REPRESENTATION OF $L^1$-VALUED CONTROLLER ON BESOV SPACES

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ABSTRACT. This paper will show that the relation

$$L^1(\Omega) \subset C_0(\overline{\Omega}) \subset H_{p,q}$$

if $1/p' - 1/n(1 - 2/q') < 0$ where $p' = p/(p - 1)$ and $q' = q/(q - 1)$ where $H_{p,q} = (W^{1,p}_0, W^{-1,p})_{1/q, q}$. We also intend to investigate the control problems for the retarded systems with $L^1(\Omega)$-valued controller in $H_{p,q}$

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. Let $A(x, D_x)$ be an elliptic differential operator of second order as follows.

$$A(x, D_x) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial}{\partial x_i}) + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

where $\{a_{i,j}(x)\}$ is a positive definite symmetric matrix for each $x \in \Omega$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$.

If we put that $A_0 u = -A(x, D_x)u$ then it is known that $A_0$ generates an analytic semigroup in $W^{-1,p}(\Omega)$ where $W^{-1,p}(\Omega)$ is the dual

Received May 19, 2003
2000 Mathematics Subject Classification: Primary 35B37, Secondary 93C20
Key words and phrases. Besop spaces, Sobolev's embedding theorem, retarded system, approximate controllability, observability
space of $W_0^{1,p'}(\Omega), p' = p/(p-1)$ as is seen in [5]. Therefore, from the interpolation theory it is easily seen that the operator $A_0$ generates an analytic semigroup in $H_{p,q} = (W_0^{1,p}, W^{-1,p'})_{1/q,q}$. In the section 4, we will show that the relation

$$L^1(\Omega) \subset C_0(\Omega) \subset H_{p,q}$$

if $1/p' - 1/n(1 - 2/q') < 0$ where $p' = p/(p-1)$ and $q' = q/(q-1)$.

Hence we intend to investigate the control problem for the following retarded system with $L^1(\Omega)$-valued controller:

$$\begin{align*}
(1.1) & & L^1(\Omega) \subset C_0(\Omega) \subset H_{p,q} \\
(1.2) & & \frac{d}{dt} u(t) = A_0 u(t) + A_1 u(t-h) + \int_{-h}^0 a(s) A_2 u(t+s) ds + \Phi_0 w(t), \quad t \in (0,T] \\
(1.3) & & u(0) = g^0, \quad u(s) = g^1(s) \quad s \in [-h,0),
\end{align*}$$

in the space $H_{p,q}$. Here, $A_\iota u = -A_\iota(x,D_x) u, \iota = 1, 2$, where $A_\iota(x,D_x)$ are second order linear differential operators with real coefficients. The kernel $a(\cdot)$ belongs to $L^q'(-h,0)$ where $h$ is a fixed positive number and the controller $\Phi_0$ is a bounded linear operator from some Banach space $U$ to $L^1(\Omega)$. The initial data $g^0, g^1$ are given functions so that needed for the construction of solution semigroup for (1.2) and (1.3) and of $L^1(\Omega)$-valued controller. From the relation (1.1) we shall deal with approximate controllability and observability for the system (1.2) and (1.3) in the space $H_{p,q}$ choosing $p$ and $q$ such that $1/p' - 1/n(1 - 2/q') < 0$.

In view of Sobolev’s embedding theorem we may also consider $L^1(\Omega) \subset W^{-1,p}(\Omega)$ if $1 < p < n/(n-1)$ as is seen in [5]. Hence, we can investigate the system (1.2) and (1.3) in the space $W^{-1,p}(\Omega)$ considering $\Phi_0$ as an operator into $W^{-1,p}(\Omega)$. Furthermore, it is known that $W^{-1,p}(\Omega)$ is $\zeta$-convex and the initial value problem

$$\begin{align*}
(1.4) & & \frac{d}{dt} u(t) = A_0 u(t) + f(t), \quad t \in (0,T], \\
& & u(0) = u_0
\end{align*}$$
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has a unique solution $u \in L^q(0, T; W_0^{1,p}(\Omega) \cap W^{1,q}(0, T; W^{-1,p}(\Omega))$ for any $u_0 \in H_{p,q}$ and $f \in L^q(0, T; W^{-1,p}(\Omega))$ (see Theorem 3.1 in [5]). Thereafter, we can apply the method of G. D. Blasio, K. Kunisch and E. Sinestrari [3] to the system (1.2) and (1.3) to show the existence and uniqueness of the solution

$$u \in L^q(0, T; W_0^{1,p}(\Omega) \cap W^{1,q}(0, T; W^{-1,p}(\Omega)) \subset C([0, T]; H_{p,q})$$

