THE BESOV SPACES OF M-HARMONIC FUNCTIONS

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ABSTRACT. We extend the characterization for the analytic Besov space obtained by Nowak to the invariant harmonic Besov space.

1. Introduction

Let \( H(B) \) and \( h(B) \) denote the spaces of holomorphic functions and of invariant harmonic functions on the unit ball \( B \) of \( C^n \), respectively. For \( 0 < p < \infty \), the Bergman space \( L^p_\alpha(B) \), the Hardy space \( H^p(B) \) and the Besov space \( B^p_p(B) \) in the unit ball of \( C^n \) are defined respectively as

\[
L^p_\alpha(B) = \{ f \in H(B) : \| f \|_{L^p_\alpha}^p = \int_B |f(z)|^p d\nu(z) < \infty \},
\]

\[
H^p(B) = \{ f \in H(B) : \| f \|_{H^p}^p = \sup_{0 < r < 1} \int_S |f(r\zeta)|^p d\sigma(\zeta) < \infty \}
\]

and

\[
B^p_p(B) = \{ f \in H(B) : \| f \|_{B^p_p}^p = \int_B (\hat{Q}f)^p(z) d\lambda(z) < \infty \}
\]

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where $\nu$ is the normalized Lebesgue measure on $B$, $S$ is the boundary of $B$, $\sigma$ is the normalized Lebesgue measure on $S$, $\lambda$ is the invariant measure on $B$ and $\hat{Q}f$ is the maximal derivative of $f$ with respect to the Bergman metric on $B$. We use the notations $L^p_n(B)$, $h^p(B)$ and $MB_p(B)$ in cases of invariant harmonic functions.

K. Stroethoff gave the following characterization for the Besov space on the unit disc $D$.

**Theorem A** [6]. If $2 < p < \infty$, then for an analytic function $f$ on $D$,

$$f \in B_p \iff \int_D \int_D \left| \frac{f(z) - f(w)}{z - w} \right|^p \left(1 - |z|^2\right)^{\frac{p}{2}} \left(1 - |w|^2\right)^{\frac{p}{2}} d\lambda(w) d\lambda(z) < \infty.$$ 

Recently M. Nowak ([3]) extended the theorem to $n \geq 2$ case. In this paper we shall give an $M$-harmonic version of the above characterization in the unit ball. The main result of the paper is as follows:

**Theorem B.** Assume that $f \in C^1(B)$ and $2n < p < \infty$. Then $f \in MB_p$ if and only if

$$\int_B \int_{E(z,r)} \left| \frac{f(z) - f(w)}{1 - \langle z, w \rangle} \right|^p \left(1 - |z|^2\right)^{\frac{p}{2}} \left(1 - |w|^2\right)^{\frac{p}{2}} d\lambda(w) d\lambda(z) < \infty.$$ 

2. Notations and Preliminaries

We introduce a few facts that we need in the sequel, most of which are well known. See [1] and [5] for details. For each $a \in B$, the M"obius transformation $\varphi_a : B \to B$ is defined by

$$(2.1) \quad \varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle}, \quad z \in B,$$

where $s_a = \sqrt{1 - |a|^2}$, $P_a$ is the orthogonal projection from $C^n$ onto the subspace generated by $a$ and $Q_a = I - P_a$ i.e., $P_a z = \frac{\langle a, z \rangle}{|a|^2} a$. 
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For \( a \in B \) and \( z \in \bar{B} \),

\[
1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - <z, a>|^2}.
\]

The determinant \( J_R \varphi_a(z) \) of the real Jacobian matrix of \( \varphi_a \) satisfies the following:

\[
J_R \varphi_a(z) = |J_C \varphi_a(z)|^2 = \left(\frac{1 - |a|^2}{|1 - <z, a>|^2}\right)^{n+1} = \left(\frac{1 - |\varphi_a(z)|^2}{1 - |z|^2}\right)^{n+1}.
\]

The following transformation formula holds under \( \varphi_a \in Aut(B) \)

\[
d\nu(\varphi_a(z)) = |J_C \varphi_a(z)|^2 d\nu(z).
\]

For \( 0 < r < 1 \), \( \varphi_a \) is a biholomorphic map from the ball \( rB = B(0, r) \) onto the ellipsoid \( E(a, r) := \{z \in B : |\varphi_a(z)| < r\} \). The invariant measure is given by

\[
d\lambda(z) = \frac{1}{(1 - |z|^2)^{n+1}} d\nu(z).
\]

