbb{R})^+$ is unique, the map $f$ is well-defined and is injective. Clearly, $f$ is surjective. To complete the proof, it remains that $f$ is a homomorphism from $\Gamma^+ \Gamma^-$ to $\text{SL}(2, \mathbb{R})^+$.

Let $g, h \in \Gamma^+ \Gamma^-$ be of the form (8). Then by (9) we have

\[ f(gh) = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -l & 1 \end{bmatrix}, \]

where

\[ u = \frac{\lambda^2 x' - xy x' + x}{1 - yx'} \geq 0 \]
\[ l = \frac{\lambda' y - yx'y' + y'}{1 - yx'} \leq 0 \]
\[ \rho = \frac{\lambda \lambda'}{1 - yx'} > 0. \]

Since

\[ f(g) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix} \]
\[ f(h) = \begin{bmatrix} 1 & x' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda' & 0 \\ 0 & \frac{1}{\lambda'} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y' & 1 \end{bmatrix}, \]

we have

\[ f(g)f(h) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \]

where

\[ b = \frac{\lambda}{\lambda'} x' - \frac{1}{\lambda \lambda'} xy x' + \frac{1}{\lambda \lambda'} x \geq 0 \]
\[ c = -\frac{\lambda'}{\lambda} y + \frac{1}{\lambda \lambda'} yx'y - \frac{1}{\lambda \lambda'} y' \geq 0 \]
\[ d = \frac{1 - yx'}{\lambda \lambda'} > 0. \]
Since $d \neq 0$,

$$f(g)f(h) = \begin{bmatrix} 1 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d^{-1} & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d^{-1}c & 1 \end{bmatrix}$$

and $bd^{-1} = u, d^{-1}c = -l$ and $d^{-1} = \rho$. The proof now is complete □

REFERENCES


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