Since $Q$ is a mapping into $W^{-1,p}(\Omega)$ not into $H_{p,q}$, we cannot express the solution $u(t; 0, Q \circ w)$ of system of (1.2) and (1.3) with $g = 0$ using the solution semigroup $S(t)$. Here, the solution semigroup for the system (1.2) and (1.3) is defined by

$$S(t)g = (u(t; g, 0), u_t(\cdot; g, 0))$$

where $g = (g^0, g^1) \in Z_{p,q} = H_{p,q} \times L^q(-h, 0; W_0^{1,p}(\Omega))$, $u(t; g, 0)$ is the solution of (1.2) and (1.3) with $Q = 0$ and $u_t(\cdot; g, 0)$ is the function $u_t(s; g, 0) = u(t + s; g, 0)$ defined in $[-h, 0]$. Therefore, we have to define the approximate controllability and observability in $W^{-1,p}(\Omega)$ using the fundamental solution as is seen in definition 5.1 in [5]. Here, we note that in order to existing of fundamental solution of system (1.2) and (1.3), we must need the assumption that $\alpha(\cdot)$ is Hölder continuous.

In this paper, assuming that $\alpha(\cdot)$ has only to belong to $L^q(-h, 0)$, with the aid of the relation (1.1) we can define the approximate controllability and observability in $H_{p,q}$ without using the fundamental solution. We define the set of attainability by

$$R = \{ \int_0^t S(t - \tau)Qw(t)d\tau : w \in L^q(0, t; U), \ t > 0 \}$$

where $Qw = (Q_0w, 0)$. We say that the system (1.2) and (1.3) is approximately controllable if $R$ is dense in $Z_{p,q}$ and the adjoint system is observability if $\phi \in Z_{p,q}'$, $Q_0^*\phi(t; \phi) = 0$ implies $\phi = 0$. where $v(t; \phi)$ is a solution the following adjoint system of (1.2) and (1.3).

$$\frac{d}{dt}v(t) = A_0^*v(t) + A_1^1v(t - h) + \int_{-h}^0 a(s)A_2u(t + s)ds,$$

$$v(0) = \phi^0, \quad v(s) = \phi^1(s), \quad s \in [-h, 0].$$
where \( \phi = (\phi^0, \phi^1) \in Z_{p',q'} \). The structural operator \( F : Z_{p,q} \to Z_{p',q'} \) is defined by

\[
Fg = (g^0, A_1 g^1 (-h - s) + \int_{-h}^0 a(\tau) A_2 g^1 (\tau - s) d\tau).
\]

We will show that if \( F \) is an isomorphism, then the approximate controllability of (1.2) and (1.3) is equivalent to the observability of (1.4) and (1.5). Finally, we remark that in the space \( W^{-1,p}(\Omega) \) we cannot define the attainability set using solution semigroup \( S(t) \) and it is said that the system (1.4) and (1.5) is observable if \( \phi \in Z_{p',q'} \), \( \Phi^0_\phi v(t; \phi) \equiv 0 \) almost everywhere implies \( \phi = 0 \).

2. Notations

Let \( \Omega \) be a region in an \( n \)-dimensional Euclidean space \( \mathcal{R}^n \) and closure \( \overline{\Omega} \). \( C^m(\Omega) \) is the set of all \( m \)-times continuously differential functions on \( \Omega \). \( C^m_0(\Omega) \) will denote the subspace of \( C^m(\Omega) \) consisting of these functions which have compact support in \( \Omega \). \( W^{m,p}(\Omega) \) is the set of all functions \( f = f(x) \) whose derivative \( D^\alpha f \) up to degree \( m \) in distribution sense belong to \( L^p(\Omega) \). As usual, the norm is then given by

\[
||f||_{m,p,\Omega} = (\sum_{\alpha \leq m} ||D^\alpha f||^p_{p,\Omega})^{1/p}, \quad 1 \leq p < \infty
\]

\[
||f||_{m,\infty,\Omega} = \max_{\alpha \leq m} ||D^\alpha f||_{\infty,\Omega};
\]

where \( D^0 f = f \). In particular, \( W^{0,p}(\Omega) = L^p(\Omega) \) with the norm \( || \cdot ||_{p,\Omega} \). Let \( p' = p/(p - 1) \), \( 1 < p < \infty \). \( W^{-1,p}(\Omega) \) stands for the dual space \( W^{1,p'}_0(\Omega) \) of \( W^{1,p}_0(\Omega) \) whose norm is denoted by \( || \cdot ||_{-1,p,\infty} \).

If \( X \) is a Banach space and \( 1 < p < \infty \), \( L^p(0,T; X) \) is the collection of all strongly measurable functions from \( (0,T) \) into \( X \) the \( p \)-th powers of norms are integrable. \( C^m([0,T]; X) \) will denote the set of all \( m \)-times continuously differentiable functions from \( [0,T] \) into \( X \).
If $X$ and $Y$ are two Banach spaces, $B(X, Y)$ is the collection of all bounded linear operators from $X$ into $Y$, and $B(X, X)$ is simply written as $B(X)$. For an interpolation couple of Banach spaces $X_0$ and $X_1$, $(X_0, X_1)_{\theta,p}$ and $[X_0, X_1]_{\theta}$ denote the real and complex interpolation spaces between $X_0$ and $X_1$, respectively. Let $B^s_{p,q}(\Omega)$ denote the Besov space and $B^s_{p,q}(\Omega)$ will be denote the subspace of $B^s_{p,q}(\Omega)$ consisting of those functions which have compact support in $\Omega$.