The invariant Laplacian \( \tilde{\Delta} \) on \( B \) is given by

\[
\tilde{\Delta} f(z) = \frac{1}{n+1} \Delta (f \circ \varphi_z)(0) = \frac{4}{n+1} (1 - |z|^2) \sum_{i,j=1}^{n} (\delta_{i,j} - z_i \bar{z}_j) \frac{\partial^2 f(z)}{\partial z_i \partial \bar{z}_j}, \quad f \in C^2(B),
\]

where \( \Delta = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} \) is the usual Laplacian. An invariant harmonic or simply M-harmonic function is a function in \( C^2(B) \) which is annihilated by \( \tilde{\Delta} \) in \( B \). For a \( C^1 \)-function \( f \) the invariant gradient \( \tilde{\nabla} \) is the vector field on \( B \) defined by

\[
\tilde{\nabla} f(z) = \nabla (f \circ \varphi_z)(0) = \frac{2}{n+1} (1 - |z|^2) \sum_{i,j=1}^{n} [\delta_{i,j} - z_i \bar{z}_j] \left( \frac{\partial f}{\partial \bar{z}_i} \frac{\partial}{\partial z_j} + \frac{\partial f}{\partial z_j} \frac{\partial}{\partial \bar{z}_i} \right)
\]
where $\nabla$ is the real gradient in $\mathbb{R}^{2n}$. Then

$$|\hat{\nabla} f(z)|^2 = \frac{2}{n+1}(1-|z|^2) \sum_{i,j=1}^{n} [\delta_{i,j} - z_i \bar{z}_j] \left( \frac{\partial f}{\partial z_i} \frac{\partial \bar{f}}{\partial z_j} + \frac{\partial f}{\partial z_j} \frac{\partial \bar{f}}{\partial \bar{z}_i} \right).$$

The Laplacian $\tilde{\Delta}$ and the gradient $\hat{\nabla}$ are both invariant under the automorphisms of $B$. The Bergman metric $\beta : B \times \mathbb{C}^n \to \mathbb{R}$ is defined by the differential form:

$$\beta^2(z, \xi) = \sum_{i,j=1}^{n} b_{ij}(z) \xi_i \bar{\xi}_j,$$

(2.6)

$$= \frac{(1-|z|^2)|\xi|^2 + |<z, \xi>|^2}{(1-|z|^2)^2}, \quad z, \xi \in B$$

where

$$b_{ij}(z) = \frac{\partial^2 \log K(z, \bar{z})}{\partial z_i \partial \bar{z}_j}, \quad (i,j = 1, \cdots, n)$$

and

$$K(z, w) = \frac{1}{(1 - <z, w>)^{n+1}}, \quad z, w \in B$$

denotes the Bergman kernel of $B$. It follows from (2.6) that

(2.7) $\frac{|\xi|}{\sqrt{1-|z|^2}} \leq \beta(z, \xi) \leq \frac{|\xi|}{1 - |z|^2}, \quad z, \xi \in B.$

**Definition.** Let $f \in C^1(B)$ and $\xi \in \mathbb{C}^n$. The maximal derivative of $f$ with respect to the Bergman metric $\beta$ on $B$ is defined by

$$\hat{\nabla} f(z) = \sup_{|\xi|=1} \frac{|df(z) \cdot \xi|}{\beta(z, \xi)}, \quad z \in B$$

where

$$df(z) \cdot \xi = \sum_{i=1}^{n} \left[ \frac{\partial f}{\partial z_i}(z) \xi_i + \frac{\partial f}{\partial \bar{z}_i}(z) \bar{\xi}_i \right] = \partial f(z) \cdot \xi + \bar{\partial} f(z) \cdot \bar{\xi}.$$
The following identities are easily verified. For a $C^1$-function $f$ in $B$ and $\varphi \in Aut(B)$,

\[\hat{Q}(f \circ \varphi) = (\hat{Q}f) \circ \varphi,\]
\[\frac{1}{2} \sqrt{\Delta|f|^2} \leq \hat{Q}f = 2|\hat{\nabla}f| \leq \sqrt{\Delta|f|^2}.\] (2.8)

