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Characterization of Some Classes of Distributions Related to Operator Semi-stable Distributions[1)]

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Abstract

For a positive integer m, operator m-semi-stability and the strict operator m-semi-stability of probability measures on R^d are defined. The operator m-semi-stability is a generalization of the definition of operator semi-stability with exponent Q. Characterization of strictly operator m-semi-stable distributions among operator m-semi-stable distributions is given. Translation of strictly operator m-semi-stable distribution is discussed.

Keywords : Operator semi-stability, Semi-stability, Operator stability, Strictly semi-stability

1. Introduction

Let m be a positive integer. In [3], the classes of m-semi-stable and strictly m-semi-stable distributions on R^d were studied. In one dimension, they were first investigated by Lévy [8]. The characterization of these classes on R was developed by Linnik [9], Shimizu [16], Ramachandran and Rao [10], and others. Extension to multidimension was done by Krapavickaitė [5,6] and Choi [3]. Here we extend those classes to linear operator cases.

Let $I(R^d)$ be the collection of infinitly divisible distributions on R^d. The characteristic function of $\mu\in I(R^d)$ is denoted by $\widehat{\mu}(z)$, $z\in R^d$. Let $M_+(R^d)$ be the class of linear operators on R^d all

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of whose eigenvalues have positive real parts. Let $0<b_l<1$, $Q\in M_+(R^d)$ and m a positive integer in this paper throughout. We call a distribution μ on R^d operator m-semi-stable if $\mu\in I(R^d)$ and there exist real numbers b_l, c_l, $l=1,2,\cdots,m$, and a vector $\gamma\in R^d$ satisfying

$$c_l>0,\ \sum_{l=1}^{m} c_l>1, \text{and } \sum_{l=1}^{m} b_l c_l=1$$

such that

$$\widehat{\mu}(z)=e^{\langle\gamma,z\rangle}\Pi_{l=1}^{m}\widehat{\mu}(b_l^{Q'}z)^{c_l}. \qquad (1.1)$$

Here $\langle,\rangle$ is the Euclidean inner product in R^d and Q' is the adjoint of Q. The class of distributions satisfying (1.1) is denoted by $OSS(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$. We call distributions in $OSS(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ $(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)-$*semi*$-$ *stable*. If $\mu\in OSS(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ with $Q=I$, then μ is a strictly m- semi-stable distribution on R^d in the sense of [3].

Further, a distribution μ on R^d is strictly operator m-semi-stable if $\mu\in OSS(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ satisfying

$$\widehat{\mu}(z)=\Pi_{l=1}^{m}\widehat{\mu}(b_l^{Q'}z)^{c_l}. \qquad (1.2)$$

The class of distributions satisfying (1.2) is denoted by $OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$. We call distributions in $OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ strictly $(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ $-$*semi*$-$ *stable*. We note that $\mu\in OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ with $Q=I$ is a strictly m-semi-stable distribution on R^d in the sense of [3]. In one dimension ($d=1$), the class of $OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ is that of strictly m-semi-stable distributions on R, which is extended to more general case by Shimizu [17], Shimizu and Davies [18] and others.

For some $0<b<1$, let $OSS(b,Q)$ be the class of $\mu\in I(R^d)$ such that

$$\widehat{\mu}(z)=e^{\langle\gamma,z\rangle}\widehat{\mu}(b^{Q'}z)^{c} \qquad (1.3)$$

for some $c>0$ and $\gamma\in R^d$. Distributions in $OSS(b,Q)$ are called $(b,Q)-$*semi*$-$ *stable*.

For some $0<b<1$, $\mu\in OSS_0(b,Q)$ means that $\widehat{\mu}(z)=\widehat{\mu}(b^{Q'}z)^{c}$ for some $c>0$. Distributions in $OSS_0(b,Q)$ are called *strictly-* (b,Q)*- semi- stable*. We note that operator $1-$semi-stable distribution is (b,Q)-semi-stable distribution. Also we note that, for every $b\in(1,\infty)$, the distribution satisfying (1.3) is operator stable distribution of Sharpe [15], which is an extension of stable distribution. Stable distributions were introduced in the 1920s by Lévy. It is well-known that Gaussian distributions are special cases of stable distributions. See [11] for details and further references.