3. Preliminaries

Let $\Omega$ be a bounded differential domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. Consider an elliptic differential operator of second order

$$A(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} (a_{i,j}(x) \frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

where $\{a_{i,j}(x)\}$ is a positive definite symmetric matrix for each $x \in \Omega$. The operator

$$A'(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} (a_{i,j}(x) \frac{\partial}{\partial x_i}) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (b_i(x) \cdot) + c(x)$$

is the formal adjoint of $A$.

For $1 < p < \infty$ we denote the realization of $A$ in $L^p(\Omega)$ under the Dirichlet boundary condition by $A_p$

$$D(A_p) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega),$$
$$A_p u = Au \quad \text{for} \quad u \in D(A_p).$$

For $p' = p/(p - 1)$, we can also define the realization $A'$ in $L^{p'}(\Omega)$ under Dirichlet boundary condition by $A'_p$

$$D(A'_p) = W^{2,p'}(\Omega) \cap W^{1,p'}_0(\Omega),$$
$$A'_p u = A' u \quad \text{for} \quad u \in D(A'_p).$$
It is known that $-A_p$ and $-A^{'}_p$, generate analytic semigroups in $L^p(\Omega)$ and $L^p(\Omega)$, respectively, and $A^*_p = A^{'}_p$. From the result of R. Seeley [10] (see also H. Triebel [14;p. 321]) we obtain that

$$[D(A_p), L^p(\Omega)]^{1/2} = W_0^{1,p}(\Omega)$$

and hence, may consider that

$$D(A_p) \subset W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,p}(\Omega) \subset D(A^{'}_p)^*.$$ 

Let $(A^{'}_p)^{'}$ be the adjoint of $A_p$, considered as a bounded linear operator from $D(A^{'}_p)$ to $L^p(\Omega)$. Let $\tilde{A}$ be the restriction of $(A^{'}_p)^{'}$ to $W_0^{1,p}(\Omega)$. Then by the interpolation theory, the operator $\tilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$. Similarly, we consider that the restriction $\tilde{A}'$ of $(A_p)^{'} \in B(L^p(\Omega), D(A_p)^*)$ to $W_0^{1,p}(\Omega)$ is an isomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$. Furthermore, as is seen in proposition 3.1 in J. M. Jeong [5], it is known that $\tilde{A}$ and $\tilde{A}'$ generate analytic semigroups in $W^{-1,p}(\Omega)$ and $W^{-1,p}(\Omega)$, respectively, and the inequality

$$||\overline{(\tilde{A})^s}||_{B(W^{-1,p}(\Omega))} \leq C e^{\gamma |s|}, \quad -\infty < s < \infty,$$

holds for some constants $C > 0$ and $\gamma \in (0, \pi/2)$. We set

$$H_{p,q} = (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1}{q},q},$$

for $q \in (1, \infty)$. Since $\tilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ onto $W^{-1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ are $\zeta$-convex spaces, it is easily seen that $H_{p,q}$ is also $\zeta$-convex. From the interpolation theory and definitions of the operator $\tilde{A}$ and the space $H_{p,q}$ we can see the following results.
PROPOSITION 3.1. The operator $\tilde{A}$ and $\tilde{A}'$ generate analytic semigroups in $H_{p,q}$ and $H_{p',q'}$, respectively.

Proof. Since $-A_p$ and $-\tilde{A}$ generate analytic semigroup in $L^p(\Omega)$ and $W^{-1,p}(\Omega)$, respectively (see Proposition 3.1 in J. M. Jeong [5]), there exists an angle $\gamma \in (0, \frac{\pi}{2})$ such that

\begin{align}
\Sigma &= \{ \lambda : \gamma \leq \arg \lambda \leq 2\pi - \gamma \} \subset \rho(A_p) \cap \rho(\tilde{A}), \\
(3.2) &\quad ||(\lambda - A_p)^{-1}||_{B(L^p(\Omega))} \leq C/|\lambda|, \quad \lambda \in \Sigma, \\
(3.3) &\quad ||(\lambda - \tilde{A})^{-1}||_{B(W^{-1,p}(\Omega))} \leq C/|\lambda|, \quad \lambda \in \Sigma,
\end{align}

where $\rho(A_p)$ is the resolvent set of $A_p$. In view of (3.3)

\[
||A_p(\lambda - A_p)^{-1}u||_{p,\Omega} = ||(\lambda - A_p)^{-1}A_p u||_{p,\Omega} \leq \frac{C}{|\lambda|} ||A_p u||_{p,\Omega},
\]

for any $u \in D(A_p)$, we have

\begin{align}
(3.5) &\quad ||(\lambda - A_p)^{-1}||_{B(D(A_p))} \leq \frac{C}{|\lambda|}.
\end{align}

From (3.3) and (3.5) it follows that

\begin{align}
(3.6) &\quad ||(\lambda - \tilde{A})^{-1}||_{B(W^{-1,p}(\Omega))} \leq \frac{C}{|\lambda|}
\end{align}

and, hence from (3.4), (3.6) and the definition of the space $H_{p,q}$ we have that

\[
|| (\lambda - \tilde{A})^{-1} ||_{B(H_{p,q})} \leq \frac{C}{|\lambda|}.
\]

Therefore we have shown that $-\tilde{A}$ generates an analytic semigroup in $H_{p,q}$. 

