EMBEDDING OF WEIGHTED $L^p$ SPACES AND THE $\bar{\partial}$-PROBLEM

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ABSTRACT. Let $D$ be a bounded domain in $\mathbb{C}^n$ with $C^2$ boundary. In this paper, we prove the following inequality

$$
\|u\|_{p_2, \alpha_2} \lesssim \|u\|_{p_1, \alpha_1} + \|\bar{\partial} u\|_{p_1, \alpha_1+p_1/2},
$$

where $1 \leq p_1 < p_2 < \infty$, $\alpha_2 > 0$, $(n + \alpha_1)/p_1 = (n + \alpha_2)/p_2$, and $1/p_2 \geq 1/p_1 - 1/2n$.

1. Introduction and statement of results

Let $D$ be a bounded domain in $\mathbb{C}^n$ with $C^2$ boundary. For $z \in D$ let $\delta(z)$ denote the distance from $z$ to $\partial D$. For $\alpha > 0$, we define a weighted measure $dV_\alpha$ on $D$ by $dV_\alpha = C_\alpha \delta^{\alpha-1} dV$ where $dV$ is the volume element and $C_\alpha$ is chosen so that $dV_\alpha$ is a probability measure. As $\alpha \to 0^+$ the measures $dV_\alpha$ converges as measures on $\partial D$ to the normalized surface measure on $\partial D$ which we denote $d\sigma$. We will denote the $L^p$ space with respect to $dV_\alpha$ by $L^p_\alpha$, and the associated norm by $\| \cdot \|_{p, \alpha}$. We will denote by $A^p_\alpha(D) = L^p_\alpha(D) \cap \mathcal{O}(D)$ the subspace of $L^p_\alpha(D)$ consisting of...
functions which are holomorphic on $D$. In particular, $A^p_0(D)$ is the Hardy class usually denoted by $H^p(D)$, which we identify in the usual way with a subspace of $L^p_0(D) = L^p(\partial D; d\sigma)$ (see [16]). Beatrous [6] proved the following embedding theorem.

**Theorem 1.1** ([6]). Let $D$ be a bounded domain in $\mathbb{C}^n$ with $C^2$ boundary and assume that $0 < p_1 \leq p_2 < \infty$, $\alpha_j > 0$, and $(n+\alpha_1)/p_1 = (n+\alpha_2)/p_2$. Then $A^p_{\alpha_1}(D) \subset A^p_{\alpha_2}(D)$ and the inclusion is continuous.

In the case $\alpha_1 = 0$ the embedding in Theorem 1.1 will be

$$H^{p_1}(D) \subset A^{p_2}_{\alpha_2}(D), \quad \text{where} \quad n/p_1 = (n + \alpha_1)/p_2.$$  

This is a generalization of a well-known result of Hardy-Littlewood in the unit disc (see [13, p.87]). Beatrous [7] proved that the case $\alpha_1 = 0$ holds if $D$ is strictly pseudoconvex domains. Recently, the author proved the case $\alpha_1 = 0$ in convex domains of finite type (see [9]). Moreover, it is proved that the case $\alpha_1 = 0$ holds in bounded domains with $C^2$ boundary ([10], [11]).

In this paper, we extend Theorem 1.1 for $L^p_0(D)$ functions $u$ with some growth condition of $\partial u$, and give some consequences for the $\bar{\partial}$-problem.

**Theorem 1.2.** Let $D$ be a bounded domain in $\mathbb{C}^n$ with $C^2$ boundary and assume that $1 \leq p_1 \leq p_2 < \infty$, $\alpha_j > 0$, $(n+\alpha_1)/p_1 = (n+\alpha_2)/p_2$, and $1/p_2 \geq 1/p_1 - 1/2n$. Let $u \in L^{p_1}_{\alpha_1}(D)$. Then $u$ belongs to $L^{p_2}_{\alpha_2}(D)$ under the extra condition that $\bar{\partial} u \in L^{p_1}_{\alpha_1+p/2}(D)$.