Operator semi-stable distributions are defined as limit distributions of subsequences via $\{k_j\}$ with $\frac{k_j}{k_{j+1}} \to b$ for some $b \in (0,1)$, of operator normalizations of partial sums Y_k of sequences of independent identically distributed random vectors. The operator normalization of Y_k here means $A_k Y_k + c_k$ with vectors c_k in R^d and sequences of invertible linear operators A_k acting in R^d. Under the condition of fullness(that is, the support of μ is not contained in any $(d-1)$-dimensional hyperplane in R^d), Jajte [7] shows that μ is operator semi-stable if and only if $\mu \in OSS(b, Q)$ for some $0 < b < 1$ and $Q \in M_+(R^d)$ satisfying the condition that the real parts of all eigenvalues are more than or equal to $1/2$. Without the assumption of the fullness of μ, the class of operator semi-stable distributions is strictly bigger than $OSS(b, Q)$, which is given in [2]. The operator semi-stable distributions are an extension of operator stable distributions on one hand and of semi-stable distributions on the other. See [13,14] for review on operator stable distributions and see [11] for review on semi-stable distributions.

The main purpose of this paper is to obtain characterization of translations of strictly $(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$-semi-stable distributions and to discuss relations between translation of strictly operator semi-stable distribution and translation of strictly $(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$-semi-stable distribution.

After some preliminaries in Section 2, we give a necessary and sufficient condition for $(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$ -semi-stable purely non-Gaussian distribution in Section 3. In Section 4, we get complete characterization of $\mu \in OSS_0(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$ and give a representation of translation of strictly $(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$-semi-stable distribution. Using these results, we investigate relations between translation of strictly operator semi-stable distribution and strictly $(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$-semi-stable distribution.

2. Preliminaries

We begin with some notation. Let $B(R^d)$ be the collection of Borel sets in R^d. Let θ_j, $1 \le j \le q+2r$ denote all distinct eigenvalues of Q such that $\theta_1, \cdots, \theta_q$, are real if $q \ge 1$ and that $\theta_{q+1}, \cdots, \theta_{q+2r}$, are non-real and $\theta_j = \overline{\theta_{j+r}}$ for $q+1 \le j \le q+r$ if $r \ge 1$. Let $\theta_j = \alpha_j + i\beta_j$, where α_j and β_j are real numbers. Let $f(\zeta)$ be the minimal polynomial of Q

with

$$f(\zeta)=f_1(\zeta)^{n_1}\cdots f_{q+r}(\zeta)^{n_{q+r}},$$

where $f_j(\zeta)=\zeta-\alpha_j$ for $1\le j\le q$ and $(\zeta-\alpha_j)^2+\beta_j^2$ for $q+1\le j\le q+r$. We write W_j for the kernel of $f_j(Q)^{n_j}$ in R^d, $1\le j\le q+r$. We denote the kernel of $(Q-\theta_j)^{n_j}$ in C^d, $1\le j\le q+2r$, by V_j. Let T_j be the projector onto V_j. We denote

$$D_j=\{(Q-\theta_j)v : v\in V_j\} \text{ in } C^d,\ 1\le j\le q+2r.$$

Let P_j be the projector onto D_j in C^d, $1\le j\le q+2r$. We easily show the following proposition.

Proposition 2.1. Suppose that 1 is not an eigenvalue of $\sum_{l=1}^{m} c_l b_l^Q$. Then any $\mu\in OSS(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ is a translation of a strictly $(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$-semi- stable distribution.