PROPOSITION 3.2. There exists a constant $C > 0$ such that

$$||\tilde{A}^{i\varepsilon}||_{B(H_{p,q})} \leq Ce^{\gamma |\lambda|}, s \in \mathcal{R},$$

where $\gamma$ is the constant in (3.2).

Proof. From Theorem 1 of Seeley [9] and Proposition 3.2 of J. M. Jeong [5] there exists a constant $C > 0$ such that

$$(3.7)\quad ||(A_p)^{\varepsilon+i\varepsilon}||_{B(L^p(\Omega))} \leq Ce^{\gamma |s|},$$

$$(3.8)\quad ||\tilde{A}^{\varepsilon+i\varepsilon}||_{B(W^{-1,p}(\Omega))} \leq Ce^{\gamma |s|},$$

for any $s \in \mathcal{R}$ and $\varepsilon > 0$. From (3.7) it follows

$$(3.9)\quad ||(A_p)^{\varepsilon+i\varepsilon}||_{B(D(A_p))} \leq Ce^{\gamma |s|},$$

and hence, from (3.7) and (3.9) we obtain

$$(3.10)\quad ||\tilde{A}^{\varepsilon+i\varepsilon}||_{B(W^{-1,p}(\Omega))} \leq Ce^{\gamma |s|}. $$

Hence from (3.1), (3.8) and (3.10) we have shown that

$$||\tilde{A}^{\varepsilon+i\varepsilon}||_{B(H_{p,q})} \leq Ce^{\gamma |s|}.$$

So the proof is complete.

If $1 < p < n/(n-1)$, then we see that $L^1(\Omega) \subset W^{-1,p}(\Omega)$ from Sobolev's embedding theorem. Furthermore, we will show that $L^1(\Omega) \subset H_{p,q}$ in the section 4. Therefore, from Propositions 3.1 and 3.2 we can apply Theorem 3.1 and Proposition 4.1 of J. M. Jeong [5] to the abstract Cauchy problem in the space $H_{p,q}$. Hence, the following equation may be considered as an equation in both $H_{p,q}$ and $W^{-1,p}(\Omega)$.

$$\frac{d}{dt}u(t) = A_0u(t) + A_1u(t - h) + \int_{-h}^{0} a(s)A_2u(t + s)ds + \Phi_0w(t), \quad t \in (0,T]$$

$$(3.11)\quad u(0) = g^0, \quad u(s) = g^1(s) \quad s \in [-h, 0],$$

$$(3.12)\quad u(0) = g^0, \quad u(s) = g^1(s) \quad s \in [-h, 0].$$
where $A_0 = -\tilde{A}$, $A_i u$ ($i = 1, 2$) are the restriction $W_0^{1,p}(\Omega)$ of the following linear differential operators $-A_i$ ($i = 1, 2$) with real coefficient:

$$A(x, D_x) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{i,j}(x) \frac{\partial}{\partial x_i}) + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

where $a_{i,j}^t = a_{j,i}^t \in C^1(\bar{\Omega})$, $b_i^t \in C^1(\bar{\Omega})$, $c^t \in L^\infty(\Omega)$. The kernel $a(\cdot)$ belongs to $L^q (-h, 0)$ and the controller $\Phi_0$ is a bounded linear operator from some Banach space $U$ to $L^1(\Omega)$. For $q \in (1, \infty)$ we set

$$Z_{p,q} = H_{p,q} \times L^q(-h, 0; W_0^{1,p}(\Omega)).$$

Let $g = (g^0, g^1) \in Z_{p,q}$ and $w \in L^q(0, T; U)$. Then as is seen in Proposition 4.1 of J.M. Jeong [5] a solution of the equation (3.11) and (3.12) exists and is unique for each $T > 0$, moreover, we have

$$||u||_{L^q(0,T,W_0^{1,p}(\Omega))} \leq c(||g^0||_{H_{p,q}} + ||g^1||_{L^q(-h,0,W_0^{1,p}(\Omega))} + ||w||_{L^q(0,T,U)}),$$

where $c$ is a constant. Thus, we can define the solution semigroup for the system (3.11) and (3.12) as follows [3, Theorem 4.1]:

$$S(t) = (u(t; g, 0), u_t(\cdot; g, 0))$$

where $g = (g^0, g^1) \in Z_{p,q}$, $u(t; g, 0)$ is a solution of (3.11), (3.12) with $\Phi_0 = 0$ and $u_t(\cdot; g, 0)$ is the function $u_t(s; g, 0) = u(t + s; g, 0)$ defined in $[-h, 0]$. It is also known that $S(t)$ is a $C_0$-semigroup in $Z_{p,q}$.

