SL(2, C)-REPRESENTATION VARIETIES OF PERIODIC LINKS

SANG YOUL LEE

Abstract In this paper, we characterize SL(2, C)-representations of an n-periodic link \( \hat{L} \) in terms of SL(2, C)-representations of its quotient link \( L \) and express the SL(2, C)-representation variety \( R(\hat{L}) \) of \( \hat{L} \) as the union of \( n \) affine algebraic subsets which have the same dimension. Also, we show that the dimension of \( R(\hat{L}) \) is bounded by the dimensions of affine algebraic subsets of the SL(2, C)-representation variety \( R(L) \) of its quotient link \( L \).

1. Introduction

Let \( L \) be a tame link in the 3-sphere \( S^3 \) and let \( G = \pi_1(S^3 - L) \) be the fundamental group of the complement \( S^3 - L \). Let \( R(G) \) denote the set of all representations of \( G \) in the 2 \times 2 \) special linear group SL(2, C) with entries in the field C of complex numbers. Suppose we fix a finite system of generators of \( G \), say \((g_1, \ldots, g_m)\). Then a representation \( \rho : G \to SL(2, C) \) is uniquely determined by specifying the \( m \)-tuple \( (\rho(g_1), \ldots, \rho(g_m)) \). We define \( R(G) = \{(\rho(g_1), \ldots, \rho(g_m)) \in SL(2, C)^m \mid \rho \in R(G)\} \). Then \( R(G) \) carries with it the structure of an affine algebraic set in \( C^{4m} \). Throughout this paper we shall call it the SL(2, C)-representation variety of \( L \) and denote it by \( R(L) \). SL(2, C)-representation varieties of knots and links and their applications have

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been studied extensively by many mathematicians. For examples, see [2, 4, 5, 6, 7, 13, 14, 15] and therein.

A link \( \tilde{L} \) in \( S^3 \) is said to have period \( n \) (\( n \geq 2 \)) if there exists an \( n \)-periodic homeomorphism \( \phi \) from \( S^3 \) onto itself such that \( \tilde{L} \) is invariant under \( \phi \) and the fixed point set \( \tilde{K}_1 \) of the \( \mathbb{Z}_n \)-action induced by \( \phi \) is homeomorphic to a 1-sphere in \( S^3 \) disjoint from \( \tilde{L} \). By the positive solution of the Smith Conjecture [9], \( \tilde{K}_1 \) is unknotted and so the homeomorphism \( \phi \) is conjugate to one point compactification of the \( \mathbb{Z}_n \)-rotation about the z-axis in \( \mathbb{R}^3 \). Hence the quotient map \( q : S^3 \rightarrow S^3/\mathbb{Z}_n \) is an \( n \)-fold cyclic covering branched along the unknot \( q(\tilde{K}_1) = K_1 \). Set \( L = q(\tilde{L}) \). Then the link \( L_1 = K_1 \cup L \) in the orbit space \( S^3/\mathbb{Z}_n \approx S^3 \) is called the quotient link of \( \tilde{L} \). Some authors showed that a certain properties of periodic links can be characterized by their quotient links [3, 5, 8, 11, 12]. In this paper we are interested in studying the \( SL(2, \mathbb{C}) \)-representation variety \( \mathcal{R}(\tilde{L}) \) of an \( n \)-periodic link \( \tilde{L} \) in \( S^3 \) in terms of \( SL(2, \mathbb{C}) \)-representations of its quotient link \( L_1 \) in \( SL(2, \mathbb{C}) \).

The paper is organized as follows. In Section 2, we review a few basic terminologies concerning affine algebraic sets. In Section 3, we consider the \( SL(2, \mathbb{C}) \)-representation variety \( \mathcal{R}(L_1) \) of a link \( L_1 = K_1 \cup L \) with unknotted component \( K_1 \). In Section 4, we show that \( SL(2, \mathbb{C}) \)-representations of an \( n \)-periodic link \( \tilde{L} \) are completely determined by the \( SL(2, \mathbb{C}) \)-representations of its quotient link \( L_1 \) and express the \( SL(2, \mathbb{C}) \)-representation variety \( \mathcal{R}(\tilde{L}) \) of \( \tilde{L} \) as the union of \( n \) affine algebraic subsets which have the same dimension. As a consequence, we show that the dimension of \( \mathcal{R}(\tilde{L}) \) is bounded by the dimensions of algebraic subsets of the \( SL(2, \mathbb{C}) \)-representation variety \( \mathcal{R}(L_1) \) of its quotient link \( L_1 \).

2. Representation variety of knots and links

Let \( \mathbb{C} \) be the field of complex numbers. An (affine) algebraic set in the affine space \( \mathbb{C}^n (n \geq 1) \) is the set of zeros of some finite set of polynomials \( f_1, \cdots, f_s \) in \( \mathbb{C}[X_1, \cdots, X_n] \). We denote it by \( V(f_1, \cdots, f_s) \). 

or simply by $\mathcal{V}$, i.e., $\mathcal{V}(f_1, \ldots, f_s) = \\
\{(a_1, \ldots, a_n) \in \mathbb{C}^n \mid f_1(a_1, \ldots, a_n) = 0, \forall i = 1, 2, \ldots, s\}$.

If $\mathcal{U}$ is the ideal of $\mathbb{C}[X_1, \ldots, X_n]$ generated by $f_1, \ldots, f_s$, then the set of all zeros of $f_i$'s is equal to the set of all zeros of every $g \in \mathcal{U}$ and so we will denote $\mathcal{V}(f_1, \ldots, f_s)$ also by $\mathcal{V}(\mathcal{U})$. A non-empty affine algebraic set is said to be irreducible if it cannot be expressed as the union of two proper algebraic subsets. An irreducible algebraic subset $\mathcal{V} = \mathcal{V}(f_1, \ldots, f_s)$ of $\mathbb{C}^n$ is called an affine variety defined by $f_1, \ldots, f_s$.

Every affine algebraic set may be written canonically as a finite union of affine varieties, called its irreducible components. An affine algebraic set $\mathcal{V}$ has a well-defined (complex) dimension, denoted by $\dim(\mathcal{V})$. If $\mathcal{V} \subset \mathbb{C}^m$ and $\mathcal{W} \subset \mathbb{C}^n$ are affine algebraic sets, a map $\phi: \mathcal{V} \rightarrow \mathcal{W}$ is said to be regular if it is the restriction of some map from $\mathbb{C}^m$ to $\mathbb{C}^n$ which is defined by $n$ polynomials in $m$ variables. 

Let $M(2, \mathbb{C})$ be the set of all $2 \times 2$ matrices with entries in $\mathbb{C}$. Throughout this paper, we shall identify $M(2, \mathbb{C})$ with $\mathbb{C}^4$ by simply writing down the rows of each matrix one after the other and so, for example, $M(2, \mathbb{C})^m$ is identified with $\mathbb{C}^{4m}$. The general linear group $GL(2, \mathbb{C})$ is the group of all members of $M(2, \mathbb{C})$ with nonzero determinant and the special linear group $SL(2, \mathbb{C})$ is the subgroup of $GL(2, \mathbb{C})$ with determinant 1.

