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A Stochastic Differential Equation Model for Software Reliability Assessment and Its Goodness-of-Fit

Shigeru Yamada * **Akio Nishigaki**
Department of Social Systems Engineering
Faculty of Engineering, Tottori University
4-101 Minami, Koyama-cho, Tottori-shi, 680-8552 Japan

Mitsuhiro Kimura
Department of Industrial and Systems Engineering
Faculty of Engineering, Hosei University
3-7-2, Kajino-cho, Koganei-shi, 184-8584 Japan

Abstract. Many software reliability growth models (SRGM's) based on a nonhomogeneous Poisson process (NHPP) have been proposed by many researchers. Most of the SRGM's which have been proposed up to the present treat the event of software fault-detection in the testing and operational phases as a counting process. However, if the size of the software system is large, the number of software faults detected during the testing phase becomes large, and the change of the number of faults which are detected and removed through debugging activities becomes sufficiently small compared with the initial fault content at the beginning of the testing phase. Therefore, in such a situation, we can model the software fault-detection process as a stochastic process with a continuous state space.

In this paper, we propose a new software reliability growth model describing the fault-detection process by applying a mathematical technique of stochastic differential equations of an Itô type. We also compare our model with the existing SRGM's in terms of goodness-of-fit for actual data sets.

Key Words : *software reliability growth model, stochastic differential equations, instantaneous MTBF, cumulative MTBF, goodness-of-fit.*

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* Corresponding author.
E-mail address: yamada@sse.tottori-u.ac.jp

1. INTRODUCTION

At present, computer systems have been used all over the world. We depend on their benefit more and more to make our life comfortable. Therefore if such a computing system which plays an important role in our infrastructure breaks down, the impact of the accident will be terrible. Since software systems are mainly developed by human work, it is impossible to avoid the introduction of software faults during the development phase of software systems. Therefore, the assessment problem of software reliability has become very important.

In this paper, we propose new SRGM's describing a fault-detection process by applying a mathematical technique of stochastic differential equations of an Itô type (see Arnold (1974)). First, we derive a probability distribution of the software fault-detection process from the SRGM based on an NHPP (see Yamada and Osaki (1985)) by applying the mathematical technique of stochastic differential equations. Second, we derive several software reliability assessment measures based on our new models. Finally, by applying a method of maximum-likelihood, we estimate the unknown parameters of the models. Furthermore, we compare our models based on stochastic differential equations with the existing SRGM's, i.e., the exponential, delayed S-shaped, and inflection S-shaped SRGM's (see Yamada (2002)) in terms of goodness-of-fit for actual data sets.

2. MODEL DESCRIPTION

2.1 NHPP Model

Let $H(t)$ be the expected number of faults detected up to testing time t, which is called a mean value function. We can obtain the following fundamental equation based on an NHPP:

$$\frac{dH(t)}{dt} = b_0(t)\{a_0 - H(t)\} \quad (b_0(t) > 0,\ t \geq 0), \tag{2.1}$$

where a_0 is the expected number of initial inherent faults and $b_0(t)$ a fault-detection rate per unit time per fault at testing time t and a non-negative function.

2.2 Stochastic Differential Equation Modeling

Let $N(t)$ be a random variable which represents the number of software faults detected in the software system up to testing time $t (t \geq 0)$. Suppose that $N(t)$ takes on continuous real value. The NHPP models have treated the software fault-detection process in the testing phase as the discrete state space. However, if the size of the software system is large, the number of software faults detected during the testing phase becomes large, and the change of the number of faults which are detected and removed through debugging activities becomes sufficiently small compared with the initial fault content at the beginning of the testing phase. Therefore, in such a situation, we can model the software fault-detection process as a stochastic process with a continuous state space.

Since the latent faults in the software system are detected and eliminated during the testing phase, the number of faults remaining in the software system gradually decreases as the testing procedures go on. Thus, under common assumptions for software reliability growth modeling, we consider the following linear differential equation:

$$\frac{dN(t)}{dt} = b(t)\{a - N(t)\}, \tag{2.2}$$

where $b(t)$ is a fault-detection rate per unit time per fault at testing time t and assumed to be a non-negative function.