We introduce the adjoint problem of (3.11) and (3.12):

$$\frac{d}{dt} v(t) = A^*_0 v(t) + A^*_1 v(t - h)$$

$$+ \int_{-h}^0 a(s)A^*_2 v(ts)ds, \quad t \in (0, T],$$

$$v(0) = \phi^0, \quad v(s) = \phi^1(s), \quad s \in [-h, 0),$$

$$Z_{p,q} = H_{p,q} \times L^q(-h, 0; W_0^{1,p}(\Omega)).$$
where $A_0^* = -\tilde{A}'$ and $A_i^* (i = 1, 2)$ are the adjoint operators of $A_i$. It is easily seen that $A^*_i \in B(W_0^{1,p'}(\Omega), W^{-1,p'}(\Omega))$ for $i = 0, 1, 2$ and hence (3.14) and (3.15) is an equation in both $H_{p',q'}$ and $W^{-1,p'}$. We can also define the solution semigroup $S_T(t)$ of (3.14) and (3.15) by

$$S_T(t) \phi = (v(t; \phi), v_t(\cdot, \phi))$$

for $\phi = (\phi^0, \phi^1) \in Z_{p',q'}$, where $v(t; \phi)$ is the solution of (3.14) and (3.15).

The structural operator $F$ is defined by

$$F g = ([F g]^0, [F g]^1),$$

$$[F g]^0 = g^0,$$

$$[F g]^1(s) = A_1 g^1(-h-s) + \int_{-h}^0 a(\tau) A_2 g^1(\tau - s) d\tau$$

for $g = (g^0, g^1) \in Z_{p,q}$. It is easy to see that $F \in B(Z_{p,q}, Z_{p',q'})$, $F^* \in B(Z_{p',q'}, Z_{p,q}^*)$. As in [5,7] we have that

$$F S(t) = S_T^*(t) F^*, \quad F^* S_T(t) = S^*(t) F^*.$$  

4. Representation of $H_{p,q}$ into Besov spaces

Let $Y_0$ and $Y_1$ be two Banach spaces contained in a Banach space $\mathcal{Y}$ such that the identity mapping of $Y_i$ ($i = 0, 1$) in $\mathcal{Y}$ is continuous and norms will be denoted by $|| \cdot ||_i$. The algebraic sum $Y_0 + Y_1$ of $Y_0$ and $Y_1$ is the space of all elements $a \in \mathcal{Y}$ of the form $a = a_0 + a_1$, $a_0 \in Y_0$ and $a_1 \in Y_1$. The intersection $Y_0 \cap Y_1$ and the sum $Y_0 + Y_1$ are Banach spaces with the norms

$$||a||_{Y_0 \cap Y_1} = \max \{||a||_0, ||a||_1\}$$

and

$$||a||_{Y_0 + Y_1} = \inf_a \{||a_0||_0 + ||a_1||_1\}, \quad a = a_0 + a_1, \quad a_i \in Y_i,$$

respectively.
DEFINITION (Lions-Peetre) 4.1. We say an intermediate space $Y$ of $Y_0$ and $Y_1$ belongs to

(i) the class $K_{\theta}(Y_0, Y_1)$, $0 < \theta < 1$, if for any $a \in Y_0 \cap Y_1$,

$$||a||_Y \leq c||a||^{1-\theta}_0 ||a||^\theta_1$$

where $c$ is a constant;

(ii) the class $\overline{K}_{\theta}(Y_0, Y_1)$, $0 < \theta < 1$, if for any $a \in Y$ and $t > 0$ there exist $a_i \in Y_i$ ($i = 1, 2$) such that $a = a_0 + a_1$ and

$$||a_0|| \leq ct^{-\theta}||a||_Y, \quad ||a_1||_1 \leq ct^{1-\theta}||a||_Y$$

where $c$ is a constant;

(iii) the class $K_{\theta}(A_0, A_1)$, $0 < \theta < 1$, if the space $Y$ belongs to both $K_{\theta}(Y_0, Y_1)$ and $\overline{K}_{\theta}(Y_0, Y_1)$.

Here, we note that by replacing $t$ with $t^{-1}$ the condition in (ii) rewrite as follows:

$$||a_0||_0 \leq ct^{\theta}||a||_Y, \quad ||a_1||_1 \leq ct^{\theta-1}||a||_Y$$

The following result is obtained from Lions-Peetre theorem 2.3 in [6].

