Explicit Motion of Dynamic Systems with Position Constraints

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Although many methodologies exist for determining the constrained equations of motion, most of these methods depend on numerical approaches such as the Lagrange multiplier's method expressed in differential/algebraic systems. In 1992, Udwadia and Kalaba proposed explicit equations of motion for constrained systems based on Gauss's principle and elementary linear algebra without any multipliers or complicated intermediate processes. The generalized inverse method was the first work to present explicit equations of motion for constrained systems. However, numerical integration results of the equation of motion gradually veer away from the constraint equations with time. Thus, an objective of this study is to provide a numerical integration scheme, which modifies the generalized inverse method to reduce the errors. The modified equations of motion for constrained systems include the position constraints of index 3 systems and their first derivatives with respect to time in addition to their second derivatives with respect to time. The effectiveness of the proposed method is illustrated by numerical examples.

Key Words : Generalized Inverse, Numerical Integration, Numerical Error, Position Constraint

1. Introduction

The motion of mechanical or structural systems is often constrained by given trajectories or conditions. The constrained motion requires the constraint force provided by the nature for satisfying the given constraints. Gauss's Principle defines the constraint force as the minimum force of all forces that are required to satisfy the constraints or pull the state variables into the prescribed trajectories.

The constraint force must be explicitly calculated and provided such that the state variables do not violate the constraints. But most of the methods for describing the constrained motion depend on numerical approaches such as the Lagrange multiplier's method expressed in a differential/algebraic system (Gear and Petzold, 1984; Gear, Leimkuhler, and Gupta, 1985; Gear, 1986; Gear, 1988). Mathematically, the equations of motion for constrained systems based on the Lagrangian formulation can be expressed in differential/algebraic systems $\mathbf{F}(t, \mathbf{y}, \dot{\mathbf{y}})=0$, where $\mathbf{F}$, $\mathbf{y}$ and $\dot{\mathbf{y}}$ are n-dimensional vectors. They also involve the Lagrange multiplier functions. The formulations are based on an overdetermined system of equations including time derivatives of the constraints and stabilization with respect to the differential constraints via additional Lagrange multipliers. These methods require much efforts in numerically determining the multipliers.

Gibbs-Appell (Appell, 1911; Gibbs, 1879) method requires a felicitous choice of quasi-coor-

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dinates and is also difficult to use, when dealing with systems having several tens of degrees of freedom and several non-integrable constraints. Kane (1983) introduced an analytical method for nonholonomic systems based upon the development of Lagrange equations from D'Alembert's Principle. Though his method is usually less tedious than the computation associated with Lagrange multipliers, it is difficult to compute components of the acceleration vector. It also gets rapidly more complicated with increasing numbers of degrees of freedom. The effectiveness of Kane's method was compared with other approaches in the study of the constrained motion of multibody systems (Park et al., 2000 ; Park et al., 1997). Passerello and Huston (1973) introduced a computer-oriented method similar to the method of the orthogonal components of the matrix associated with the constraint equations, which reduces the dimension of the dynamical equations by eliminating the constraint forces. In 1992, Udwadia and Kalaba (1992) proposed explicit equations of motion for constrained mechanical and structural systems. The generalized inverse method by Udwadia and Kalaba was the first work to present the explicit equations of motion for constrained systems since Lagrange. This method has advantages in that it does not require any linearization process for the control of nonlinear systems and can explicitly describe the motion of holonomically and/or nonholonomically constrained systems.

The constrained motion can be described by numerically integrating the differential equations by Udwadia and Kalaba, and the numerical results must satisfy the constraints during the integration. However, the numerical results gradually veer away from the given constraints with time. From the viewpoint of numerical integration, it is necessary to devise numerical methods to pull the deviated state variables into the given paths. Because the generalized inverse method was based on the only second derivatives with respect to time of the position constraints, the errors in the satisfaction of constraints are caused by the neglect of the position constraints as well as their first derivatives with respect to time. Accordingly, an objective of this paper is to present a numerical method which modifies the generalized inverse method to reduce the errors in the satisfaction of the constraints. The modified equations of motion for constrained systems include the effects of the position constraints, their first and second derivatives with respect to time in the differential equations. Numerical examples illustrate the effectiveness of the proposed numerical method.