3. Proof of Theorem B

Proof. [Proof of the necessity of Theorem B]. The mean value property of $f \in h(B)$ implies that

\[f(z) = \frac{1}{r^{2n}} \int_{rB} f(w)K_B(\frac{z}{r}, \frac{w}{r})d\nu(w), \quad z \in rB\]

for the reproducing kernel $K_{rB}(z, w) = \frac{1}{r^{2n}}K_B(\frac{z}{r}, \frac{w}{r})$ of $rB$. Using similar arguments as in the proof of (1.6a) [1], we have that

\[|\nabla f(0)| \leq C\|f\|_{L^p(rB, \nu)}\]

where $C$ is a positive constant independent of $f$. Replacing $f$ by $f \circ \varphi_a - f(a)$ yields

\[|\nabla (f \circ \varphi_a)(0)| \leq C\left(\int_{rB} |(f \circ \varphi_a)(w) - f(a)|^p d\nu(w)\right)^{1/p}.\]

By (2.5) and (2.8),

\[(\hat{Q}f)(a) = C \left(\int_{rB} |(f \circ \varphi_a)(w) - f(a)|^p d\nu(w)\right)^{1/p} \]

(a change of variable)

\[= C \left(\int_{E(a, r)} |f(w) - f(a)|^p d\nu(\varphi_a(w))\right)^{1/p} \]

(by (2.3) and (2.4))

\[= C \left(\int_{E(a, r)} |f(w) - f(a)|^p \frac{(1 - |a|^2)^{n+1}}{|1 - <w, a>|^{2n+2}} d\nu(w)\right)^{1/p} \]

\[\leq C \left(\int_{E(a, r)} |f(w) - f(a)|^p \frac{1}{|1 - <w, a>|^{n+1}} d\nu(w)\right)^{1/p}.\]
Then we have
\[(\hat{Q}f)^p(a) \leq C \int_{E(a,r)} \frac{|f(w) - f(a)|^p}{|1 - < w, a >|^{n+1}} d\nu(w).\]

Integrating both sides of the inequality and using the fact that for \( w \in E(a, r) \)
\[
\frac{(1 - |w|^2)^{n+1}}{|1 - < w, a >|^{n+1-p}} \approx (1 - |w|^2)^\frac{p}{2} \left(1 - |a|^2\right)^\frac{p}{2},
\]

\[
\int_B \frac{f(w) - f(a)}{J_B} d\nu(a) \leq C \int_B \int_{E(a,r)} \frac{|f(w) - f(a)|^p}{|1 - < w, a >|^{n+1-p}} d\nu(w) d\lambda(a)
\]
\[
= C \int_B \int_{E(a,r)} \frac{|f(w) - f(a)|^p}{|1 - < w, a >|^{n+1-p}} \cdot \frac{(1 - |w|^2)^{n+1}}{|1 - < w, a >|^{n+1-p}} d\lambda(w) d\lambda(a)
\]
\[
\leq \int_B \int_{E(a,r)} \frac{|f(w) - f(a)|^p}{|1 - < w, a >|^p} \left(1 - |w|^2\right)^\frac{p}{2} \left(1 - |a|^2\right)^\frac{p}{2} d\lambda(w) d\lambda(a).
\]

\(\square\)

The following lemmas will be needed to prove the converse part of Theorem B.

**Lemma 3.1.** For \( f \in C^1(B) \), \( 1 < p < \infty \)
\[
\int_B \frac{|f(z) - f(0)|^p}{|z|^p} (1 - |z|)^{\frac{p}{2} - n-1} d\nu(z)
\]
\[
\leq \int_B \frac{(\hat{Q}f)^p(z)(1 - |z|)^{\frac{p}{2} - n-1}}{|z|^{n+p-2}} d\nu(z).
\]
Proof. For \( z \in B \) and a \( C^1 \)-function \( f \), we have by the proof of Lemma 1.1 in [1]

\[
|f(z) - f(0)| = \left| \int_0^1 \frac{df(tz)}{dt} \, dt \right| \\
= \left| \int_0^1 \frac{df(tz)}{\beta(tz, z)} \cdot \frac{z}{\beta(tz, z)} \, \beta(tz, z) \, dt \right| \\
\leq \int_0^1 (\hat{Q}f)(tz) \cdot \beta(tz, z) \, dt \\
(by (2.7)) \leq \int_0^1 (\hat{Q}f)(tz) \frac{|z|}{1 - |tz|} \, dt.
\]