Let $G$ be a finitely presented group. A homomorphism $\rho: G \rightarrow SL(2, \mathbb{C})$ is called a representation of $G$ in $SL(2, \mathbb{C})$. Two representations $\rho$ and $\rho'$ are equivalent, denoted by $\rho \equiv \rho'$, if $\rho' = \Lambda \rho$, where $\Lambda$ is an inner automorphism of $SL(2, \mathbb{C})$. Let $R(G)$ denote the set of all representations of $G$ in $SL(2, \mathbb{C})$. Then it can be parametrized by points of an affine algebraic subset of $\mathbb{C}^{4m}$ for some positive integer $m$ as follows. Let $P = \langle x_1, \ldots, x_m \mid r_j(x_1, \ldots, x_m), j = 1, 2, \ldots, n \rangle$ be a group presentation of $G$. Define $R(G, P) = \\
\{(P) \in SL(2, \mathbb{C})^m \mid R_j(P) - I = O, j = 1, 2, \ldots, n\}$, where $R_j(P)(j = 1, 2, \ldots, n)$ denotes the matrix $r_j(A_1, \ldots, A_m)$ obtained from the relator $r_j(x_1, \ldots, x_m)$ by substituting $A_i$ for $x_i$. $I$ denotes the $2 \times 2$ identity matrix and $O$ denotes the $2 \times 2$ zero matrix. Then $R(G, P)$ is an affine algebraic subset of $\mathbb{C}^{4m}$. For each point $P = (A_1, \ldots, A_m) \in R(G, P)$, we define a representation $\rho_P$. 


$G \rightarrow \text{SL}(2, \mathbb{C})$ by $\rho_P(x_i) = A_i (1 \leq i \leq m)$ Then $\rho_P$ becomes a representation of $G$ in $\text{SL}(2, \mathbb{C})$. Conversely, for an arbitrary given representation $\rho : G \rightarrow \text{SL}(2, \mathbb{C})$, the point $P = (\rho(x_1), \ldots, \rho(x_m))$ is an element of $\mathcal{R}(G, \mathcal{P})$ such that $\rho_P = \rho$. Therefore there is a natural 1-1 correspondence between the points of $\mathcal{R}(G, \mathcal{P})$ and $\mathcal{R}(G)$. If $Q$ is an another presentation of $G$, then there exists a canonical isomorphism $\phi : \mathcal{R}(G, \mathcal{P}) \rightarrow \mathcal{R}(G, \mathcal{Q})$ as affine algebraic sets. We shall identify points in $\mathcal{R}(G, \mathcal{P})$ with the corresponding representations. Although $\mathcal{R}(G; \mathcal{P})$ is not a variety in general, we call $\mathcal{R}(G, \mathcal{P})$ the $\text{SL}(2, \mathbb{C})$-representation variety of $G$ associated to $\mathcal{P}$.

Now let $L = K_1 \cup \cdots \cup K_\mu$ be an oriented tame link in $S^3$ of $\mu$ components($\mu \geq 1$) and let $G = \pi_1(S^3 - L)$ be the link group of $L$, i.e., the fundamental group of the complement $S^3 - L$ with a finite presentation $\mathcal{P}$. Then in what follows the variety $\mathcal{R}(G, \mathcal{P})$ is called the $\text{SL}(2, \mathbb{C})$-representation variety of the link $L$ associated to $\mathcal{P}$ and denoted by $\mathcal{R}(L, \mathcal{P})$. Note that the isomorphism class $\mathcal{R}(L)$ of $\mathcal{R}(L, \mathcal{P})$ is an invariant of the link type $L$.

3. Representation variety of a link with one trivial component

Let $L_1 = K_1 \cup K_2 \cup \cdots \cup K_\mu$ be an oriented link in $S^3$ of $\mu$ components($\mu \geq 2$) such that $K_1$ is unknotted For each $2 \leq i \leq \mu$, let $\lambda_1 = \text{lkd}(K_1, K_i)$, the linking number of $K_1$ and $K_i$. Let $N_i(i = 1, \cdots, \mu)$ be a small open tubular neighborhood of $K_i$ in $S^3$ whose boundary $\partial N_i = T_i$ is a torus in $S^3$. Let $(m_i, l_i)$ be a meridian-longitude pair of $T_i$ Then $\pi_1(T_i)$ is a free abelian group generated by $m_i$ and $l_i$ and it has a presentation $\pi_1(T_i) = \langle x_i, \xi_i : x_i \xi_i x_i^{-1} \xi_i^{-1} \rangle$, where $x_i$ and $\xi_i$ represent $m_i$ and $l_i$, respectively This presentation is called a canonical presentation of $\pi_1(T_i)$.

For our simplicity, we assume that $\mu = 2$ and $\lambda_{12} \neq 0$. Applying an isotopy deformation if necessary, we can choose an oriented diagram $D = D_1 \cup D_2$ in $\mathbb{R}^2$ of the link $L_1 = K_1 \cup K_2$ which is of the form as shown in Figure 1, where $D_1(z = 1, 2)$ denotes a diagram representing the component $K_1$. 
Using Wirtinger presentation and Tietz transformations if necessary, we obtain a deficiency one presentation $P'$ of the group $G = \pi_1(S^3 - L_1)$ which contains a canonical presentation of $\pi_1(\mathbb{T}_1)$, which is of the form (cf. [1])

$$P' = \langle z_1, \ldots, z_a, w_1, \ldots, w_b, \xi_1 \mid r'_1, s' \rangle,$$

$$r'_1(1 \leq i \leq a - 1), r'_2(1 \leq i \leq b - 1),$$

where the generators $z_i$ and $w_j$ correspond to the $i$-th and $j$-th branch of the component $D_1$ and $D_2$ of $D$, respectively, and $\xi_1$ represents a longitude $l'_1$ of $D_1$ and

$$r' = z_1 \xi_1 z_1^{-1} \xi_1^{-1},$$

$$s' = \xi_1 (w_{j_1+1} w_{j_2+1} \ldots w_{j_r+1} w_{j_{r+1}}^{-1} \ldots w_{j_{a-1}}^{-1} w_{b}^{-1})^{-1}.$$

The relators $r'_1$ and $r'_2$ correspond to the crossings in $D$. The relators $r'_4$ correspond to the crossings incident to the component $D_1$, which have the form (cf. Figure 1)
The relators \( r'^j \) correspond to the self crossings of the component \( D_2 \), which have the form:

\[
 r'^j_{a-2} = w_{j_{a-2}} w_{j_{a-2}}^{-1} z_{j_{a-2}^{-1}}, \quad r'^j_{a-1} = w_{j_{a-1}} w_{j_{a-1}}^{-1} z_{j_{a-1}^{-1}},
\]

\[
 r'^j_j = w_j z_j w_j^{-1} z_j^{-1}, \quad \cdots \quad r'^j_{jr} = w_j z_j w_j^{-1} z_j^{-1},
\]

\[
 r'^j_{jr+1} = z_1 w_{jr+1} z_1^{-1} w_{jr+1}^{-1}, \quad \cdots \quad r'^j_{jr+1} = z_1 w_{jr+1} z_1^{-1} w_{jr+1}^{-1}.
\]

where \( w^j_q \) is a certain generator \( w_j (1 \leq j \leq b) \) and \( \epsilon_q = \pm 1 \).