PROPOSITION 4.1. For $0 < \theta_0 < \theta < \theta_1 < 1$, if the spaces $X_0$ and $X_1$ belong to the space $K_{\theta_0}(Y_0, Y_1)$ and the space $K_{\theta_1}(Y_0, Y_1)$, respectively, then

$$(X_0, X_1)^{\theta_0-\theta_1, p} = (Y_0, Y_1)_{\theta, p}.$$ 

In particular, if the space $X_1$ belongs to $K_{\theta_1}(Y_0, Y_1)$ then for $0 < \theta < \theta_1 < 1$

$$(Y_0, X_1)^{\theta_1, p} = (Y_0, Y_1)_{\theta, p}.$$ 

If the space $X_0$ belongs to $K_{\theta_0}(Y_0, Y_1)$, then $0 < \theta_0 < \theta < 1$

$$(X_0, Y_1)^{\theta_0-\theta, p} = (Y_0, Y_1)_{\theta, p}.$$ 

Let $A = -A(x, D_x)$ as in section 3. Then the operator $A$ is an isomorphism from $W_0^{1, p}(\Omega)$ to $W^{-1, p}(\Omega)$. 
**Lemma 4.1.** For any $t > 0$, there exists a constant $C$ such that

$$||(t + A)^{-1}||_{B(W^{-1,p}(\Omega), L^p(\Omega))} \leq Ct^{-\frac{1}{2}},$$

and

$$||(t + A)^{-1}||_{B(L^p(\Omega), W^{1,p}_0(\Omega))} \leq Ct^{-\frac{1}{2}}.$$  

**Proof.** For $t > 0$ since $(t + A'_p)^{-1}$ is an isomorphism from $L^p(\Omega)$ to $D(A'_p)$, the resolvent $((t + A'_p)^{-1})'$ is an isomorphism from $D(A'_p)^*$ onto $L^p(\Omega)$. It is not difficult to see that

$$((t + A'_p)^{-1})' = (t + (A'_p)'^{-1})^{-1}$$

and

$$(t + (A'_p)'^{-1})^{-1}|_{W^{-1,p}(\Omega)} = (t + A)^{-1}.$$  

Therefore, we have

$$||(t + A)^{-1}||_{B(D(A'_p)^*, L^p(\Omega))} \leq C$$

where $C$ is a constant. Combining (3.3) and (4.3) we obtain the inequality of (4.1). The proof of (4.2) is similar.

**Theorem 4.1.** For $1 < p < \infty$, the space $L^p(\Omega)$ belongs to the class $K_{1/2}(W^{1,p}_0(\Omega), W^{-1,p}(\Omega))$.

**Proof.** For any $u \in W^{1,p}_0(\Omega)$ and $t > 0$, from Lemma 4.1 and

$$u = A(t + A)^{-1}u + t(t + A)^{-1}u = (t + A)^{-1}Au + t(t + A)^{-1}u,$$

it follows

$$||u||_{p, \Omega} \leq ||(t + A)^{-1}||_{B(W^{-1,p}(\Omega), L^p(\Omega))}||Au||_{-1,p, \Omega}$$

$$+ t||(t + A)^{-1}||_{B(W^{-1,p}(\Omega), L^p(\Omega))}||u||_{-1,p, \Omega}$$

$$\leq Ct^{-\frac{1}{2}}||u||_{1,p, \Omega} + Ct^\frac{1}{2}||u||_{-1,p, \Omega}.$$
By choosing $t > 0$ such that $t^{-1/2}||u||_{1,p,\Omega} = t^{1/2}||u||_{-1,p,\Omega}$, we obtain
\[ ||u||_{p,\Omega} \leq C||u||_{1,p,\Omega}^{1/2}||u||_{-1,p,\Omega}^{1/2}. \]
Therefore, $L^p(\Omega)$ belongs to the class $\overline{K}_{1/2}(W_0^1,p(\Omega), W^{-1,p}(\Omega))$. Put $u_0 = t(t + A)^{-1}u$ and $u_1 = A(t + A)^{-1}u$ for any $u \in L^p(\Omega)$. Then $u = u_0 + u_1$, and we obtain that
\[ ||u_0||_{1,p,\Omega} \leq t||(t + A)^{-1}u||_{B(L^p(\Omega), W_0^1,p(\Omega))} ||u||_{p,\Omega} \leq C t^{\frac{1}{p}} ||u||_{p,\Omega} \]
\[ ||u_1||_{-1,p,\Omega} \leq C||(t + A)^{-1}u||_{1,p,\Omega} \leq C t^{\frac{1}{p}} ||u||_{p,\Omega}. \]
Therefore $L^p(\Omega)$ belongs to the class $\overline{K}_{1/2}(W_0^1,p(\Omega), W^{-1,p}(\Omega))$, and hence, it belongs to $K_{1/2}(W_0^1,p(\Omega), W^{-1,p}(\Omega))$.