2. Equations of Motion of Constrained Systems

The equations of motion of a system modeled by an n-degree-of-freedom lumped mass-spring-dashpot system can be written as

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{E}\mathbf{f}(t) \quad (1)$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are, respectively, the $n\times n$ mass, damping, and stiffness matrices, $\mathbf{x}(t)$ is the n-dimensional displacement vector, and $\mathbf{f}(t)$ is an r-vector representing applied load or external excitation. The $n\times r$ matrix $\mathbf{E}$ is location matrix which defines locations of the excitation.

Assume that the n-degree-of-freedom system is constrained by the m consistent constraints of the form

$$\phi_i(\mathbf{x}, t)=0, \qquad i=1, 2, \cdots, m \quad (2)$$

and $m<n$. The constrained motion requires the constraint force such that the state variables satisfy the constraint sets. Therefore, the general equations of motion at time t of the constrained system can be expressed as

$$\mathbf{M}\ddot{\mathbf{x}}=\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)+\mathbf{F}^c(\mathbf{x}, \dot{\mathbf{x}}, t) \quad (3)$$

where $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)=-\mathbf{C}\dot{\mathbf{x}}(t)-\mathbf{K}\mathbf{x}(t)+\mathbf{E}\mathbf{f}(t)$, and $\mathbf{F}^c(\mathbf{x}, \dot{\mathbf{x}}, t)$ is the n-dimensional constraint force vector.

Assuming that the constraint equations are sufficiently smooth, a proper differentiation of Eq. (2) with respect to time t leads to the linear set of equations

$$\mathbf{A}(\mathbf{x}, \dot{\mathbf{x}}, t)\ddot{\mathbf{x}}=\mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t) \quad (4)$$

where $\mathbf{A}$ is an $n\times n$ matrix, and $\mathbf{b}$ is an $m\times 1$ vector. Using Gauss's Principle and elementary linear algebra, and combining Eqs. (1) and (4), the generalized inverse method gives constrained equations of motion given by

$$\ddot{\mathbf{x}}=\mathbf{a}+\mathbf{M}^{-1/2}(\mathbf{A}\mathbf{M}^{-1/2})^{+}(\mathbf{b}-\mathbf{A}\mathbf{a}) \qquad (5)$$

where $\mathbf{a}=\mathbf{M}^{-1}\mathbf{F}(\mathbf{x},\ \dot{\mathbf{x}},\ t)$. This is the first work that presents explicit equations of motion for constrained systems that do not require any linearization process for the control of nonlinear systems and can explicitly describe the motion of holonomically and/or nonholonomically constrained systems.

However, the numerical integration results of the differential equations (5) veer away from the constraints. The integration of the constrained equations of motion that involve the second derivatives of position constraints with respect to time leads to the errors in the satisfaction of the constraints caused by the neglect of the position constraints and their first derivatives in time. Thus, starting from the generalized inverse method, this study presents a modified equation of motion for more accurate results.

3. Errors in the Satisfaction of Constraints

To investigate the errors developed during numerical integration of the differential equation, consider a three-DOF system subjected to a constraint. As shown in Fig. 1, the state variable vector, which describes the configuration space of the system, is denoted by $\mathbf{q}=[q_1\ \ q_2\ \ q_3]^T$. The unconstrained equations of motion for this system is given by