We set

$$J=\{j : 1\le j\le q+2r \textit{ satisfying } \sum_{l=1}^{m} c_l b_l^{\theta_j}=1 \text{ and } \alpha_j>1/2\},$$

$$K=\{j : 1\le j\le q+2r \textit{ satisfying } \sum_{l=1}^{m} c_l b_l^{\theta_j}\neq 1 \text{ and } \alpha_j>1/2\},$$

and

$$\Gamma=\{j : 1\le j\le q+r \textit{ satisfying } \alpha_j>1/2\}.$$

Let $W_\Gamma=\oplus_{j\in\Gamma}W_j$, and let $S_\Gamma=\{\xi\in W_\Gamma : |\xi|=1, |u^Q\xi|>1 \text{ for } all\ u>1\}$. Then any $x\in W_\Gamma$ is uniquely expressed as $x=u^Q\xi$ with $\xi\in S_\Gamma$ and $u\in(0,\infty)$.

Any $\mu\in I(R^d)$ has the Lévy representation (A,ν,γ), which means

$$\hat{\mu}(z)=\exp\left[i\langle\gamma,z\rangle-\frac{1}{2}\langle Az,z\rangle+\int_{R^d}G(z,x)\nu(dx)\right],$$

with $G(z,x)=e^{i\langle z,x\rangle}-1-i\langle z,x\rangle(1+|x|^2)^{-1}$. Here $\gamma\in R^d$, A(called the Gaussian covariance of μ) is a symmetric nonnegative-definite operator on R^d, and ν (called the Lévy measure of μ) is a Lévy measure satisfying $\nu\{0\}=0$ and

$\int_{R^d-\{0\}} |x|^2(1+|x|^2)^{-1}\nu(dx)<\infty$. These A, ν and γ are uniquely determined by μ. When $\nu=0$, we call μ a Gaussian distribution. If $A=0$, then we call μ a purely non-Gaussian distribution.

3. Purely non-Gaussian operator m-semi-stable distributions

For any $\rho>0$, $A_m(0)$ and $A_m(\rho)$ are respectively the sets of all m-tuples $(b_1,\cdots,b_m)$ with $0<b_j<1$, $j=1,\cdots,m$, satisfying the following conditions.

$A_m(0)$: for some l and i, $\log b_l/\log b_i$ is an irrational number,

$A_m(\rho)$: $\log b_l/\log b_i$ is a rational numbers for every l and i, and $m_l=-\log b_l/\rho$, $l=1,\cdots,m$, are positive integers with their greatest common factor equal to one.

The following Theorem 3.1 characterizes the class of all purely non-Gaussian operator m-semi-stable distributions. A related paper is Chorny [4]. In this paper, we do not treat the whole structure of Gaussian operator m-semi-stable distributions. The complete description of Gaussian operator stable distributions and Gaussian operator semi-stable distributions is respectively obtained by Sato [13] and Sato and Yamazato[14], and Choi [2].

Theorem 3.1. Let μ be a $(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$-semi-stable distribution on R^d with Lévy representation $(0,\nu,\gamma)$. Then, $\mu\in OSS(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ if and only if

$$\nu(E)=\int_{S_r}\lambda(d\xi)\int_0^\infty I_E(u^Q\xi)d(-H_\xi(u)u^{-1}),\qquad E\in B(R^d),$$

where

(i) λ is a finite measure on S_r.

(ii) $H_\xi(u)$ is a real-valued function being right-continuous in $u\in(0,\infty)$ and measurable in $\xi\in S_r$ such that $H_\xi(u)u^{-1}$ is decreasing(in the wide sense allowing flatness), $H_\xi(1)=1$ for any u and ξ and in addition, one of the following (a) and (b):

(a) $(b_1,\cdots b_m)\in A_m(0)$, $H_\xi(u)=1$,

(b) $(b_1,\cdots b_m)\in A_m(\rho)$, $H_\xi(bu)=H_\xi(u)$ and $b=e^{-\rho}$.

This λ is uniquely determined by ν, called *spherical component* of ν and $H_\xi(u)$ called the *Q-radial component* of ν.