Let \( q \) be the conjugate exponent of \( p \). By Hölder inequality, for \( 1/p + 1/q = 1 \), we have

\[
\frac{|f(z) - f(0)|}{|z|^q} \leq \int_0^1 (\hat{Q}f)(tz) \frac{1}{1 - |tz|} \, dt \\
\leq \left\{ \int_0^1 (\hat{Q}f)^p(tz) \left( \frac{1}{1 - |tz|} \right) \frac{1}{2} \, dt \right\}^{\frac{1}{p}} \times \left\{ \int_0^1 \left( \frac{1}{1 - |tz|} \right) \frac{1}{2} \, dt \right\}^{\frac{1}{q}}.
\]

Thus

\[
\frac{|f(z) - f(0)|^p}{|z|^p} \leq \int_0^1 (\hat{Q}f)^p(tz) \left( \frac{1}{1 - |tz|} \right) \frac{1}{2} \, dt \times \left\{ \int_0^1 \left( \frac{1}{1 - |tz|} \right) \frac{1}{2} \, dt \right\} \frac{p}{q}.
\]

By an elementary calculation in the second integral,

\[
\left\{ \int_0^1 \left( \frac{1}{1 - |tz|} \right) \frac{1}{2} \, dt \right\} \frac{p}{q} \leq C \left( \frac{1}{|z|} \right)^{\frac{p}{q}} (1 - |z|)^{-\frac{q+2}{2}} \frac{p}{q}.
\]

Then

\[
\frac{|f(z) - f(0)|^p}{|z|^p} \leq C \left( \frac{1}{|z|} \right)^{\frac{p}{q}} (1 - |z|)^{-\frac{q+2}{2}} \frac{p}{q} \int_0^1 (\hat{Q}f)^p(tz) \left( \frac{1}{1 - |tz|} \right) \frac{1}{2} \, dt.
\]
In integrating both sides on the unit ball,
\[
\int_B \frac{|f(z) - f(0)|^p}{|z|^p} (1 - |z|)^\frac{p}{q} - n - 1 \, d\nu(z)
\]
\[
\leq C \int_B \int_0^1 \frac{1}{|z|_0} \frac{1}{(1 - |z|)^{\frac{q-p}{p}}} (\hat{Q}f)^p(tz) \left(\frac{1}{1 - |tz|}\right)^\frac{p}{q} (1 - \frac{1}{|z|})^{\frac{p}{2} - n - 1} \, dt \, d\nu(z)
\]
\[
= \int_0^1 \int_{B_t} \frac{(\hat{Q}f)^p(z)(1 - \frac{|z|}{t})^{\frac{p}{2} - n - 1}}{|z|_t \left(1 - \frac{|z|}{t}\right)^{\frac{q-p}{p}} \cdot (1 - |z|)^{\frac{p}{2}}} \, d\nu(z) \left(\frac{1}{t}\right)^{2n} \, dt
\]
\[
= \int_B \frac{(\hat{Q}f)^p(z)}{|z|_q (1 - |z|)^{\frac{p}{q}}} \, d\nu(z) \int_0^1 \left(\frac{1}{t}\right)^{2n} (1 - \frac{|z|}{t})^{\frac{p}{q} - n - 1} \frac{\left(\frac{1}{t}\right)^{\frac{p}{q}}}{\left(\frac{1}{t}\right)^{\frac{p}{q}} - n} \, dt.
\]