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{P}(t) \qquad (6)$$

Assuming that this system is constrained by a constraint

$$\phi_1=q_1-3q_2=0 \qquad (7)$$

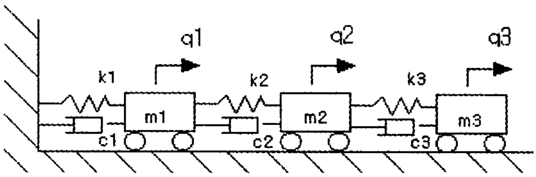


Fig. 1 A three-DOF system

and differentiating Eq. (7) twice and expressing $\mathbf{A}\ddot{\mathbf{x}}=\mathbf{b}$ of Eq. (4), we obtain

$$\ddot{\phi}_1=\ddot{q}_1-3\ddot{q}_2=0 \qquad (8)$$

The physical values for the numerical application selected are

$$m_1=m_2=3units,\ m_3=1unit,\quad k_1=300units,\ k_2=200units,\ k_3=100units \qquad (9)$$

The damping coeffients are selected so that the damping ratio of each mode is 0.02 and the external excitation vector used is $\mathbf{P}(t)=[300\sin 6t\ \ 500\cos 3t\ \ 0]^T$. For numerical integration of the constrained equations of motion, the local tolerance for the Runge-Kutta scheme is set at 10^{-6}. When the differential equations are integrated by a numerical integration scheme, the state variables must satisfy the constraint equation (7) at all times. To investigate the errors in the satisfaction of the constraints, which are the position constraint and its first derivative with respect to time, we define the errors as

$$\text{Error 1}=q_1-3q_2 \text{ and Error 2}=\dot{q}_1-3\dot{q}_2 \qquad (10)$$

Figure 2 shows the errors given by Eq. (10). It can be observed that the numerical solutions of the differencial equations proposed by Udwadia and Kalaba are found to gradually veer away from the constraints, and the errors increase with time. Recognizing that the generalized inverse method involves only the second derivatives of the position constraints with respect to time, it can be interpreted that the errors are caused by the neglect of the position constraints and their first derivatives with respect to time. The errors

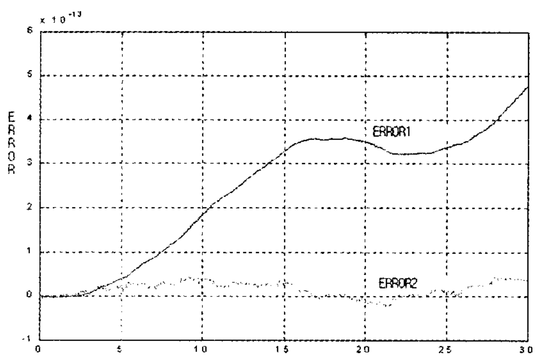


Fig. 2 Errors in the satisfaction of the constraints

can be reduced by the action of an additional force which is needed to pull the deviated state variables into the position constraints and their first derivatives with respect to time. Thus, the modified equations of motion consider the effects of the position constraints and their first derivatives with respect to time.

4. Numerical Integration Scheme

Assuming that the constraint equations are sufficiently smooth and taking the total derivatives of the set (2), and using the chain rule, we obtain the following equations

$$\dot{\phi}_i=\sum_{j=1}^{n}\frac{\partial\phi_i}{\partial x_j}\dot{x}_j+\frac{\partial\phi_i}{\partial t}=0, \qquad i=1,\ 2,\ \cdots,\ m \quad (11)$$

These equations are differentiated, provided the functions $\frac{\partial\phi_i}{\partial x_j}$ and $\frac{\partial\phi_i}{\partial t}$ are sufficiently smooth, to yield the set of equations

$$\ddot{\phi}_i=\sum_{j=1}^{n}\frac{\partial\phi_i}{\partial x_j}\ddot{x}_j+\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_i}{\partial x_j}\right)\dot{x}_k\dot{x}_j +\sum_{j=1}^{n}\frac{\partial}{\partial t}\left(\frac{\partial\phi_i}{\partial x_j}\right)\dot{x}_j+\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_i}{\partial t}\right)\dot{x}_k+\frac{\partial^2\phi_i}{\partial t^2}=0 \quad (12)$$