We modify the presentation \( \mathcal{P}' \) of \( G \) as follows. Since \( H_1 (S^3 - L_1) = G_1 / [G, G] \) is generated by \( z_1, w_1 \), we have that \( z_i \equiv z_1 \pmod{[G, G]}, i = 2, \cdots, a \), and \( w_i \equiv w_1 \pmod{[G, G]}, j = 2, \cdots, b \), and \( \xi_1 \equiv w_1^{a-2r} \pmod{[G, G]} \). Introduce new generators \( x_i = z_i, x_i = z_i z_1^{-1} (2 \leq i \leq a), y_i = w_i, y_i = w_i y_i^{-1} (2 \leq i \leq b) \), and \( \ell_i = \xi_1 y_1^{-1} \). Using these generators, we obtain a new deficiency one presentation \( \mathcal{P} \) of \( G \)

\[
 \mathcal{P} = \langle x_1, \cdots, x_a, y_1, \cdots, y_b, \ell_1 \mid r, s, \quad r_{1i} (1 \leq i \leq a-1), r_{2j} (1 \leq j \leq b-1) \rangle,
\]

where \( r, s, r_{1i}, \) and \( r_{2j} \) are obtained from \( r', s', r'_1, \) and \( r'_2 \) by rewriting in terms of the new generators \( x_i, y_i, \) and \( \ell_i \). Precisely,
Now let $\mathcal{R}(L_1, \mathcal{P})$ be the $\text{SL}(2, \mathbb{C})$-representation variety of $L_1$ associated to the presentation $\mathcal{P}$ in (1)

Let $A_i = \begin{pmatrix} X_{4(i-1)+1} & X_{4(i-1)+2} \\ X_{4(i-1)+3} & X_{4i} \end{pmatrix}$, $B_j = \begin{pmatrix} X_{4(a+j-1)+1} & X_{4(a+j-1)+2} \\ X_{4(a+j-1)+3} & X_{4(a+j)} \end{pmatrix}$, $C_1 = \begin{pmatrix} X_{4(a+b)+1} & X_{4(a+b)+2} \\ X_{4(a+b)+3} & X_{4(a+b+1)} \end{pmatrix} \in \text{M}(2, \mathbb{C})$ for $i = 1, 2, \ldots, a$, $j = 1, 2, \ldots, b$ A point $P = (A_1, A_2, \ldots, A_a, B_1, \ldots, B_b, C_1) \in \text{M}(2, \mathbb{C})^{a+b+1}$ lies in $\mathcal{R}(L_1, \mathcal{P})$, i.e., the map defined by $x_i \mapsto A_i(1 \leq i \leq a), y_j \mapsto B_j(1 \leq j \leq b), \ell_1 \mapsto C_1$ is a representation of $G$ in $\text{SL}(2, \mathbb{C})$ if and
only if

(3) \( \det(A_1) = 1 \)
(4) \( \det(A_i) = 1, \det(B_j) = 1, \det(C_i) = 1, 2 \leq i \leq a, 1 \leq j \leq b, \)
(5) \( R(P) - I = O, S(P) - I = O, R_{1i}(P) - I = O, 1 \leq i \leq a - 1, \)
(6) \( R_{2j}(P) - I = O, 1 \leq j \leq b - 1. \)

On the other hand, a presentation \( \mathcal{P}_* \) of \( G_* = \pi_1(S^3 - K_2) \) is obtained from \( \mathcal{P} \) by adding one relator \( x_1 = 1 \). Let \( \mathcal{R}(K_2, \mathcal{P}_*) \) be the \( \text{SL}(2, \mathbb{C}) \)-representation variety of \( K_2 \) associated to the presentation \( \mathcal{P}_* \).

**Proposition 3.1**. \( \mathcal{R}(K_2, \mathcal{P}_*) \) is an affine algebraic subset of \( \mathcal{R}(L_1, \mathcal{P}) \).

**Proof**. A point \( P = (A_1, \ldots, A_a, B_1, \ldots, B_b, C_1) \in M(2, \mathbb{C})^{a+b+1} \)

lies in \( \mathcal{R}(K_2, \mathcal{P}_*) \) if and only if it satisfies the equations (3), (4), (5), (6) and the equation \( A_1 = I \), i.e.,

(7) \( \mathcal{R}(K_2, \mathcal{P}_*) = \{(A_1, A_2, \ldots, A_a, B_1, \ldots, B_b, C_1) \in \mathcal{R}(L_1, \mathcal{P}) \mid A_1 = I \} \)

This implies that \( \mathcal{R}(K_2, \mathcal{P}_*) \) is an affine algebraic set defined by the defining polynomials of \( \mathcal{R}(L_1, \mathcal{P}) \), together with the polynomials \( X_1 - 1 = 0, X_2 = 0, X_3 = 0 \) and \( X_4 - 1 = 0 \). \( \square \)

Let \( \mathcal{U}(\mathcal{P}) \) be the ideal of \( \mathbb{C}[X_1, X_2, X_3, X_4, X_5, \ldots, X_{4(a+b+1)}] \) generated by the polynomials in (3) and (4) and the entries of the left hand side of the matrix equations in (5) and (6). Note that \( \mathcal{R}(L_1, \mathcal{P}) = V(\mathcal{U}(\mathcal{P})) \). Let \( \pi_4 : M(2, \mathbb{C})^{a+b+1} \to M(2, \mathbb{C})^{a+b} \) be the projection map which sends \( (A_1, A_2, \ldots, A_a, B_1, \ldots, B_b, C_1) \) to \( (A_2, \ldots, A_a, B_1, \ldots, B_b, C_1) \) and let \( \mathcal{U}_4(\mathcal{P}) = \mathcal{U}(\mathcal{P}) \cap \mathbb{C}[X_5, \ldots, X_{4(a+b+1)}] \) be the 4-th elimination ideal of \( \mathcal{U}(\mathcal{P}) \). Then it is well known that the projection \( \pi_4(\mathcal{R}(L_1, \mathcal{P})) \) is given by

\[
\pi_4(\mathcal{R}(L_1, \mathcal{P})) = \{(A_2, \ldots, A_a, B_1, \ldots, B_b, C_1) \in V(\mathcal{U}_4(\mathcal{P})) \mid \exists A_1 \in M(2, \mathbb{C}) \text{ s.t. } (A_1, A_2, \ldots, A_a, B_1, \ldots, B_b, C_1) \in \mathcal{R}(L_1, \mathcal{P})\}
\]