In this paper, we suppose that $b(t)$ in equation (2.2) has the irregular fluctuation. That is, we extend equation (2.2) to the following equation:

$$\frac{dN(t)}{dt} = \{b(t) + \sigma\gamma(t)\}\{a - N(t)\}, \tag{2.3}$$

where σ is a positive constant representing a magnitude of the irregular fluctuation and $\gamma(t)$ a standardized Gaussian white noise. We assume that $b(t)$ in equation (2.3) is approximately equal to $b_0(t)$ in equation (2.1) as follows:

$$b(t) = \frac{\frac{dN(t)}{dt}}{a - N(t)} \doteq b_0(t) = \frac{\frac{dH(t)}{dt}}{a - H(t)}. \tag{2.4}$$

We extend equation (2.3) to the following stochastic differential equation of an Itô type:

$$dN(t) = \{b(t) - \frac{1}{2}\sigma^2\}\{a - N(t)\}dt + \sigma\{a - N(t)\}dw(t), \tag{2.5}$$

where $w(t)$ is a one-dimensional Wiener process which is formally defined as an integration of the white noise $\gamma(t)$ with respect to time t. The Wiener process is a Gaussian process and it has the following properties:

$$\Pr[w(0) = 0] = 1, \tag{2.6}$$

$$\mathrm{E}[w(t)] = 0, \tag{2.7}$$

$$\mathrm{E}[w(t)w(t')] = \min[t, t']. \tag{2.8}$$

By using the Itô's formula, we can obtain the solution process as follows:

$$N(t) = a\left[1 - \exp\left\{-\int_0^t b(s)ds - \sigma w(t)\right\}\right]. \tag{2.9}$$

Thus we can derive new software reliability assessment models from the existing NHPP ones by using equation (2.4). We show three models based on the exponential, delayed S-shaped, and inflection S-shaped SRGM's in the following.

In the case of the exponential SRGM, we can obtain the following equations (see Yamada et al. (1994)):

$$\begin{aligned}
H_e(t) &= a\{1-\exp(-bt)\} \qquad (a>0, b>0), && (2.10)\\
h_e(t) &= ab\cdot\exp(-bt), && (2.11)\\
b_e(t) &= \frac{\frac{dN_e(t)}{dt}}{a-N_e(t)} \doteq \frac{\frac{dH_e(t)}{dt}}{a-H_e(t)} = \frac{h_e(t)}{a-H_e(t)} = b, && (2.12)
\end{aligned}$$

where $H_e(t)$, $h_e(t)$, $b_e(t)$, and $N_e(t)$ are the mean value function of the exponential SRGM, an intensity function, a fault-detection rate per unit time per fault at testing time t, and the number of faults detected up to testing time t (random variable), respectively. By substituting equation (2.12) into equation (2.9), we can obtain the following solution process:

$$N_e(t) = a[1-\exp\{-bt-\sigma w(t)\}]. \qquad (2.13)$$

Since Wiener process $w(t)$ is a Gaussian process, $\log\{a-N_e(t)\}$ is also a Gaussian process, and its expected value and variance is calculated as follows:

$$\begin{aligned}
\mathrm{E}[\log\{a-N_e(t)\}] &= \log a - bt, && (2.14)\\
\mathrm{Var}[\log\{a-N_e(t)\}] &= \sigma^2 t. && (2.15)
\end{aligned}$$

Thus we can obtain the following equation:

$$\Pr[\log\{a-N_e(t)\} \le x] = \Phi\left(\frac{x-\log a+bt}{\sigma\sqrt{t}}\right), \qquad (2.16)$$

where $\Phi(\cdot)$ is a standardized normal distribution function which is defined as follows:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp(-\frac{y^2}{2})dy. \qquad (2.17)$$