$$i=1,\ 2,\ \cdots,\ m$$

Equation (12) can be cast into the form $\mathbf{A\ddot{x}=b}$. Considering the effects of all three constraint equations, they are combined as

$$\ddot{\phi}_i+\alpha_i\dot{\phi}_i+\beta_i\phi_i=0, \qquad i=1,\ 2,\ \cdots,\ m \quad (13)$$

or

$$\mathbf{\ddot{H}+R\dot{H}+SH=0} \quad (14)$$

where α_i's and β_i's are positive values, and

$$\mathbf{H}=[\phi_i \ \ \phi_2 \ \ \cdots \ \ \phi_m]^T$$

$$\mathbf{R}=\begin{bmatrix}\alpha_1 & 0 & \cdots & 0\\ 0 & \alpha_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \alpha_m\end{bmatrix}, \text{ and } \mathbf{S}=\begin{bmatrix}\beta_1 & 0 & \cdots & 0\\ 0 & \beta_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \beta_m\end{bmatrix} \quad (15)$$

Also, as the result of combing those constraint equation sets, $\mathbf{A\ddot{x}=b}$ is replaced by

$$\mathbf{A\ddot{x}=b-R\dot{H}-SH} \quad (16)$$

where

$$\mathbf{A}=\begin{bmatrix}\frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_n}\\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial\phi_m}{\partial x_1} & \frac{\partial\phi_m}{\partial x_2} & \cdots & \frac{\partial\phi_m}{\partial x_n}\end{bmatrix} \quad (17)$$

$$\mathbf{b}=-\begin{bmatrix}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_1}{\partial x_j}\right)\dot{x}_k\dot{x}_j+\sum_{j=1}^{n}\frac{\partial}{\partial t}\left(\frac{\partial\phi_1}{\partial x_j}\right)\dot{x}_j+\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_1}{\partial t}\right)\dot{x}_k+\frac{\partial^2\phi_1}{\partial t^2}\\ \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_2}{\partial x_j}\right)\dot{x}_k\dot{x}_j+\sum_{j=1}^{n}\frac{\partial}{\partial t}\left(\frac{\partial\phi_2}{\partial x_j}\right)\dot{x}_j+\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_2}{\partial t}\right)\dot{x}_k+\frac{\partial^2\phi_2}{\partial t^2}\\ \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_m}{\partial x_j}\right)\dot{x}_k\dot{x}_j+\sum_{j=1}^{n}\frac{\partial}{\partial t}\left(\frac{\partial\phi_m}{\partial x_j}\right)\dot{x}_j+\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_m}{\partial t}\right)\dot{x}_k+\frac{\partial^2\phi_m}{\partial t^2}\end{bmatrix} \quad (18)$$

Consequently, the original equations of motion (5) for the constrained systems are modified as

$$\mathbf{\ddot{x}=a+M^{-1/2}(AM^{-1/2})^{+}(b-R\dot{H}-SH-Aa)} \quad (19)$$

We can alternatively think of Eq. (13) as the equations of motion of m second-order dynamic systems. The α_i's and β_i's are damping coefficient and stiffness of i-th oscillator, respectively. Let us call Eq. (13) i-th dynamical error equation. The terms $\alpha_i\dot{\phi}_i+\beta_i\phi_i$ in Eq. (13) play an important role for reducing the errors in the satisfaction of the constraints. The coefficients α_i and β_i need to be selected in such a way that the errors in the satisfaction of the constraints ϕ_i and $\dot{\phi}_i$ are damped out rapidly.

Baumgarte (1972) discussed the proper choice of the values of the coefficients α_i and β_i in Eq. (13) for reducing numerical errors and suggested positive values for the parameters α_i and β_i corresponding to i-th oscillator. His method considered the dynamical error equations as decoupled equations in which the unknown coefficients involved in each of the m dynamical error equations are independently selected.