Proof of Theorem 3.1. Suppose that $\mu\in OSS(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ with the Lévy representation $(0, \nu, \gamma)$. Then we see that

$$\widehat{\mu}(b_j^{Q'} z) = \exp\left\{ \int_{R^d} G(b_j^{Q'} z, x)\nu(dx) + i\langle \gamma, b_j^{Q'} z\rangle \right\}.$$

For $E \in B(R^d)$, let $\nu_{b_j^Q}(E) = \nu(b_j^{-Q}E)$ on $R^d - \{0\}$. Then we have that

$$\nu(E) = \sum_{l=1}^{m} c_l \nu(b_l^{-Q}E). \tag{3.1}$$

For $s \in (0, \infty)$ and $B \in B(S_\Gamma)$, define $N(s, B) = \nu(\{u^Q\xi : \xi \in B,\ u > s\})$, then the condition (3.1) gives

$$N(s, B) = \sum_{l=1}^{m} c_l\, N(b_l^{-1}s, B). \tag{3.2}$$

Let $-B = \{-\xi \mid \xi \in B\}$. Then we easily check that $-B \in B(S_\Gamma)$ for $B \in B(S_\Gamma)$. By (3.2), we get

$$N(s, -B) = \sum_{l=1}^{m} c_l N(b_l^{-1}s, -B).$$

Let $g_B(u) = N(e^u, B)$ and $h_B(u) = N(e^u, -B)$. we see that $g_{-B}(u) = h_B(u)$. We also see the following by Theorem 5.4.1 (163) in [9]:

(i) If $(b_1, \cdots, b_m) \in A_m(0)$, then $N(ts, B) = tN(s, B)$ for every $t \in (0, \infty)$.

(ii) If $(b_1, \cdots, b_m) \in A_m(\rho)$, then $N(b^n s, B) = b^{-n}N(s, B)$ for every integer n, where $b = e^{-\rho}$.

For $B \in B(S_\Gamma)$, define $\lambda(B) = \nu(\{u^Q\xi \mid \xi \in B,\ u > 1\})$. For any fixed $s \in (0, \infty)$, there is a nonnegative measurable function $N_\xi(s)$ of ξ such that

$$N(s, B) = \int_B N_\xi(s)\lambda(d\xi), \qquad B \in B(S_\Gamma).$$

Further discussion is given in proof of Theorem 4.1 in [2]. Note that $N_\xi(s)$ is right-continuous in $s \in (0, \infty)$. Theorem 2.3 in [3] yields the following:

If $(b_1, \cdots, b_m) \in A_m(0)$, then $N_\xi(s) = s^{-1}N_\xi(1)$.

If $(b_1, \cdots, b_m) \in A_m(\rho)$, then $N_\xi(b^n s) = b^{-n}N_\xi(s)$ for every integer n.

Setting $N_\xi(u)=H_\xi(u)u^{-1}$, we see that $H_\xi(1)=1$ and $H_\xi(s)$ is nonnegative right-continuous function. Hence ν is the Lévy measure of (b,Q)-semi-stable. Thus by Lemma 4. 2. in [2], we have that $spt\ \nu\subset W_\Gamma$.

In the same way as in proof of Theorem 2.3 in [3] , we can prove the convers assertion.

4. Strictly operator m-semi-stable distributions

For $\xi\in S_\Gamma$, we set

$$a(b_l,u,\xi)=\frac{1}{1+|u^Q\xi|^2}-\frac{1}{1+|(\frac{u}{b_l})^Q\xi|^2}$$

and set

$$g_{j,k}(b_1,\cdots,b_m,u,\xi)=u^{\theta_j}(k!)^{-1}(\log u)^k\sum_{l=1}^{m}c_l a(b_l,u,\xi)$$

for $1\le j\le q+2r$, $k\ge 0$. For $\xi\in S_\Gamma$, define

$$g_0(b_1,\cdots,b_m,\xi)=\sum_{j\in K}\int_0^\infty\sum_{k=0}^{n_j-1}(Q-\theta_j)T_j\xi g_{j,k}(b_1,\cdots,b_m,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right),$$
$$g_1(b_1,\cdots,b_m,\xi)=\sum_{j\in J}\int_0^\infty\sum_{k=0}^{n_j-1}(Q-\theta_j)^kT_j\xi g_{j,k}(b_1,\cdots,b_m,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right).$$

Lemma 4.1. The functions $g_0(b_1,\cdots,b_m,\xi)$, $g_1(b_1,\cdots,b_m,\xi)$ are well-defined, R^d-valued, bounded, and measurable on S_Γ.