and \( V(\mathcal{U}_4(\mathcal{P})) = \pi_4(\mathcal{R}(L_1, \mathcal{P})) \), the Zariski closure of \( \pi_4(\mathcal{R}(L_1, \mathcal{P})) \) in \( \mathbb{C}^{4(a+b)} \).
Let $n$ be an integer $\geq 2$ and set $\zeta = \exp\left(\frac{2\pi \sqrt{-1}}{n}\right)$, a primitive $n$-th root of 1. Let $\mathcal{V}(L_1, \mathcal{P})$ be the affine algebraic subset of $\mathbb{C}^{4(a+b+1)}$ consisting of all points $P = (A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in M(2, \mathbb{C})^{a+b+1}$ satisfying all equations in (4), (5) and (6). For each $k = 0, 1, \cdots, n-1$, let $D^\alpha_k = \{M \in M(2, \mathbb{C}) \mid M^n = I, \det(M) = \zeta^k\}$. Then we define $V_k(L_1, \mathcal{P}), 0 \leq k \leq n-1$, to be the subset of $\mathbb{C}^{4(a+b+1)}$ given by

$$V_k(L_1, \mathcal{P}) = \mathcal{V}(L_1, \mathcal{P}) \cap (D^\alpha_k \times V(U_4(\mathcal{P})))$$

and define

$$R_n(L_1, \mathcal{P}) = \bigcup_{k=0}^{n-1} V_k(L_1, \mathcal{P}).$$

In particular, $V_0(L_1, \mathcal{P}) = \{(A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in R(L_1, \mathcal{P}) \mid A_1^n = I\}$

**Proposition 3.2.** (1) For each $k = 0, 1, \cdots, n-1$, $V_k(L_1, \mathcal{P})$ is an affine algebraic subset of $\mathbb{C}^{4(a+b+1)}$ and so is $R_n(L_1, \mathcal{P})$.

(2) If $0 \leq i \neq j \leq n-1$, then $V_i(L_1, \mathcal{P}) \cap V_j(L_1, \mathcal{P}) = \emptyset$.

(3) For each $k = 1, \cdots, n-1$, $V_k(L_1, \mathcal{P})$ is isomorphic to $V_0(L_1, \mathcal{P})$ as affine algebraic sets.

(4) $R(K_2, \mathcal{P}) \subset V_0(L_1, \mathcal{P}) \subset R(L_1, \mathcal{P})$ and $R_n(L_1, \mathcal{P}) \cap R_n(L_1, \mathcal{P}) = V_0(L_1, \mathcal{P})$.

**Proof.** Since $V(L_1, \mathcal{P}), D^\alpha_k$ and $V(U_4(\mathcal{P}))$ are all affine algebraic sets, (1) follows immediately. (2) follows from the fact that $D^\alpha_i \cap D^\alpha_j = \emptyset$ if $i \neq j$.

(3) We consider the map $\phi: V_0(L_1, \mathcal{P}) \to V_k(L_1, \mathcal{P})$ defined by

$$\phi((A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)) = (\zeta^k A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)$$

for all $(A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in V_0(L_1, \mathcal{P})$. By the definition of $V_0(L_1, \mathcal{P})$, it follows that $P = (A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in V(U_4(\mathcal{P}))$, det$(\zeta^k A_1) = \zeta^k \det(A_1) = \zeta^k$ and $(\zeta^k A_1)^n = \zeta^{nk} A_1^n = A_1^n = I$. Notice that either the relators $r, s, r_1$, and $r_2$, in (2) contain
both the generator $x_1$ and its inverse $x_1^{-1}$ exactly once or they do not contain both $x_1$ and $x_1^{-1}$ at all. This gives that

$$R(\zeta^{\frac{1}{2}} A_1, P) = R(A_1, P) = I, S(\zeta^{\frac{1}{2}} A_1, P) = S(A_1, P) = I,$$

$$R_{12}(\zeta^{\frac{1}{2}} A_1, P) = R_{11}(A_1, P) = I, R_{23}(\zeta^{\frac{1}{2}} A_1, P) = R_{22}(A_1, P) = I.$$

Hence $(\zeta^{\frac{1}{2}} A_1, P) \in \mathcal{V}_k(L_1, \mathcal{P})$. It is clear that $\phi$ is the restriction of a polynomial map from $\mathbb{C}^{4(a+b+1)}$ to itself. Thus $\phi$ is a well-defined regular mapping. Now let $\psi : \mathcal{V}_k(L_1, \mathcal{P}) \to \mathcal{V}_0(L_1, \mathcal{P})$ be a map defined by

$$\psi((A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)) = (\zeta^{-\frac{1}{2}} A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)$$

for all $(A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathcal{V}_k(L_1, \mathcal{P})$. By similar argument above, $\psi$ is a regular mapping. It is easy to check that $\psi \circ \phi = \text{id}_{\mathcal{V}_k(L_1, \mathcal{P})}$ and $\phi \circ \psi = \text{id}_{\mathcal{V}_k(L_1, \mathcal{P})}$. Therefore $\phi$ is an isomorphism.

(4) It follows from (7) and (8) shows that $\mathcal{V}(K_2, \mathcal{P}_*) \subset \mathcal{V}_0(L_1, \mathcal{P})$.

By definition, $\mathcal{V}_0(L_1, \mathcal{P}) \subset \mathcal{V}(L_1, \mathcal{P}) \cap \mathcal{V}_n(L_1, \mathcal{P})$. Now let

$$P = (A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathcal{V}(L_1, \mathcal{P}) \cap \mathcal{V}_n(L_1, \mathcal{P}).$$

Then $P$ represents a representation of $G$ into $\text{SL}(2, \mathbb{C})$ and so $P \in \mathcal{V}(L_1, \mathcal{P})$ and $\det(A_1) = 1$. Since $P \in \mathcal{V}_n(L_1, \mathcal{P})$, $A_1 \in D^+_k$ for some $k$. By (2), $\mathcal{V}_n(L_1, \mathcal{P}) = \bigcup_{k=0}^{n-1} \mathcal{V}_k(L_1, \mathcal{P})$ and hence $P \in \mathcal{V}_0(L_1, \mathcal{P})$. This completes the proof. \qed