Therefore, we can obtain the transition probability distribution of $N_e(t)$ as follows:

$$\Pr[N_e(t)\le n | N_e(0)=0] = \Phi\left(\frac{\log\frac{a}{a-n}-bt}{\sigma\sqrt{t}}\right). \qquad (2.18)$$

When we use the delayed S-shaped and the inflection S-shaped SRGM's, we can obtain the transition probability distribution by calculating in the same way with the exponential SRGM by using the following mean value functions:

$$\begin{aligned}
H_d(t) &= a[1-(1+bt)\exp(-bt)] \qquad (a>0, b>0), && (2.19)\\
H_i(t) &= \frac{a\{1-\exp(-bt)\}}{\{1+c\cdot\exp(-bt)\}} \qquad (a>0, b>0, c>0), && (2.20)
\end{aligned}$$

where $H_d(t)$ and $H_i(t)$ represent the mean value functions of the delayed S-shaped and the inflection S-shaped SRGM's, respectively, and c the inflection parameter which is defined as follows:

$$c = (1-l)/l, \tag{2.21}$$

where l is the inflection rate which indicates the ratio of the number of detectable faults to the total number of faults in the software system.

Therefore, the transition probability distribution of these two models are obtained as follows:

$$\begin{aligned} N_d(t) &= a[1-(1+bt)\exp\{-bt-\sigma w(t)\}], && (2.22)\\ N_i(t) &= a\left[1-\frac{1+c}{1+c\cdot\exp(-bt)}\exp\{-bt-\sigma w(t)\}\right], && (2.23) \end{aligned}$$

where $N_d(t)$ and $N_i(t)$ are the number of faults detected up to testing time t by using the delayed S-shaped SRGM in equation (2.19) and the inflection S-shaped SRGM in equation (2.20), respectively.

3. SOFTWARE RELIABILITY ASSESSMENT MEASURES

3.1 Instantaneous MTBF

First, we show the instantaneous mean time between software failures (instantaneous MTBF, denoted as $MTBF_I$).

Instantaneous MTBF is approximately given by:

$$MTBF_I = \frac{dt}{\mathrm{E}[dN(t)]}. \tag{3.1}$$

We consider the mean number of faults detected up to testing time t. The density function of $w(t)$ is given by:

$$f(w(t)) = \frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{w(t)^2}{2t}\right\}. \tag{3.2}$$

Thus the mean number of detected faults up to testing time t for the three new models can be obtained as follows:

$$\begin{aligned} \mathrm{E}[N_e(t)] &= \mathrm{E}[a]-\mathrm{E}[a]\exp\{-bt+\frac{\sigma^2}{2}t\}\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\exp\left[-\frac{\{w(t)+\sigma t\}^2}{2t}\right]dw(t)\\ &= a\left\{1-\exp\left(-bt+\frac{\sigma^2}{2}t\right)\right\}, && (3.3)\\ \mathrm{E}[N_d(t)] &= a\left\{1-(1+bt)\exp\left(-bt+\frac{\sigma^2}{2}t\right)\right\}, && (3.4)\\ \mathrm{E}[N_i(t)] &= a\left\{1-\frac{1+c}{1+c\cdot\exp(-bt)}\exp\left(-bt+\frac{\sigma^2}{2}t\right)\right\}. && (3.5) \end{aligned}$$

By using equation (2.5), we can obtain $\mathrm{E}[dN(t)]$ as follows:

$$\begin{aligned}\mathrm{E}[dN(t)] = \mathrm{E}[a]\{b(t) - \frac{1}{2}\sigma^2\}\exp(-bt)\mathrm{E}[\exp\{-\sigma w(t)\}]dt \\ +\sigma\mathrm{E}[a]\exp(-bt)\mathrm{E}[\exp\{-\sigma w(t)\}dw(t)],\end{aligned} \tag{3.6}$$

where since the Wiener process has the independent increment property, $w(t)$ and $dw(t)$ are statistically independent with each other and $\mathrm{E}[dw(t)] = 0$. Therefore, we have

$$\mathrm{E}[dN(t)] = \mathrm{E}[a]\{b(t) - \frac{1}{2}\sigma^2\}\exp\{-(b - \frac{1}{2}\sigma^2)t\}dt. \tag{3.7}$$