In condition of $\bar{\partial} u$, one recognizes the gain for the solution of the $\bar{\partial}$-equation in strictly pseudoconvex domains. Let $D$ be a strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^2$ boundary. Let $f \in L^{p}_{\alpha+p/2}(D)$ be a $\bar{\partial}$-closed $(0,1)$ form on $D$. In ([2], [12]) it was proved that there is a solution $u$ for $\bar{\partial} u = f$ such that

$$\|u\|_{p,\alpha} \leq C_{p,\alpha}\|f\|_{p,\alpha+p/2} \quad \text{for} \quad 1 \leq p < \infty, \alpha > 0.$$  

Thus we get the following result.
Corollary 1.3. Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^2$ boundary. Let $1 \leq p_1 \leq p_2 < \infty$, $\alpha_2 > 0$, $(n + \alpha_1)/p_1 = (n + \alpha_2)/p_2$, and $1/p_2 \geq 1/p_1 - 1/2n$. Let $f \in L^p_{\alpha_1 + p_1/2}(D)$ be a $\partial$-closed $(0,1)$ form on $D$. Then there is a solution $u \in L^p_{\alpha_2}(D)$ for $\partial u = f$.

Remark 1.4. When $1 \leq p_1 < 2$, if we take $p = p_1$, $\alpha_1 = 1 - p/2$, and $\alpha_2 = 1$, then Corollary 1.3 implies that for a $\partial$-closed $(0,1)$ form $f \in L^p(D)$ there is a solution $u \in L^q(D)$ for $\partial u = f$, where $1/q = 1/p - 1/(2n + 2)$. The result is the optimal $L^p$-estimate for $\partial$ proved in [14] when $1 \leq p < 2$. For more recent results about estimates for $\partial$ and $\partial_b$ by means of integral kernels we can refer ([1], [3], [4], [5]).

2. Proof of Theorem 1.2

We shall rely on Bonami-Sibony’s ideas [8] for the proof of Theorem 1.2. Before proceeding with the proof, we give the key lemma.

Lemma 2.1 ([8]). Let $B$ be the unit ball, $\tilde{B}$ its homothetic of radius $R_0 > 1$, let $1 \leq p \leq r < \infty$. Then there exists a constant $C > 0$ such that for any $f \in L^p(\tilde{B})$ for which $\partial f$ belongs to $L^t(\tilde{B})$ with $t \geq 1$ and $1/r \geq 1/t - 1/2n$:

$$\left( \int_B |f|^r dV \right)^{1/r} \leq C \left( \int_{\tilde{B}} |f|^p dV \right)^{1/p} + C \left( \int_{\tilde{B}} |\partial f|^t dV \right)^{1/t}.$$  

Proof of Theorem 1.2. It is enough to prove the inequality

$$\int_D |u|^{p_2} \delta^{\alpha_2 - 1} dV \lesssim \int_D |u|^{p_1} \delta^{\alpha_1 - 1} dV + \int_D |\partial u|^{p_1} \delta^{\alpha_1 + p_1/2 - 1} dV.$$  

For $p_0 \in D$ sufficiently near $\partial D$, we translate and rotate the coordinate system so that $z(p_0) = 0$ and the Im $z_1$ axis is perpendicular
to \( \partial D \). Let \( B_\epsilon(p_0) \) denote the non-isotropic ball

\[
B_\epsilon(p_0) = \left\{ \frac{|z_1|^2}{(\epsilon \delta(p_0))^2} + \sum_{j=2}^{n} \frac{|z_j|^2}{\epsilon \delta(p_0)} < 1 \right\}.
\]

Since \( \partial D \) is \( C^2 \), it follows that there is an \( \epsilon_0 > 0 \) such that for \( p_0 \) sufficiently near \( \partial D \) and \( z \in B_{\epsilon_0}(p_0) \) we have \( z \in D \) and

\[
\frac{\delta(p_0)}{2} \leq \delta(z) \leq 2\delta(p_0).
\]

There is a compact subset \( K \) of \( D \), a sequence \( \{p_j\} \) in \( D \setminus K \), and a positive integer \( N \) such that

\[
(2.1) \quad \frac{\delta(p_0)}{2} \leq \delta(z) \leq 2\delta(p_0).
\]

(2.2) the family \( \{ B_{\epsilon_0/2}(p_j) \} \) covers \( D \setminus K \), and

(2.3) each point of \( D \) lies in at most \( N \) of the sets \( B_{\epsilon_0}(p_j) \).

For brevity we denote by \( B_j \) the ball \( B_{\epsilon_0/2}(p_j) \) and by \( \tilde{B}_j \) the ball \( B_{\epsilon_0}(p_j) \).