Proof. Let C_0 be constants independent of $\xi\in S_\Gamma$ and u. Since $\sum_{l=1}^{m}c_l a(b_l,u,\xi)\le C_0|u^Q\xi|^2$, we can show the assertion in the above lemma by using the proof of Lemma 2.1 in [1].

Theorem 4.1. Let $\mu\in OSS(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$. Let λ and $H_\xi(u)$ be the spherical component and the Q-radial component of ν. Then $\mu\in OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ if and only if

$$(I-\sum_{l=1}^{m}c_l b_l^Q)T_j\gamma=\int_{S_\Gamma}(g_0+g_1)(b_1,\cdots,b_m,\xi)\lambda(d\xi),\qquad 1\le j\le q+2r. \tag{4.1}$$

Proof. Let $\mu\in OSS(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$. Then $\mu\in OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ if and only if

$$(I-\sum_{l=1}^{m}c_l b_l^Q)\gamma=\int_{S_r}\lambda(d\xi)\int_0^\infty u^Q\xi\sum_{l=1}^{m}c_l a(b_l,u,\xi)d(\frac{-H_\xi(u)}{u}).$$

By Lemma 4.1, the condition (4.1) is written as

$$(I-\sum_{l=1}^{m}c_l b_l^Q)\gamma=\int_{S_r}(g_0+g_1)(b_1,\cdots,b_m,\xi)\lambda(d\xi).$$

For $j\in J$, $\xi\in S_r$, and $T_j\xi\neq 0$, define

$$g_{j,0}(b_1,\cdots,b_m,\xi)=\int_0^\infty g_{j,0}(b_1,\cdots,b_m,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right).$$

Using the proof of Lemma 4.1, we get that

$$\int_0^\infty |g_{j,0}(b_1,\cdots,b_m,u,\xi)|d\left(\frac{-H_\xi(u)}{u}\right)<\infty,$$

which follows that $g_{j,0}(b_1,\cdots,b_m,\xi)$ is well-defined.

Theorem 4.2. Let μ be as in Theorem 3.1. Then μ is a translation of a strictly $(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$-semi-stable distribution if and only if

$$\int_{S_r}(I-P_j)g_{j,0}(b_1,\cdots,b_m,\xi)T_j\xi\lambda(d\xi)=0 \text{ for } j\in J. \tag{4.2}$$

Proof. We first prove that (4.2) is a sufficient condition for a translation of a strictly $(b_1,\cdots,b_m,Q)$-semi-stable distribution. Suppose that (4.2) holds. Since $(I-P_j)T_jg_1(b_1,\cdots,b_m,\xi)=(I-P_j)\,g_{j,0}(b_1,\cdots,b_m,\xi)T_j\xi$, the condition (4.2) gives $\int_{S_T}(I-P_j)T_jg_1(b_1,\cdots b_m,\xi)\lambda(d\xi)=0$. This is written as

$$\int_{S_T}T_jg_1(b_1,\cdots b_m,\xi)\lambda(d\xi)\in D_j.$$

Let $\hat{V}_j=\text{kernel}(Q-\theta_j)$, $j=1,\cdots,q+2r$. Let $\tilde{V}_j$ be the orthogonal complement of $\hat{V}_j$ in the decomposition $V_j=\tilde{V}_j\oplus\hat{V}_j$.

We claim that $I-\sum_{l=1}^{m} c_l b_l^Q : \widetilde{V}_j \rightarrow D_j$ is bijective map for $j \in J$.