**Theorem 4.2.** If $1 - 2\theta - 1/p \neq 0$ and $2\theta - 2 + 1/p \neq 0$ for $0 < \theta < 1$ and $1 < p$, $q < \infty$, then
\[ (W_0^1,p(\Omega), W^{-1,p}(\Omega))_{\theta,q} = \begin{cases} B_p^{1-2\theta}(\Omega) & \theta < \frac{1}{2}(1 - \frac{1}{p}), \\ B_p^{1-2\theta}(\Omega) & \theta > \frac{1}{2}(1 - \frac{1}{p}). \end{cases} \]
where $B_p^{1-2\theta}(\Omega) = \{u \in B_p^{-2\theta}(\Omega) : u|_{\partial\Omega} = 0\}$. In particular, we obtain that
\[ (W_0^1,p(\Omega), W^{-1,p}(\Omega))_{\frac{1}{2},q} = B_p^0(\Omega). \]

**Proof.** Let $0 < \theta < 1/2$. Then from Proposition 4.1 we obtain that for any $0 < \theta < 1/2$
\[ (W_0^1,p(\Omega), W^{-1,p}(\Omega))_{\theta,q} = (W_0^1,p(\Omega), L^p(\Omega))_{2\theta,q} \]
\[ = (L^p(\Omega), W_0^1,p(\Omega))_{1-2\theta,q} \]
Therefore, in view of the result of Grisvard theorem in Triebel [14, p. 321],
\[ (W_0^1,p(\Omega), W^{-1,p}(\Omega))_{\theta,q} = \begin{cases} B_p^{1-2\theta}(\Omega) & 1 - 2\theta > \frac{1}{p}, \\ B_p^{1-2\theta}(\Omega) & 1 - 2\theta < \frac{1}{p}. \end{cases} \]
Let $1/2 < \theta < 1$. Then from Proposition 4.1 it follows

$$(W_{0}^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = (L^{p}(\Omega), W^{-1,p}(\Omega))_{2\theta-1,q}$$

$$= ((L^{p}(\Omega), W_{0}^{-1,p}(\Omega))_{2\theta-1,q'})^{*}$$

where $p' = p/(p-1)$. In view of Grisvard theorem if $2\theta - 1 - 1/p' \neq 0$ then

$$(L^{p}(\Omega), W_{0}^{1,p}(\Omega))_{\theta,q} = \begin{cases} B^{2\theta-1}_{p',q}(\Omega) & 2\theta - 1 > \frac{1}{p'}, \\ B_{p',q'}^{2\theta-1}(\Omega) & 2\theta - 1 < \frac{1}{p'}. \end{cases}$$

From Theorem 4.8.2 in H. Triebel [14; p. 332], we obtain that

$$(B^{2\theta-1}_{p',q'}(\Omega))^{*} = B^{1-2\theta}_{p,q}(\Omega)$$

if $2\theta - 1 - 1/p' \neq 0$. Since $2\theta - 1 - 1/p' \neq 0$ i.e. $2\theta - 2 + 1/p \neq 0$, if $1/2 < \theta < 1$ and $2\theta - 2 + 1/p \neq 0$ then

$$(W_{0}^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = B^{1-2\theta}_{p,q}(\Omega).$$

Consequently, we obtain that

$$(W_{0}^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1-\theta}{2},q} = B_{p,q}^{\theta}(\Omega), \text{ if } 0 < \theta < \frac{1}{p}$$

and

$$(W_{0}^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1+\theta}{2},q} = B_{p,q}^{-\theta}(\Omega) \text{ if } 0 < \theta < 1 - \frac{1}{p}.$$ 

Hence, if $0 < \theta < \min\{1/p, 1 - 1/p\}$

$$(W_{0}^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1}{2},q}$$

$$= ((W_{0}^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1-\theta}{2},q}, (W_{0}^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1+\theta}{2},q})_{\frac{1}{2},q}$$

$$= (B_{p,q}^{\theta}(\Omega), B_{p,q}^{-\theta}(\Omega))_{\frac{1}{2},q} = B_{p,q}^{0}(\Omega).$$

The last equality is obtained from Theorem 1 of section 4.3.1 in H. Triebel [14; p. 317]. Hence the proof is complete.
Theorem 4.3. Let $1 < p, q < \infty$.

(i) If $2/q - 2 + 1/p \neq 0$ then

$$H_{p,q} =\begin{cases} 
B_{p,q}^{1-2/q}(\Omega) & \text{if } \frac{1}{q} < \frac{1}{2}(1 - \frac{1}{p}), \\
B_{p,q}^{1-2/q}(\Omega) & \text{if } \frac{1}{q} > \frac{1}{2}(1 - \frac{1}{p}).
\end{cases}$$

(ii) If $\frac{1}{p'} < 1/n(1 - 2/q')$ then

$$H_{p',q'} \subset C_0(\overline{\Omega}) \subset L^\infty.$$

Proof. The relation (i) follows directly from Theorem 4.2. Let $\frac{1}{p'} < 1/n(1 - 2/q')$. Then from (i)

$$H_{p',q'} = (W_0^{1,p'}(\Omega), W^{-1,p'}(\Omega))_{\frac{1}{q'},q'} = B_{p',q'}^{1-\frac{2}{q'}}(\Omega)$$

and from Sobolev-Besov's and Sobolev's embedding theorems we obtain that

$$B_{p',q'}^{1-\frac{2}{q'}}(\Omega) \subset W_{p',q'}^{1-\frac{2}{q'}} \subset C_0(\overline{\Omega})$$

Hence, the first inclusion in (ii) follows.