Each of the m dynamical error equations can be looked upon as an oscillator and shows three types of motion depending on the values of the coefficients α_i and β_i, critically damped motion, underdamped motion, and overdamped motion. The type of motion depends on the quantity $\alpha_i^2-4\beta_i$ corresponding to i-th oscillator. If $\alpha_i^2-4\beta_i<0$, an underdamped system is obtained. If $\alpha_i^2-4\beta_i=0$, a critically damped system is obtained. And if $\alpha_i^2-4\beta_i>0$, an overdamped system is obtained. Baumgarte selected the values of the unknowns corresponding to the critically damped motion with values of $\alpha_i < 20$.

In order to investigate the variations of the errors according to the values of α_i and β_i for the above system, let us define the magnitude of the errors in the satisfaction of constraints caused by numerical procedure as

$$E_1 = \frac{1}{T_f}\sqrt{\int_0^{T_f} (q_1 - 3q_2)^2 dt} \quad (20)$$

and

$$E_2 = \frac{1}{T_f}\sqrt{\int_0^{T_f} (\dot{q}_1 - 3\dot{q}_2)^2 dt} \quad (21)$$

where T_f is 30 seconds.

Figures 3 and 4 show the variations of E_1 and E_2 according to the coefficients α and β, where the values of α range from 0 to 20 in increments of 2 and the values of β corresponds to the underdamped, critically damped, and overdamped system. The minimum value of E_1 occurs at $\alpha=20$ and $\beta=200$, while the minimum value of E_2 occurs at $\alpha=18.0$ and $\beta=16.2$. The parameter values α and β for minimizing E_1 and E_2 correspond to an underdamped and an overdamped system, respectively.

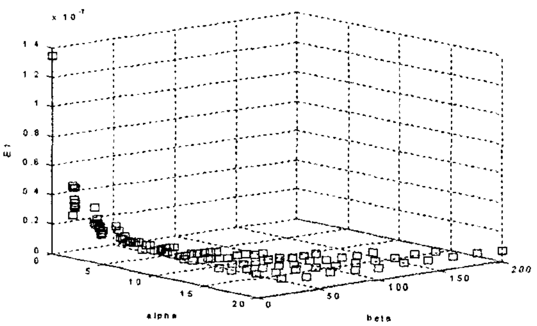


Fig. 3 Variation of the magnitude of error 1

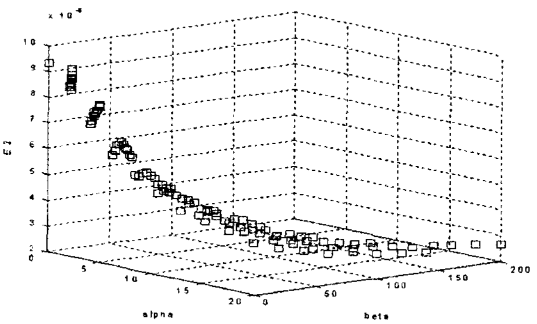
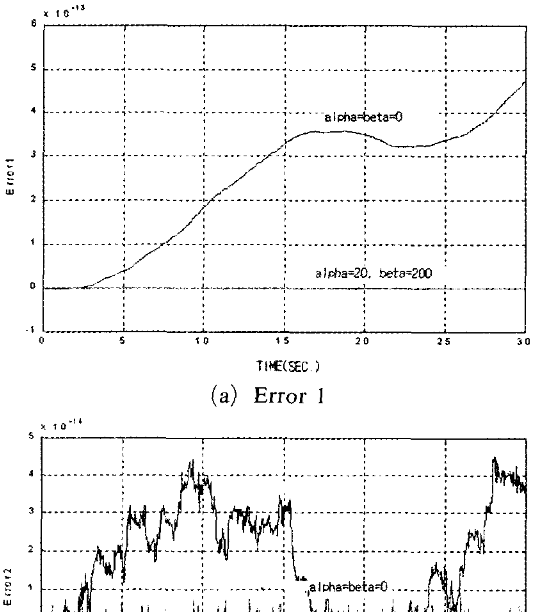