To prove this, suppose that $\widetilde{v}_j \in \widetilde{V}_j$ and $\widetilde{v} \neq 0$. Then we have that

$$(I-\sum_{l=1}^{m} c_l b_l^Q)\widetilde{v}_j = \sum_{k=1}^{L_j-1}(k!)(Q-\theta_j)^k \sum_{l=1}^{m} c_l \log b_l^k \quad \text{for } j \in J,$$

where L_j is a nonnegative integer such that $(Q-\theta_j)^{L_j}\widetilde{v}_j = 0$ and $(Q-\theta_j)^k \widetilde{v}_j \neq 0$ for all $k=0,\cdots,L_j-1$. If $(I-\sum_{l=1}^{m} c_l b_l^Q)\widetilde{v}_j = 0$, then

$$(I-\sum_{l=1}^{m} c_l b_l^Q)\widetilde{v}_j = \sum_{k=1}^{L_j-1}(k!)(Q-\theta_j)^k \sum_{l=1}^{m} c_l \log b_l^k = 0,$$

which leads to $\widetilde{v}_j = 0$. From this we conclude that $I-\sum_{l=1}^{m} c_l b_l^Q$ is injective as a map from $\widetilde{V}_j$ to D_j for $j \in J$. Using the fact that $\dim(D_j) = \dim(\widetilde{V}_j)$, we see that $I-\sum_{l=1}^{m} c_l b_l^Q$ is surjective as a map from $\widetilde{V}_j$ to D_j.

For $j \in J$, there is a unique $\gamma_j \in \widetilde{V}_j$ such that

$$(I-\sum_{l=1}^{m} c_l b_l^Q)\gamma_j = T_j \int_{S_r} g_1(b_1,\cdots,b_m,\xi)\lambda(d\xi).$$

Given $j \notin J$, we see that $(I-\sum_{l=1}^{m} c_l b_l^{\theta_j})(v_j) \neq 0$ for $v_j \neq 0$. This says that $(I-\sum_{l=1}^{m} c_l b_l^Q)$ is bijective. Hence there is a unique γ_j such that

$$(I-\sum_{l=1}^{m} c_l b_l^Q)\gamma_j = T_j \int_{S_r} g_0(b_1,\cdots,b_m,\xi)\lambda(d\xi), \quad j \notin J.$$

Let $\gamma_j = 0$, $j \notin K \cup J$, and $c = \sum_{j=1}^{q+2r} \gamma_j$. Then $c \in R^d$ and

$$(I-\sum_{l=1}^{m} c_l b_l^Q)c = (1-\sum_{l=1}^{m} c_l b_l^Q)\sum_{j=1}^{q+2r} \gamma_j = \int_{S_r}(g_0+g_1)(b_1,\cdots,b_m,\xi)\lambda(d\xi).$$

This means that $\mu * \delta_{-\gamma+c} \in OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$.

To prove the converse, suppose that $\mu * \delta_{-\gamma+c} \in OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ for some c. From (4.1),

$$(I-\sum_{l=1}^{m} c_l b_l^Q)\sum_{j\in I} T_j(\gamma+c)=\int_{S_r} g_0(b_1,\cdots,b_m,\xi)\lambda(d\xi)$$

and

$$(I-\sum_{l=1}^{m} c_l b_l^Q)\sum_{j\in J} T_j(\gamma+c)=\int_{S_r} g_1(b_1,\cdots,b_m,\xi)\lambda(d\xi).$$

This implies that $\int_{S_r} g_0(b_1,\cdots,b_m,\xi)\lambda(d\xi)\in D_j$ for $j\in J$, which gives

$$\int_{S_r}(I-P_j)g_{j,0}(b_1,\cdots,b_m,\xi)T_j\xi\lambda(d\xi)=0, \qquad j\in J.$$

Theorem 4.3. Let $OSS(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)=OSS(b,Q)$ for some b. If $\mu\in OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$, then μ is a translation of a strictly (b,Q)-semi- stable distribution.