5. Control problem for $L^1(\Omega)$-valued controller

As is seen in section 4, if $1/p' - 1/n(1 - 2/q') < 0$ then we obtained that

$$H_{p',q'} \subset C_0(\overline{\Omega}) \subset L^\infty(\Omega).$$

Thus, since

$$H_{p,q} = H^*_{p',q'} \supset C_0(\overline{\Omega})^* \supset L^1(\Omega)$$

we consider $\Phi_0$ as an operator in $B(U, H_{p,q})$. Hence it is possible to investigate the control problem for (3.11) and (3.12) in $H_{p,q}$. In what
follows in this section we fix $p$ and $q$ so that $1/p' - 1/n(1 - 2/q') < 0$. Then it immediately implies that $1 < p < n/(n - 1)$. Let $A$ be the infinitesimal generator of $S(t)$ as in section 3. Then the equation (3.11) and (3.12) can be transformed into an abstract equation as follows

$$z(t) = Az(t) + \Phi w(t),$$
$$z(0) = g$$

where $g = (g^0, g^1) \in Z_{p,q}$ and the controller operator is defined $\Phi w = (\Phi_0 w, 0)$. In this case for the solution $u$ of (3.11) and (3.12) the equation satisfied by $(u(t), u_t(\cdot))$ is an equation in $Z_{p,q}$ since $\Phi_0$ is an operator into $H_{p,q}$. Since the dual $\Phi_0^*$ of $\Phi_0$ is the operator from $L^\infty(\Omega)$ into $U^*$, the operator $\Phi_0^*$ may be considered as an operator from $H_{p',q'}$ into $U^*$. Hence with the aid of Theorem 4.3 we remarked that the condition that $\Phi_0^* \phi = 0$ almost everywhere can be rewrite to the fact that $\Phi_0^* \phi \equiv 0$ for $\phi \in Z_{p,q}$. We define the attainable set by

$$R = \{ \int_0^t S(t - \tau)\Phi w(\tau) d\tau : w \in L^q(0,t;U), \quad t \geq 0 \}.$$

**Definition 5.1.** (1) The system (3.11), (3.12) is approximately controllable if $\overline{R} = Z_{p,q}$, where $\overline{R}$ is the closure of $R$ in $Z_{p,q}$

(2) The system (3.14), (3.15) is observable if for $\phi \in Z_{p',q'}$, $\Phi_0^* [S_T(t)\phi]^0 \equiv 0$ implies $\phi = 0$.

**Theorem 5.1.** Let the structural operator $F$ is an isomorphism. Then the system (3.11) and (3.12) is approximately controllable if and only if the system (3.14) and (3.15) is observable.

**Proof.** Let the system (3.11) and (3.12) is approximately controllable. Then for $f \in Z_{p,q}^*$

$$(f, \int_0^t S(t - \tau)\Phi w(\tau) d\tau) = 0$$

for $w \in L^q(0,t;U)$ and $t > 0$ implies $f = 0$. By duality theorem it is equivalent to the fact that for any $f \in Z_{p,q}^*$, $\Phi^* S^*(t)f \equiv 0$ implies
f = 0. Since the operator $F^*$ is an isomorphism by assumption, there exists $\phi \in Z_{p',q'}$ such that $f = F^*\phi$. From (3.16) we obtain that

$$\Phi_0^*[S^*(t)f]^0 = \Phi_0^*[FS_T(t)\phi]^0 = \Phi_0^*[S_T(t)\phi]^0.$$ 

Hence, the system (3.11) and (3.12) is approximately controllable iff for any $\phi \in Z_{p',q'}$, $\Phi_0^*[S_T(t)\phi]^0 = 0$ implies $\phi = 0$. Therefore, the statement is equivalent that the system (3.14) and (3.14) is observable.

**Remark.** When we deal with the control problem of (3.11) and (3.12) in negative space $W^{-1,p}(\Omega)$, we needed a assumption that the kernel $a(\cdot)$ is Hölder continuous for using of the properties of fundamental solution since $\Phi_0$ is not operator into $H_{p,q}$. If we assume that $a(\cdot)$ is Hölder continuous then the fundamental solution of (3.11) and (3.12) exists (see in [13]). By fixing $p, q$ so that $1 < p < n/(n-1)$ and $1/p' < 1/n(1 - 1/q')$, we can obtain the Theorem 5.1 using the solution semigroup without requirement of fundamental solution of (3.11) and (3.12). Hence, the kernel $a(\cdot)$ need not be Holder continuous but has only to belong to $L^1(-h, 0)$ for wellposedness and regularity for equation (3.11) and (3.12).

**REFERENCES**


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