4. Representation variety of an $n$-periodic link

Let $L_1 = K_1 \cup K_2$ be an oriented link in $S^3$ with 2 components such that $K_1$ is unknotted. Let $\mu$ be the greatest common divisor of $n$ and $\lambda_{12}$. For any integer $n \geq 2$, let $\pi : S^3 \to S^3$ be the $n$-fold cyclic cover branched along $K_1$. Then $K_2$ is covered by $\mu$ knots $\bar{K}_1, \cdots, \bar{K}_\mu$ in $S^3$. We give orientations to $\bar{K}_1, \cdots, \bar{K}_\mu$ inherited from $K_2$. Then the oriented link $\bar{L} = \pi^{-1}(K_2) = \bar{K}_1 \cup \cdots \cup \bar{K}_\mu$ is the $n$-periodic link in $S^3$ with $L$ as its quotient link. Note that every periodic links arises in this way.
Let $\tilde{G} = \pi_1(S^3 - \tilde{L})$ be the link group of $\tilde{L}$. Then from the choice of the generators in the presentation $P$ of $G = \pi_1(S^3 - L)$ as given in (1), the group $\tilde{G}$ has a presentation $\tilde{P}$ of the form (cf. [11])

$$\tilde{P} = \langle x_{ik}, y_{jk}, z_k \big| 1 \leq i \leq a - 1, 1 \leq j \leq b, 1 \leq k \leq n \rangle \big| r_k, s_k, r_{1i}^k, r_{2j}^k \big| 1 \leq i \leq a - 1, 1 \leq j \leq b - 1, 1 \leq k \leq n \rangle.$$ \hspace{1cm} (9)

where

$$x_{ik} = x_i^{-1} x_{i+1}^{-1} x_i^{(k-1)}, \quad y_{jk} = x_j^{-1} y_j x_j^{(k-1)}, \quad z_k = x_i^{-1} x_i^{(k-1)},$$

$$x_i^n = 1, \quad x_i^k \neq 1 \text{ for all } k = 1, \ldots, n - 1,$$

and

$$r_k = x_i^{-1} x_i^{(k-1)} s_k = x_i^{-1} x_i^{(k-1)},$$

$$r_{1i}^k = x_i^{-1} r_{1i} x_i^{(k-1)}, \quad r_{2j}^k = x_i^{-1} r_{2j} x_i^{(k-1)},$$

or equivalently, for each $k = 1, \ldots, n$,

$$r_k = z_{k+1}^{\lambda_{12}} y_{j+1}^{\lambda_{12}} y_{j+1}^{\lambda_{12}} z_{k}^{-1},$$

$$s_k = z_k y_{j+1}^{\lambda_{12}} y_{j+1}^{\lambda_{12}} y_{j+1}^{\lambda_{12}} y_{j+1}^{\lambda_{12}} \cdots,$$

$$r_{1i}^k = y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} x_{k}^{-1},$$

$$r_{12}^k = y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} x_{2k}^{-1},$$

$$r_{1r}^k = y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} x_{r}^{-1},$$

$$r_{1r+1}^k = y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} x_{r+1}^{-1},$$

$$r_{2j}^k = y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} x_{2j+1}^{-1},$$

$$r_{1a}^{-1} = y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1},$$

and

$$r_{2j+1}^k = y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} y_{j+1}^{-1} x_{2j+1}^{-1}.$$ \hspace{1cm} (12)
\[
\begin{align*}
\tau_{2j}^k &= y_{2j}y_1y_{2j+1}^{-1}y_{2j+1}^{-1}, \\
\tau_{2j+1}^k &= y_{2j+1}y_1y_{2j+1}^{-1}y_{2j+1}^{-1}.
\end{align*}
\]

(13)

\[
\begin{align*}
\tau_{2j_a-1}^k &= y_{2j_a-1}y_1y_{2j_a-1}^{-1}y_{2j_a-1}^{-1} \\
\tau_{2q}^k &= (w_q)^{\epsilon(q)}y_q y_1 y_q^{-1} y_q^{-1}.
\end{align*}
\]

We shall introduce some notations for the following theorem. Let 
\( P_1 = (M_{11}, \ldots, M_{m1}), P_2 = (M_{12}, \ldots, M_{m2}), \ldots, P_n = (M_{1n}, \ldots, M_{mn}) \) be \( n \) points in \( M(2, \mathbb{C})^m \), where \( m \) is an integer \( \geq 1 \) and \( M_{ij} \in M(2, \mathbb{C}) \). Then \( (P_1, P_2, \ldots, P_n) \) denotes the point \( (M_{11}, \ldots, M_{m1}, M_{12}, \ldots, M_{m2}, \ldots, M_{1n}, \ldots, M_{mn}) \) in \( M(2, \mathbb{C})^m \). For a matrix \( N \in M(2, \mathbb{C}) \) and an integer \( k, N^k P_j N^{-k} (1 \leq j \leq n) \) denotes the point \( (N^k M_{1j} N^{-k}, \ldots, N^k M_{mj} N^{-k}) \) in \( M(2, \mathbb{C})^m \).

**Theorem 4.1.** Let \( L_1 = K_1 \cup K_2 \) be an oriented link in \( S^3 \) such that \( K_1 \) is unknotted and \( \lambda_{12} =lk(K_1, K_2) \neq 0 \) and let \( \mathcal{P} \) be the presentation of \( G = \pi_1(S^3 - L_1) \) as given in (1). For any integer \( n \geq 2 \), let \( \bar{L} \) be an \( n \)-periodic link in \( S^3 \) with the quotient link \( L_1 \) and let \( \mathcal{R}(\bar{L}, \mathcal{P}) \) be the \( SL(2, \mathbb{C}) \)-representation variety of \( \bar{L} \) associated to the presentation \( \mathcal{P} \) in (9). Then a point \( P = (P_1, P_2, \ldots, P_n) \in M(2, \mathbb{C})^{(a+b)n} \) lies in \( \mathcal{R}(\bar{L}, \mathcal{P}) \) if and only if \( P_1 \in V(U_1(\mathcal{P})) \) and for each \( k = 2, \ldots, n, P_k = M^{-k} P_1 M^{-(k-1)} \) for some matrix \( M \in GL(2, \mathbb{C}) \) such that \( (M, P_1) \in \mathcal{R}_n(L_1, \mathcal{P}) \).

**Proof.** Let

\[
\begin{align*}
P_1 &= (A_{11}, \ldots, A_{a-11}, B_{11}, \ldots, B_{b1}, C_1), \\
P_2 &= (A_{12}, \ldots, A_{a-12}, B_{12}, \ldots, B_{b2}, C_2), \\
& \vdots \\
P_n &= (A_{1n}, \ldots, A_{a-1n}, B_{1n}, \ldots, B_{bn}, C_n).
\end{align*}
\]

Suppose that \( P = (P_1, P_2, \ldots, P_n) \) is a point of \( \mathcal{R}(\bar{L}, \mathcal{P}) \), i.e., the mapping defined by \( x_{ik} \mapsto A_{ik}, y_{jk} \mapsto B_{jk}, z_k \mapsto C_k \) is a representation
of $G$ in $\text{SL}(2, \mathbb{C})$. Then

\begin{equation}
\mathsf{det}(A_{ik}) = 1, \mathsf{det}(B_{jk}) = 1, \mathsf{det}(C_k) = 1,
\end{equation}

\begin{equation}
R_k(P) - I = O, \quad S_k(P) - I = O,
\end{equation}

\begin{equation}
R_{k_1}^{k_2}(P) - I = O, \quad R_{k_2}^{k_1}(P) - I = O
\end{equation}

for all $1 \leq i \leq a - 1, 1 \leq j \leq b$ and $1 \leq k \leq n$.