By homogeneity, it follows from Lemma 2.1 that

\[
(2.4) \quad \left( \int_{B_j} |u|^{p_2} \, dV \right) \delta(p_j)^{-\frac{n+1}{p_2}} \leq \left( \int_{B_j} |u|^{p_1} \, dV \right)^{\frac{p_2}{p_2/p_1}} \delta(p_j)^{-\frac{(n+1)p_2}{p_1}} + \left( \int_{\tilde{B}_j} |\delta u|^{p_1} \, dV \right)^{\frac{p_2}{p_2/p_1}} \delta(p_j)^{-\frac{(n+1)p_2}{p_1}}.
\]

By (2.1), (2.2), and (2.3), we have to give a bound to

\[
\int_D |u|^{p_2} \delta^{\alpha_2-1} \, dV \sim \sum_j \int_{B_j} |u|^{p_2} \delta^{\alpha_2-1} \, dV
\]

\[
\sim \sum_j \left( \int_{B_j} |u|^{p_2} \, dV \right) \delta(p_j)^{\alpha_2-1},
\]
where the summation is a finite sum.

Using (2.4), it is enough to show that

\[(2.5) \sum \left( \int_{B_j} |u|^{p_1} \, dV \right) \delta(p_j)^{- (n+1)(1-p_1/p_2) + (\alpha_2-1)p_1/p_2} < \infty \]

and

\[(2.6) \sum \left( \int_{B_j} |\partial u|^{p_1} \, dV \right) \delta(p_j)^{p_1/2 - (n+1)(1-p_1/p_2) + (\alpha_2-1)p_1/p_2} < \infty.\]

We note that \(-(n+1)(1-p_1/p_2) + (\alpha_2-1)p_1/p_2 = \alpha_1 - 1.\) Hence the inequalities (2.5) and (2.6) follows from (2.3) and growth conditions of \(u\) and \(\partial u.\)

\[\square\]

3. An example

In this section we give an example to show that the embedding in Theorem 1.2 is the optimal result in some sense for strictly pseudo-convex domains. We restrict ourselves to the unit ball \(B_2\) in \(\mathbb{C}^2\).

**Lemma 3.1 ([15]).** For \(z \in B_2, c \) real, \(\eta > -1,\) define

\[J_{c, \eta}(z) = \int_{B_2} \frac{(1 - |\zeta|^2)^\eta}{1 - \zeta \cdot z |n+1+\eta+c}|dV(\zeta).\]

When \(c < 0,\) then \(J_{c, \eta}\) is bounded in \(B_2.\) When \(c > 0,\) then \(J_{c, \eta}(z) \approx (1 - |z|^2)^{-c.}\) Finally, \(J_{0, \eta} \approx - \log(1 - |z|^2).\)

**Theorem 3.2.** Let \(1 \leq p_1 \leq p_2 < \infty, \alpha_j > 0,\) and \((2 + \alpha_1)/p_1 = (2 + \alpha_2)/p_2.\) For any \(\epsilon > 0\) there is \(u_{p_1, \alpha_1, \epsilon} \in L_{\alpha_1}^{p_1}(B_2)\) such that \(u_{p_1, \alpha_1, \epsilon}\) does not belong to \(L_{\alpha_2}^{p_2+\epsilon}(B_2)\) or \(L_{\alpha_2-\epsilon}^{p_2}(B_2),\) while \(\partial u_{p_1, \alpha_1, \epsilon}\) belongs to \(L_{\alpha_1+p_1/2}^{p_1}(B_2).\)
Proof. If \( dp_1 - \alpha_1 - p_1/2 = 2 - \mu \) then \( dp_2 - \alpha_2 - p_2/2 + \mu p_2/p_1 = 2 \). So, we can choose \( d > 0 \) such that \( dp_1 - \alpha_1 - p_1/2 < 2, \ dp_2 - \alpha_2 - p_2/2 + \varepsilon > 2 \), and \( dp_2 - \alpha_2 - p_2/2 + \varepsilon(d - 1/2) > 2 \).