Since
\[
\int_0^1 \left(\frac{1}{t}\right)^{2n} (1 - \frac{|z|}{t})^{\frac{p}{q} - n - 1} \frac{\left(\frac{1}{t}\right)^{\frac{p}{q}}}{\left(\frac{1}{t}\right)^{\frac{p}{q}} - n} \, dt
\]
\[
\leq \left(\frac{1}{|z|}\right)^{n-1} \int_0^1 \left(t - |z|\right)^{\frac{p}{q} - n - 1} \, dt
\]
\[
= C \left(\frac{1}{|z|}\right)^{n-1} (1 - |z|)^{\frac{p}{q} - n},
\]
we have
\[
\int_B \frac{|f(z) - f(0)|^p}{|z|^p} (1 - |z|)^\frac{p}{q} - n - 1 \, d\nu(z)
\]
\[
\leq \int_B \frac{(\hat{Q}f)^p(z)}{|z|_q (1 - |z|)^{\frac{p}{q}}} \left(\frac{1}{|z|}\right)^{n-1} (1 - |z|)^{\frac{p}{q} - n} \, d\nu(z)
\]
\[
= \int_B \frac{(\hat{Q}f)^p(z)(1 - |z|)^{\frac{p}{q} - n - \frac{p}{q}}}{|z|^{\frac{p}{q} + n - 1}} \, d\nu(z)
\]
\[
= \int_B \frac{(\hat{Q}f)^p(z)(1 - |z|)^{\frac{p}{q} - n - 1}}{|z|^{n+p-2}} \, d\nu(z).
\]
Let $2n < p < \infty$ and $-1 < \alpha < \infty$ and let $\gamma$ be a positive number such that
\[
\max\left(1 - \frac{1}{p}, 1 + \frac{\alpha - 1}{p}\right) < \gamma < 1 + \frac{\alpha}{p}.
\]

**Lemma 3.2** [3]. For the Bergman kernel $K(z, w)$, there exists a constant $C > 0$ such that for each $z \in B$
\[
\int_B \frac{|K(z, w)|^2 \left(1 - |\varphi_w(z)|\right)^\alpha}{|\varphi_w(z)|^{2n + \alpha + p(1 - \gamma) - 1}} \, d\nu(w) = CK(z, z).
\]

**Lemma 3.3.** Let $2n < p < \infty$. Then there exists a positive constant $C$ such that for every $f \in C^1(B)$
\[
\int_B \int_B \frac{|(f \circ \varphi_z)(w) - f(z)|^p}{|w|^p} \left(1 - |w|\right)^{\frac{p}{2} - n - 1} \, d\nu(w) \, d\lambda(z)
\leq C \int_B (\hat{Q}f)^p(w) \, d\lambda(w).
\]

**Proof.** Let $f \in C^1(B)$. Lemma 3.1 with $f$ replaced by $f \circ \varphi_z$ implies
\[
\int_B \frac{|(f \circ \varphi_z)(w) - f(z)|^p}{|w|^p} \left(1 - |w|\right)^{\frac{p}{2} - n - 1} \, d\nu(w)
\leq C \int_B \hat{Q}(f \circ \varphi_z)^p(w) \left(1 - |w|\right)^{\frac{p}{2} - n - 1} \, d\nu(w).
\]

By Lemma 3.2, we can proceed analogously to the proof of Lemma 3.6 in [2].
Proof. [Completion of the proof of Theorem B]. Let \( f \in MB_p \). Then

\[
\int_B \int_{B(z, r)} \left| \frac{f(z) - f(w)}{1 - z, w} \right|^p (1 - |z|^2)^{\frac{p}{2}} (1 - |w|^2)^{\frac{p}{2}} d\lambda(w) d\lambda(z)
\]
\[
\leq \int_B \int_B \left| \frac{f(z) - f(w)}{w - Pwz - sqwz} \right|^p (1 - |z|^2)^{\frac{p}{2}} (1 - |w|^2)^{\frac{p}{2}} d\lambda(w) d\lambda(z)
\]
(by (2.1))
\[
= \int_B \int_B \frac{|f(z) - f(w)|^p}{\varphi_w(z)} \cdot |1 - z, w| \cdot (1 - |z|^2)^{\frac{p}{2}} (1 - |w|^2)^{\frac{p}{2}} d\lambda(w) d\lambda(z)
\]
(by (2.2))
\[
= \int_B \int_B |f(z) - f(w)|^p \left| \frac{\varphi_w(z)}{1 - |\varphi_w(z)|^2} \right|^p (1 - |\varphi_w(z)|^2)^{\frac{p}{2}} d\lambda(w) d\lambda(z)
\]
\[
= \int_B \int_B \left| \frac{(f \circ \varphi_w)(u) - f(w)}{|u|^p} \right|^p (1 - |u|^2)^{\frac{p}{2}} d\nu(u) d\lambda(w)
\]
\[
= \int_B \int_B \left| \frac{\hat{Q}f}(w) \right|^p (1 - |u|^2)^{\frac{p}{2}} d\nu(u) d\lambda(w)
\]
\[
\leq C \int_B \hat{Q}f^p(w) d\lambda(w).
\]
\[\square\]

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