By (10), it follows that for all $i, j$ and $k$, $A_{ik} = M^{k-1}A_{i1}M^{-(k-1)}, B_{jk} = M^{k-1}B_{j1}M^{-(k-1)}, C_k = M^{k-1}C_1M^{-(k-1)}$ for some matrix $M \in \text{GL}(2, \mathbb{C})$ such that $M^n = I$, i.e., for each $k = 1, \cdots, n$,

\begin{equation}
P_k = M^{k-1}P_1M^{-(k-1)} = MP_{k-1}M^{-1}.
\end{equation}

From (12) and (13), it follows that for each $k = 1, \cdots, n$, the relators $r_k, s_k, r_1^k$, and $r_2^k$ in $\mathcal{P}$ consist of the generators $x_{ik}, x_{ik+1}, y_{jk}, y_{jk+1}, z_k$ or $z_{k+1}$, where $1 \leq i \leq a - 1$ and $1 \leq j \leq b$. So all entries of the matrices $R_k(P_i), S_k(P_i), R_{k_1}^{k_2}(P)$ and $R_{k_2}^{k_1}(P)$ are polynomials with indeterminants which are the entries of the matrices $A_{ik}, A_{ik+1}, B_{jk}, B_{jk+1}, C_k$ and $C_{k+1}$.

Hence we obtain that for each $k = 1, \cdots, n$,

\begin{equation}
R_k(P) = r_k(P_1, P_2, \cdots, P_n) = r_k(P_k, P_{k+1}),
\end{equation}

\begin{equation}
S_k(P) = s_k(P_1, P_2, \cdots, P_n) = s_k(P_k, P_{k+1}),
\end{equation}

\begin{equation}
R_{k_1}^{k_2}(P) = r_{k_1}^{k_2}(P_1, P_2, \cdots, P_n) = r_{k_1}^{k_2}(P_k, P_{k+1}),
\end{equation}

\begin{equation}
R_{k_2}^{k_1}(P) = r_{k_2}^{k_1}(P_1, P_2, \cdots, P_n) = r_{k_2}^{k_1}(P_k, P_{k+1})
\end{equation}

By (10), we have that $x_{i1} = x_{i+1}, y_{j1} = y_j, z_1 = \ell_1$, where $x_{i+1}, y_j$ and $\ell_1$ are the generators of the presentation $\mathcal{P}$ in (1) and so it follows from (2), (12) and (13) that

\begin{equation}
r_1(P_1, P_2) = r_1(P_1, MP_1M^{-1}) = r(M, P_1),
\end{equation}

\begin{equation}
s_1(P_1, P_2) = s_1(P_1, MP_1M^{-1}) = s(M, P_1),
\end{equation}

\begin{equation}
r_1^{k_1}(P_1, P_2) = r_1^{k_1}(P_1, MP_1M^{-1}) = r_{1k}(M, P_1),
\end{equation}

\begin{equation}
r_2^{k_1}(P_1, P_2) = r_2^{k_1}(P_1, MP_1M^{-1}) = r_{2k}(M, P_1),
\end{equation}

where $r, s, r_1, \text{ and } r_2$ are the relators of the presentation $\mathcal{P}$ of $G = \pi_1(S^3 - L)$ in (1). By (16), (17) and (19), it follows that $r(M, P_1) = I, s(M, P_1) = I, r_{11}(M, P_1) = I, r_{22}(M, P_1) = I$ and hence $P_1 \in V(U_4(P))$ and $(M, P_1) \in R_n(L_1, P)$. 
Conversely, let $P = (F_1, MP_1M^{-1}, \ldots, M^{n-1}P_1M^{-(n-1)})$ be a point of $M(2, \mathbb{C})^{a+b}$ satisfying the conditions. Since $P_1 \in V(\mathcal{U}_4(P))$ and the ideal $\mathcal{U}_4(P)$ contains all polynomials in $M(2, \mathbb{C})^{a+b}$ and so $M^{k-1}P_1M^{-(k-1)}$ in $SL(2, \mathbb{C})^{a+b}$ for all $k = 2, \ldots, n$ Hence $P \in SL(2, \mathbb{C})^{a+b}$. Since $(M, P_1) \in R_n(L_1, P)$, it follows from (19) and (20) that $R_1(P) = I, S_1(P) = I, R_{11}(P) = I, R_{12}(P) = I$. Then by (12), (13) and (19), we obtain that for each $k = 2, \ldots, n$,

$$R_k(P) = \tau_k(P_k, P_{k+1})$$
$$= \tau_k(M^{k-1}P_1M^{-(k-1)}, M^{k-1}P_2M^{-(k-1)})$$
$$= M^{k-1}R_1(P_1, P_2)M^{-(k-1)}$$
$$= M^{k-1}R_1(P)M^{-(k-1)}$$
$$= I.$$

Similarly, $S_k(P) = I, R_{11}^k(P) = I$ and $R_{2j}^k(P) = I$ for all $i, j$ and $k = 2, \ldots, n$ Therefore $P \in R(\tilde{L}, \tilde{P})$. This completes the proof. □

Let $\eta : \tilde{G} \to SL(2, \mathbb{C})$ be a representation of $\tilde{G}$ in $SL(2, \mathbb{C})$ and let $\Theta : \tilde{G} \to \tilde{G}$ denote the $n$-periodic automorphism of $\tilde{G}$ defined by $\Theta(x_{ik+1}, y_{jk}, z_k) = y_{jk+1}$ and $\Theta(z_k) = z_{k+1}$. Then it is immediate that $\eta \circ \Theta$ is also a representation of $\tilde{G}$ in $SL(2, \mathbb{C})$.

**Theorem 4.2.** Let $\mathcal{F}(\tilde{L}, \tilde{P})$ denote the set of all points $P = (P_1, P_2, \ldots, P_n)$ in $R(\tilde{L}, \tilde{P})$ such that $\eta_P \circ \Theta = \eta_{\tilde{P}}$, where $\eta_P$ denotes the representation of $\tilde{G}$ corresponding to the point $P$. Then

1. $\mathcal{F}(\tilde{L}, \tilde{P})$ is an affine algebraic subset of $R(\tilde{L}, \tilde{P})$

2. $\mathcal{F}(\tilde{L}, \tilde{P}) = \{(P_1, P_1, \ldots, P_n) \in R(\tilde{L}, \tilde{P}) | \exists M \in GL(2, \mathbb{C}) \text{ s.t.} (M, P_1) \in R_n(L_1, P), MP_1 = P_1M\}$.