Let \( u_{p_1, \alpha_1, \varepsilon}(z_1, z_2) = \bar{z}_2/(1 - z_1)^d \). For simplicity of notation, we write \( u = u_{p_1, \alpha_1, \varepsilon} \) and \( r_{z_1} = \sqrt{1 - |z_1|^2} \). Then we have

(3.1)

\[
\|u\|_{p_1, \alpha_1} \lesssim \int_{B_2} \frac{|z_2|^{p_1}(1 - |z_1|^2 - |z_2|^2)^{\alpha_1 - 1}}{|1 - z_1|^{d_{p_1}}} dV
\]

\[
\lesssim \int_{|z_1| < 1} \frac{dA(z_1)}{|1 - z_1|^{d_{p_1}}} \int_{|z_2| < r_{z_1}} |z_2|^{p_1}(1 - |z_1|^2 - |z_2|^2)^{\alpha_1 - 1} dA(z_2).
\]

By the polar coordinate change \( |z_2|^2 = r e^{i\theta} \), we have

(3.2)

\[
I(z_1) = \int_{|z_1| < r_{z_1}} |z_2|^{p_1}(1 - |z_1|^2 - |z_2|^2)^{\alpha_1 - 1} dA(z_2)
\]

\[
= 2\pi (1 - |z_1|^2)^{\alpha_1 - 1} \int_0^{r_{z_1}} r^{p_1 + 1} \left( 1 - \frac{r^2}{1 - |z_1|^2} \right)^{\alpha_1 - 1} dr
\]

\[
= 2\pi (1 - |z_1|^2)^{\alpha_1 + p_1/2} \int_0^1 (1 - s^2)^{\alpha_1 - 1}s^{p_1 + 1} ds,
\]

where we set \( s = r/\sqrt{1 - |z_1|^2} \). Note that

\[
\int_0^1 (1 - s^2)^{\alpha_1 - 1}s^{p_1 + 1} ds = \frac{1}{2} B \left( \frac{p_1}{2} + 1, \alpha_1 \right),
\]

where \( B(\cdot, \cdot) \) is the beta function. By (3.1), (3.2) and Lemma 3.2, it follows that

\[
\|u\|_{p_1, \alpha_1} \lesssim \int_{|z_1| < 1} \frac{dA(z_1)}{|1 - z_1|^{d_{p_1} - \alpha_1 - p_1/2}}
\]

\[
= \lim_{r \to 1^-} \int_{|z_1| < 1} \frac{dA(z_1)}{|1 - z_1 r|^{d_{p_1} - \alpha_1 - p_1/2}}
\]

\[
\lesssim 1, \ \text{since} \ dp_1 - \alpha_1 - \frac{p_1}{2} < 2.
\]
Similarly, we can prove that \( \|\bar{\partial}u\|_{p_1, \alpha_1 + p_1/2} \leq 1 \).

Now we have

\[
\|u\|_{p_2 + \varepsilon, \alpha_2} = \int_{|z_1|<1} \frac{dA(z_1)}{|1 - z_1|^{d(p_2 + \varepsilon)}} \int_{|z_2|<r_{z_1}} |z_2|^{p_2 + \varepsilon} (1 - |z_1|^2 - |z_2|^2)^{\alpha_2 - 1} dA(z_2)
\]

\[
= 2\pi \int_{|z_1|<1} \frac{(1 - |z_1|^2)^{\alpha_2 + (p_2 + \varepsilon)/2}}{|1 - z_1|^{d(p_2 + \varepsilon)}} \int_{0}^{1} (1 - s^2)^{\alpha_2 - 1} s^{p_2 + 1 + \varepsilon} ds
\]

\[
\approx \lim_{r \to 1} \int_{|z_1|<1} \frac{(1 - |z_1|^2)^{\alpha_2 + (p_2 + \varepsilon)/2}}{|1 - z_1 r|^{d(p_2 + \varepsilon)}} dA(z_1)
\]

\[
= \lim_{r \to 1} \frac{1}{(1 - r^2)^{dp_2 - \alpha_2 - p_2/2 + \varepsilon(d-1/2)/2}} = \infty,
\]

since \( dp_2 - \alpha_2 - (p_2 + \varepsilon)/2 = dp_2 - \alpha_2 - p_2/2 + \varepsilon \) and \( dp_2 - \alpha_2 - p_2/2 + \varepsilon > 2 \).

Similarly, we can show that \( \|u\|_{p_2, \alpha_2 - \varepsilon} \) is divergent since \( dp_2 - \alpha_2 - p_2/2 + \varepsilon > 2 \). Thus we get the result. \( \square \)

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