Proof. Suppose that $\mu\in OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ with $(b_1,\cdots b_m)\in A_m(0)$. Then $H_\xi(u)=1$, which means that this μ is an operator stable distribution with exponent Q. Assume that $\theta_1=1$. Let N_1,N_2, and k be integers, we see that

$$\begin{aligned}\int_0^\infty u^{-1}a(b_l,u,\xi)du &= \lim_{N_1\to-\infty,N_2\to\infty}\int_{[b_l^{N_2+1},b_l^{N_1})} a(b_l,u,\xi)u^{-1}du\\ &= \lim_{N_1\to-\infty,N_2\to\infty}\sum_{k=N_1}^{N_2}\int_{[b_l^{k+1},b_l^k)} u^{-1}a(b_l,u,\xi)du\\ &= \lim_{N_2\to\infty}\int_{[b_l,1)}\frac{u^{-1}}{1+|b_l^{N_2Q}u^Q\xi|^2}du-\lim_{N_1\to-\infty}\int_{[b_l,1)}\frac{u^{-1}}{1+|b_l^{(N_1-1)Q}u^Q\xi|^2}du\\ &= \int_{[b_l,1)}\frac{1}{u}du\\ &= \log b_l,\end{aligned}$$

which yields that

$$g_{1,0}(b_1,\cdots,b_m,\xi)=\int_0^\infty u^{-1}\sum_{l=1}^{m}c_l a(b_l,u,\xi)du=\sum_{l=1}^{m}c_l\int_{[b_l,1)}u^{-1}du=\sum_{l=1}^{m}c_l\log b_l.$$

Hence the condition (4.2) is written as

$$\int_{S_r}(I-P_1)T_1\xi\sum_{l=1}^{m}c_l\log b_l\lambda(d\xi)=0.$$

This gives $\int_{S_r}(I-P_1)T_1\xi\lambda(d\xi)=0$. From Theorem 3.1 in [12], this means that μ is a translation of strictly (1,Q)-stable.

Suppose that $\mu\in OSS_0(b_1,\cdots,b_m,c_1,\cdots,c_m,Q)$ with $(b_1,\cdots b_m)\in A_m(\rho)$. Then $b_l=b^{m_l}$ and $H_\xi(bu)=H(u)$. Let $J_1=\{j\in J: b^{\theta_j}=b\}$. Using the tool used in Lemma 2.2 in [1], for $j\in J_1$, we can show that

$$\int_0^\infty u^{\theta_j}c_la(b_l,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right)=c_l\int_{[b_l,1)}u\cos\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)+ic_l\int_{[b_l,1)}u\sin\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)$$

for some integer n, where $\xi\in S_r$. Since

$$\begin{aligned}\int_{[b_l,1)}u\cos\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)&=\sum_{i=0}^{m_l-1}c_l\int_{[b^{i+1},b^i)}u\cos\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)\\&=\sum_{l=1}^{m}c_lm_l\int_{[b,1)}u\cos\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)\end{aligned}$$

and

$$\begin{aligned}\int_{[b_l,1)}u\sin\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)&=\sum_{i=0}^{m_l-1}c_l\int_{[b^{i+1},b^i)}u\sin\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)\\&=\sum_{l=1}^{m}c_lm_l\int_{[b,1)}u\sin\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right),\end{aligned}$$

we have that

$$\begin{aligned}&\int_0^\infty u^{\theta_j}\sum_{l=1}^{m}c_la(b_l,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right)\\&=\sum_{l=1}^{m}c_l\int_{[b^{m_l},1)}u\cos\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)+i\sum_{l=1}^{m}c_l\int_{[b^{m_l},1)}u\sin\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)\\&=\sum_{l=1}^{m}c_lm_l\int_{[b,1)}u\cos\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)+i\sum_{l=1}^{m}c_lm_l\int_{[b,1)}u\sin\left(\frac{2n\pi}{\log b}\log u\right)d\left(\frac{-H_\xi(u)}{u}\right)\end{aligned}$$