**Proof.** (1) Let $P = (P_1, P_2, \ldots, P_n)$ be a point of $\mathcal{F}(\tilde{L}, \tilde{P})$, where $P_1, P_2, \ldots, P_n$ are points of $M(2, \mathbb{C})^{a+b}$ as given in (14). Since $P = (P_1, P_2, \ldots, P_n)$ lies in $R(\tilde{L}, \tilde{P})$, $P$ satisfies the matrix equations in (15) and (17). It is clear that $\eta_P \circ \Theta = \eta_{\tilde{P}}$ if and only if

$$A_{ik+1} - A_{ik} = O, B_{jk+1} - B_{jk} = O, C_{k+1} - C_k = O$$

(21)
for all $1 \leq i \leq a-1, 1 \leq j \leq b$ and $k = 1, 2, \cdots, n$. This shows that $P$ is a zero of the polynomials in (15) and the polynomials which are the entries of the left hand side matrix of the equations in (17) and (21).

(2) Let $P = (P_1, P_2, \cdots, P_n)$ be a point of $\mathcal{R}(\bar{L}, \bar{P})$ such that $\eta_P \circ \Theta = \eta_P$, where $P_1, P_2, \cdots, P_n$ are points of $M(2, \mathbb{C})^{a+b}$ as given in (14). By Theorem 4.1, $P_1 \in \mathcal{U}_d(P)$ and for each $k = 2, 3, \cdots, n$, $P_k = M^{k-1}P_1M^{-(k-1)}$ for some matrix $M \in \text{GL}(2, \mathbb{C})$. Since $\eta_P \circ \Theta = \eta_P$, it follows that $A_k, B_{jk+1} = B_{jk}, C_{k+1} = C_k$ and so $MA_kM^{-1} = A_k, MB_{jk}M^{-1} = B_{jk}, MC_kM^{-1} = C_k$ for all $k = 1, 2, \cdots, n-1$. Therefore $P_1 = MP_1M^{-1}$. Conversely, if $P = (P_1, P_2, \cdots, P_n)$ is a point of $\mathcal{R}(\bar{L}, \bar{P})$ such that $P_1 = P_2 = \cdots = P_n$, then it is clear that the corresponding representation $\eta_P$ satisfies that $\eta_P \circ \Theta = \eta_P$. This completes the proof.

Let $n$ be an integer $\geq 2$ and set $\zeta = \exp\left(\frac{2\pi \sqrt{-1}}{n}\right)$. For each $k = 0, 1, \cdots, n-1$, we define $\mathcal{R}_k(\bar{L}, \bar{P})$ to be the subset of $\mathbb{C}^{4n(a+b)}$ given by

$$\{(P, MP^{-1}M^{-1}, \cdots, M^{n-1}P^{-1}M^{-(n-1)}) \in \mathcal{R}(\bar{L}, \bar{P}) \mid \det(M) = \zeta^k\}$$

and define $\phi_k \colon \mathcal{V}_k(L_1, P) \rightarrow M(2, \mathbb{C})^{n(a+b)}(= \mathbb{C}^{4n(a+b)})$ to be the mapping given by

$$\phi_k((M, P)) = (P, MP^{-1}M^{-1}, \cdots, M^{n-1}P^{-1}M^{-(n-1)})$$

for all $(M, P) \in \mathcal{V}_k(L_1, P)$

**Lemma 4.3**

(1) For each $k = 0, 1, \cdots, n-1$, $\phi_k$ is a regular map from $\mathcal{V}_k(L_1, P)$ onto $\mathcal{R}_k(\bar{L}, \bar{P})$, i.e., $\phi_k(\mathcal{V}_k(L_1, P)) = \mathcal{R}_k(\bar{L}, \bar{P})$

(2) $\mathcal{R}(\bar{L}, \bar{P}) = \bigcup_{k=0}^{n-1} \mathcal{R}_k(\bar{L}, \bar{P})$

(3) $\phi_0(\mathcal{R}(K_2, \mathcal{P})) \subset \mathcal{F}(\bar{L}, \bar{P}) \subset \bigcap_{k=0}^{n-1} \mathcal{R}_k(\bar{L}, \bar{P})$. 


Proof. (1) Let \( M = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \), \( P = (M_2, \ldots, M_{a+b}) \in M(2, \mathbb{C})^{a+b-1} \),
where \( M_i = \begin{pmatrix} X_{4(i-1)+1} & X_{4(i-1)+2} \\ X_{4(i-1)+3} & X_4 \end{pmatrix} \) for each \( i = 2, \ldots, a+b \).

Suppose that \((M, P) \in \mathcal{V}_k(L_1, \mathcal{P})\). By Theorem 4.1, \( \phi_k((M, P)) = (P, MPM^{-1}, \ldots, M^{n-1}PM^{-(n-1)}) \in \mathcal{R} (\hat{L}, \hat{\mathcal{P}}) \). Since \( \det(M) = \zeta^k \), it follows that \( \phi_k((M, P)) \in \mathcal{R}_k (\hat{L}, \hat{\mathcal{P}}) \). It is easy to see that \( \phi_k(\mathcal{V}_k(L_1, \mathcal{P})) = \mathcal{R}_k (\hat{L}, \hat{\mathcal{P}}) \).

Now since \( M^{-1} = \zeta^{-k} \begin{pmatrix} X_4 & -X_2 \\ -X_3 & X_1 \end{pmatrix} \), we have the following equations: for each \( 1 \leq m \leq n-1, 2 \leq t \leq a+b \), \( M^m M_t M^{-m} = \)
\[
\frac{1}{\zeta^{km}} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}^m \begin{pmatrix} X_{4(i-1)+1} & X_{4(i-1)+2} \\ X_{4(i-1)+3} & X_4 \end{pmatrix} \begin{pmatrix} X_4 & -X_2 \\ -X_3 & X_1 \end{pmatrix}^m.
\]

This shows that all entries of the matrix \( M^m M_t M^{-m} \) are polynomials in \( X_1, X_2, X_3, X_4, X_{4(i-1)+1}, X_{4(i-1)+2}, X_{4(i-1)+3} \) and \( X_4 \). Therefore each \( \phi_k \) is a regular map.

(2) Let \( P = (P_1, \ldots, P_n) \) be a point of \( \mathcal{R}(\hat{L}, \hat{\mathcal{P}}) \). By Theorem 4.1, \( P_1 \in \mathcal{V}(\mathcal{U}_4(\mathcal{P})) \) and there exists a matrix \( M \in \text{GL}(2, \mathbb{C}) \) such that \( M^n = I \) and \( P = (P_1, MPM^{-1}, \ldots, M^{n-1}PM^{-(n-1)}) \). Since \( M^n = I, \det(M)^n = 1 \). So \( \det(M) \) must be a \( n \)-th root of unity, i.e., \( \det(M) = \zeta^k \) for some \( k \) \((0 \leq k \leq n-1) \). Thus \( P \in \mathcal{R}_k (\hat{L}, \hat{\mathcal{P}}) \) for some \( k \) \((0 \leq k \leq n-1) \).