Fig. 4 Variation of the magnitude of error 2

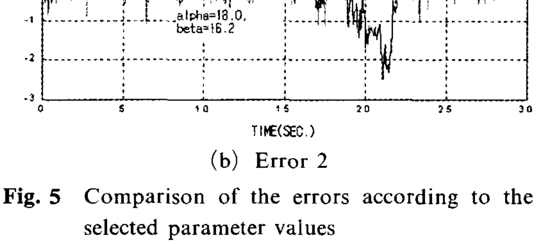
Figure 5 compares the error E_1 to be taken for $\alpha=\beta=0$ and $\alpha=20$, $\beta=200$ and E_2 for $\alpha=\beta=0$ and $\alpha=18.0$, $\beta=16.2$. For $\alpha=\beta=0$, the original equations of motion for constrained systems are obtained, and E_1 and E_2 do not show the minimum at the same values. From this plot, it is observed that the additional force caused by the position constraints and their first derivatives with respect to time leads to decreased errors. Also, it is exhibited that the error is not totally damped out and the reduction of the errors largely depends on the selection of the parameter values.

In order to reduce the errors in the satisfaction of multiple constraints, assume that the above system is constrained by an additional constraint

$$\phi_2 = q_1 + q_2 + q_3 = 0 \quad (22)$$



(a) Error 1



(b) Error 2

Fig. 5 Comparison of the errors according to the selected parameter values

By properly differentiating two constraints (7) and (22) with respect to time t, .these constraint equations in the form of Eg. (14) can be expressed as

$$\mathbf{A}=\begin{bmatrix} 1 & -3 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$
$$\mathbf{b}-\mathbf{R}\dot{\mathbf{H}}-\mathbf{S}\mathbf{H}=\begin{bmatrix} -\alpha_1(\dot{q}_1-3\dot{q}_2)-\beta_1(q_1-3q_2) \\ -\alpha_2(\dot{q}_1+\dot{q}_2+\dot{q}_3)-\beta_2(q_1+q_2+q_3) \end{bmatrix} \tag{23}$$

where the parameters α_1, α_2, β_1, and β_2 have positive values.

Figure 6 shows the magnitude of the errors defined by Eqs. (20) and (21), and

$$E_3=\frac{1}{T_f}\sqrt{\int_0^{T_f}(q_1+q_2+q_3)^2dt} \tag{24}$$

and $$E_4=\frac{1}{T_f}\sqrt{\int_0^{T_f}(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2dt} \tag{25}$$