for $j \in J_1$. Thus the condition (4.2) is written as

$$\int_{S_r} (I-P_j) T_j \xi \sum_{l=1}^{m} c_l m_l \left(\int_{[b,1)} u \cos\left(\frac{2n\pi}{\log b} \log u\right) d\left(\frac{-H_\xi(u)}{u}\right) + i \int_{[b,1)} u \sin\left(\frac{2n\pi}{\log b} \log u\right) d\left(\frac{-H_\xi(u)}{u}\right) \right) \lambda(d\xi) = 0$$

for $j \in J_1$. This shows that

$$\int_{S_r} (I-P_j) T_j \xi \left(\int_{[b,1)} u \cos\left(\frac{2n\pi}{\log b} \log u\right) d\left(\frac{-H_\xi(u)}{u}\right) + i \int_{[b,1)} u \sin\left(\frac{2n\pi}{\log b} \log u\right) d\left(\frac{-H_\xi(u)}{u}\right) \right) \lambda(d\xi) = 0, \quad j \in J_1.$$

By Theorem 2.2. in [1], this says that μ is a translation of a strictly (b, Q)-semi-stable distribution.

References

[1] Choi, G. S. (2001). Characterization of strictly operator semi-stable distributions, *J. Korean Math. Soc.*, Vol. 38, 101-123.

[2] Choi, G. S. (2000). Representation of operator semi-stable distributions, *Bull. Korean Math. Soc.*, Vol. 37, 135-152.

[3] Choi, G. S. (1995). Characterization of some classes of multidimensional distributions related to semi-stable distributions, *Japan. J. Math.*, Vol. 21, 335-353.

[4] Chorny, V. (1989). Characterization of operator-semi-stable distributions, *Mathematica Slovaca*, Vol. 39, 99-104.

[5] Krapavickait ė, D. (1980) Generalized semistable probability distributions, Litovsk.Mat.Sb., 20-2(1980), 57-68;English translation, *Lithuanian Math.J.*, 20, 111-118.

[6] Krapavickait ė, D. (1980) Certain classes of probability distributions, Litovsk. Mat. Sb., 20-4, 79-86;English translation, *Lithuanian Math. J.*, 20, 298-303.

[7] Jajte, R. (1977) Semi-stable probability measures on R^N, *Studia Math.*, 61, 29-39.

[8] L évy, P. (1954). *Theorie de l'addition des variables aleatoires,* 2^e ed., Gauthier Villars, Paris 1^e ed. (1937).

[9] Linnik, Y. (1943). Linear forms and statistical criteria(in Russian), *Ukrain.Math. Statist. and Pro.*, Vol. 5, 207-243 and 247- 290; English translation (1962),

Selected Translations in Math. Statist. and Pro., Vol. 3, 1-90.

[10] Ramachandran, B. and Rao, C. R. (1970). Solution of functional equations arising in some regression problems and a charaterizations the Cauchy law, *Sankhyā, Ser. A*, Vol. 32, 1-30.

[11] Sato, K.(1999). Lévy *Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge.

[12] Sato, K. (1987). Strictly operator-stable distributions, *J. Multivar. Anal.*, Vol. 22, 278-295.

[13] Sato, K. (1985). Lectures on multivariate infinitely divisible distributions and operator-stable processes, *Technical Report Series, Lab.Res. Statist. Probab. Carleton Unvi. and Univ Ottawa, No.54, 1985.*

[14] Sato, K. and Yamazato, M. (1985). Completely operator-selfdecomposable distribut-ions and operator-stable distributions, *Nagoya. Math. J.*, Vol. 97, 71-94.

[15] Sharpe, M. (1969) Operator-stable probability distributions on vector groups, *Trans. Amer. Math. Soc.* 136, 51-65.

[16] Shimizu, R. (1968). Characteristic function satisfying a functional equation I, *Ann. Inst. Statist. Math.*, Vol. 20, 187-209.

[17] Shimizu, R. (1978) Solution to a functional equation and its application to some characterization problem, *Sankhyā*, *Ser.A*, 40, 319-332.

[18] Shimizu, R and Davies, L. (1981) General characterization theorems for the Weibull and stable distributions, *Sankhyā*, *Ser.A,* 43, 282-310.

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