(3) Let \( P = (A_1, A_2, \ldots, A_a, B_1, \ldots, B_b, C_1) \) be a point of \( \mathcal{R}(K_2, \mathcal{P}_*) \). By (7), \( A_1 = I \). Set \( P_1 = \pi_4(P) = (A_2, \ldots, A_a, B_1, \ldots, B_b, C_1) \). Note that \( \phi_0(P) = (P_1, \ldots, P_1) \). By (4) of Proposition 3.2, \( P = (I, P_1) \in \mathcal{V}_0(L_1, \mathcal{P}) \subset \mathcal{R}_n (L_1, \mathcal{P}) \). By (2) of Theorem 4.2, \( \phi_0(P) \in \mathcal{F}(\hat{L}, \hat{\mathcal{P}}) \).

Now let \( P = (P_1, \ldots, P_1) \) be a point of \( \mathcal{F}(\hat{L}, \hat{\mathcal{P}}) \). By (2) of Theorem 4.2, there exists a matrix \( M \in \text{GL}(2, \mathbb{C}) \) such that \( (M, P_1) \in \mathcal{V}_n(L_1, \mathcal{P}) \subset \mathcal{R}_n (L_1, \mathcal{P}) \) and \( MP_1 \equiv P_1 M \) for some \( 0 \leq j \leq n-1 \). For each \( k = 0, 1, \ldots, n-1 \), let \( M_k = \zeta^{k+1} M \). Then \( \det(M_k) = \zeta^k \). Since \( \mathcal{F}(\hat{L}, \hat{\mathcal{P}}) \subset \mathcal{R}(\hat{L}, \hat{\mathcal{P}}) \), by Theorem 4.2 \( P_1 \in \mathcal{V}(\mathcal{U}_4(\mathcal{P})) \). It follows from (2) that \( (M_k, P_1) \) satisfies the matrix equations in (4), (5), and (6) and so \( (M_k, P_1) \in \mathcal{V}_k(L_1, \mathcal{P}) \) for each \( k \). Note that \( M_k P_1 = P_1 M_k \) for all \( k \).

Now \( P = (P_1, \ldots, P_1) = \phi_0(M_k, P_1) = \phi_0(\mathcal{V}_k(L_1, \mathcal{P})) = \mathcal{R}_k (L_1, \mathcal{P}) \).
all \(k = 0, 1, \cdots, n - 1\). Hence \(P \in \bigcap_{k=0}^{n-1} R_k(\bar{L}, \bar{P})\). Therefore \(F(\bar{L}, \bar{P}) \subseteq \bigcap_{k=0}^{n-1} R_k(\bar{L}, \bar{P})\). This completes the proof.

In view of (1) in Lemma 4.3, for each \(k = 0, 1, \cdots, n - 1\), we obtain an affine algebraic subset \(\phi_k(V_k(L_1, P))\) of \(C^{4n(a+b)}\). In the rest of this paper we denote it by \(\hat{V}_k\) for simplicity, that is, \(\hat{V}_k = \phi_k(V_k(L_1, P)) = R_k(\bar{L}, \bar{P}), 0 \leq k \leq n - 1\). Then we have the following theorem:

**Theorem 4.4.** (1) \(R(\bar{L}, \bar{P}) = \bigcup_{k=0}^{n-1} \hat{V}_k\)

(2) \(\phi_0(\mathcal{R}(K_2, P_*)) \subseteq F(\bar{L}, \bar{P}) \subseteq \bigcap_{k=0}^{n-1} \hat{V}_k\)

**Proof.** (1) By Lemma 4.3, we obtain that \(R_k(\bar{L}, \bar{P}) = \phi_k(V_k(L_1, P)) \subseteq \hat{V}_k\) and

\[R(\bar{L}, \bar{P}) = \bigcup_{k=0}^{n-1} R_k(\bar{L}, \bar{P}) \subseteq \bigcup_{k=0}^{n-1} \hat{V}_k.\]

Note that \(R_k(\bar{L}, \bar{P}) \subset R(\bar{L}, \bar{P})\) and \(R(\bar{L}, \bar{P})\) is an affine algebraic subset of \(C^{4n(a+b)}\). Since \(\hat{V}_k\) is the smallest algebraic subset of \(C^{4n(a+b)}\) containing \(\phi_k(V_k(L_1, P)) = R_k(\bar{L}, \bar{P})\), we have that \(\hat{V}_k \subset R(\bar{L}, \bar{P})\) for all \(k = 0, 1, \cdots, n - 1\). Therefore \(\bigcup_{k=0}^{n-1} \hat{V}_k \subset R(\bar{L}, \bar{P})\).

(2) By (3) of Lemma 4.3,

\[\phi_0(\mathcal{R}(K_2, P_*)) \subseteq F(\bar{L}, \bar{P}) \subseteq \bigcap_{k=0}^{n-1} R_k(\bar{L}, \bar{P}) \subseteq \bigcap_{k=0}^{n-1} \hat{V}_k.\]

This completes the proof. □

**Corollary 4.5.** (1)

\[\dim(\phi_0(\mathcal{R}(K_2, P_*))) \leq \dim(F(\bar{L}, \bar{P})) \leq \dim(R(\bar{L}, \bar{P}))\]
Proof. (1) follows from Theorem 4.2 and Theorem 4.4.

(2) Since $\mathcal{V}_0(L_1, P) \subset \mathcal{R}(L_1, P)$, $\dim(\mathcal{V}_0(L_1, P)) \leq \dim(\mathcal{R}(L_1, P))$. By Theorem 4.2, $\dim(\mathcal{F}(\tilde{L}, \tilde{P})) \leq \dim(\mathcal{R}(\tilde{L}, \tilde{P}))$ and, by Theorem 4.4, $\dim(\mathcal{R}(\tilde{L}, \tilde{P})) \leq \max\{\dim(\tilde{V}_0), \dim(\tilde{V}_1), \ldots, \dim(\tilde{V}_{n-1})\}$. Since $\phi_k : V_k(L_1, P) \to \tilde{V}_k$ is a dominating map, $\dim(\tilde{V}_k) \leq \dim(V_k(L_1, P))$ for each $k = 0, 1, \ldots, n - 1$. By (3) of Proposition 3.2, $\dim(V_0(L_1, P)) = \dim(V_k(L_1, P))$ for all $k = 1, \ldots, n - 1$. Hence $\dim(\tilde{V}_k) \leq \dim(V_0(L_1, P))$ for all $k = 0, 1, \ldots, n - 1$. Therefore $\dim(\mathcal{R}(\tilde{L}, \tilde{P})) \leq \dim(\mathcal{V}_0(L_1, P))$. This completes the proof. \hfill \square

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Department of Mathematics
Pusan National University
Pusan 609-735, Korea

E-mail: sangyou1@pusan.ac.kr