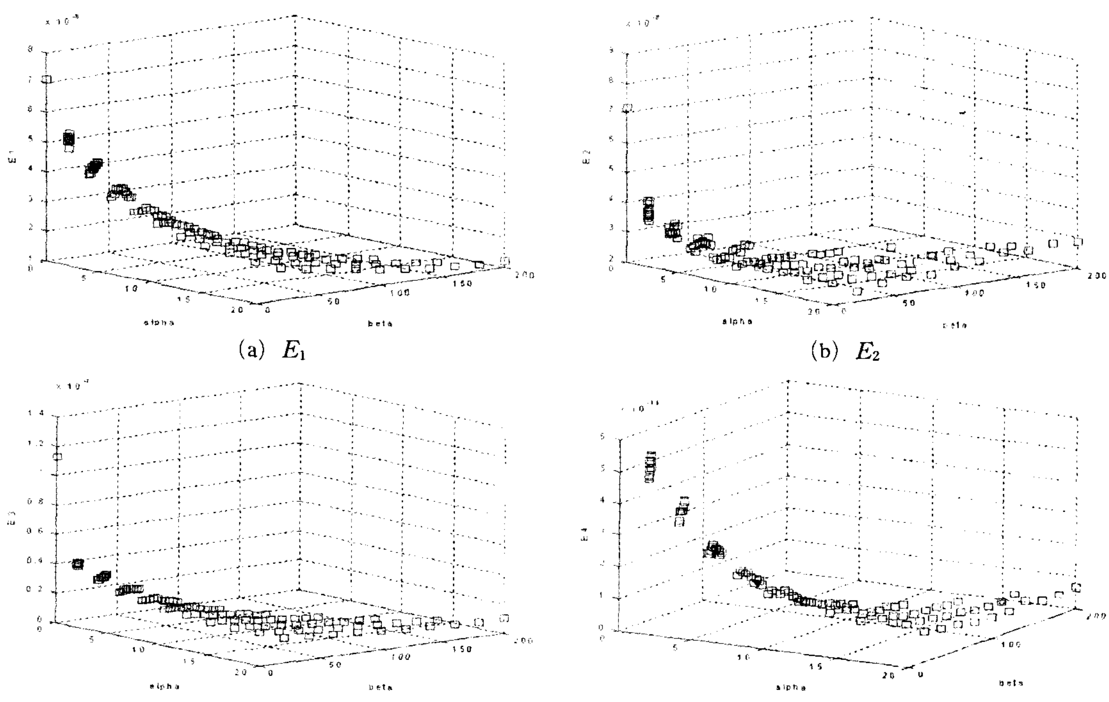
As shown by Fig. 6, the minimum values of both E_1 and E_2 occurred at the same values of $\alpha=20$ and $\beta=200$, while the minimum values of E_3 and E_4 occur at $\alpha=20$, $\beta=180$, and $\alpha=18$, $\beta=145.8$, respectively. The difference of Figs. 3 and 6(a), or Figs. 4 and 6(b) is due to the coupling of the dynamical error equations through the parameters. The errors in the satisfaction of i-th constraint are not independently affected by the parameter values α_i and β_i but in terdepently. This means that the error in the satisfaction of each constraint is a function of all the parameters present. Generalizing this scheme, the matrices $\mathbf{R}$ and $\mathbf{S}$ in Eq. (14) are replaced by

$$\mathbf{R}=\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} \end{bmatrix} \text{ and } \mathbf{S}=\begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mm} \end{bmatrix} \tag{26}$$

respectively.

The dynamical error equations are coupled by the coefficient matrices, and the matrices are selected such that the errors are rapidly damped out. Substituting $\mathbf{H}=e^{\lambda t}\mathbf{U}$ into the dynamical error equation (14), we obtain

$$(\lambda^2\mathbf{I}+\lambda\mathbf{R}+\mathbf{S})\mathbf{U}e^{\lambda t}=\mathbf{0} \tag{27}$$



(a) E_1 (b) E_2 (c) E_3 (d) E_4

Fig. 6 Comparison of the magnitude of errors where the values of λ are eigenvalues. Because $\mathbf{U} \neq \mathbf{0}$, the eigenvalues satisfying $\lambda^2\mathbf{I}+\lambda\mathbf{R}+\mathbf{S}=\mathbf{0}$ must have negative real part so that $\mathbf{H} \to \mathbf{0}$ with $t \to \infty$. It has been demonstrated that the errors in the satisfaction of the multiple constraints can be reduced by inserting the position constraints and their first derivatives with respect to time into the original equation of motion and selecting the proper coefficient matrices with the eigenvalues possessing negative real parts.

5. Conclusions

Most of the methods for describing the constrained motion depend on numerical approaches such as Lagrange multiplier's method expressed in differential/algebraic system. The equations of motion for constrained systems proposed by Udwadia and Kalaba have an advantage in that they explicitly describe the constrained motion. However, the numerical integration of the differential equations gradually veer away from the given constraints with time. The errors in the satisfaction of the constraints are caused by the neglect of the position constraints and their first derivatives with respect to time. Thus, starting from the generalized inverse method, the present study proposes a numerical method for reducing the errors by inserting the effects of the position constraints and their first derivatives with respect to time into the original constrained equations of motion. The modified equations of motion for constrained systems can more accurately describe the constrained motion by reducing the errors in the satisfaction of constraints.

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