The instantaneous MTBF of the three models are derived as follows:

$$MTBF_I^e(t) = \frac{1}{a(b - \frac{1}{2}\sigma^2)\exp\{-(b - \frac{\sigma^2}{2})t\}}, \tag{3.8}$$

$$MTBF_I^d(t) = \frac{1}{a(1 + bt)(\frac{b^2t}{1+bt} - \frac{1}{2}\sigma^2)\exp\{-(b - \frac{\sigma^2}{2})t\}}, \tag{3.9}$$

$$MTBF_I^i(t) = \frac{1}{\frac{a(1+c)}{1+c\cdot\exp(-bt)}(\frac{b}{1+c\cdot\exp(-bt)} - \frac{1}{2}\sigma^2)\exp\{-(b - \frac{\sigma^2}{2})t\}}, \tag{3.10}$$

where $MTBF_I^e(t)$, $MTBF_I^d(t)$, and $MTBF_I^i(t)$ are derived from the exponential, delayed S-shaped, and inflection S-shaped SRGM's, respectively.

3.2 Cumulative MTBF

Next, the cumulative mean time between software failures (cumulative MTBF, represented as $MTBF_C$) is approximately given by:

$$MTBF_C = \frac{t}{\mathrm{E}[N(t)]}. \tag{3.11}$$

The cumulative MTBF of the three models are respectively obtained as follows:

$$MTBF_C^e(t) = \frac{t}{a\left\{1 - \exp\left(-bt + \frac{\sigma^2}{2}t\right)\right\}}, \tag{3.12}$$

$$MTBF_C^d(t) = \frac{t}{a\left\{1 - (1 + bt)\exp\left(-bt + \frac{\sigma^2}{2}t\right)\right\}}, \tag{3.13}$$

$$MTBF_C^i(t) = \frac{t}{a\left\{1 - \frac{1+c}{1+c\cdot\exp(-bt)}\exp\left(-bt + \frac{\sigma^2}{2}t\right)\right\}}. \tag{3.14}$$

4. PARAMETER ESTIMATION

In this section the method of parameter estimation for unknown parameters in equations (2.13), (2.22), and (2.23) is presented. We use the method of maximum-likelihood to estimate for the unknown parameters. Suppose that K data pairs are observed during the system testing-phase, where the cumulative number of software failures observed in the time-interval $(0, t_j]$ is n_j $(j = 1, 2, \cdots, K)$.

Let us denote the joint probability distribution function of the process $N(t)$ as

$$\begin{aligned} & P(t_1, n_1; t_2, n_2; \cdots; t_K, n_K) \\ \equiv\ & \Pr[N(t_1) \le n_1, N(t_2) \le n_2, \cdots N(t_K) \le n_K | N(0) = 0], \end{aligned} \tag{4.1}$$

and denote its density as

$$p(t_1, n_1; t_2, n_2; \cdots; t_K, n_K) = \frac{\partial^K P(t_1, n_1; t_2, n_2; \cdots; t_K, n_K)}{\partial n_1 \partial n_2 \cdots \partial n_K}. \tag{4.2}$$

Since $N(t)$ takes on continuous values, we construct the logarithmic likelihood function L for the observed data (t_j, n_j) $(j = 1, 2, \cdots, K)$ as follows:

$$L = \log\ p(t_1, n_1; t_2, n_2; \cdots; t_K, n_K). \tag{4.3}$$

In the case of the inflection S-shaped SRGM, the maximum-likelihood estimates can be obtained as the solution of the following simultaneous equations:

$$\frac{\partial L}{\partial a} = \frac{\partial L}{\partial b} = \frac{\partial L}{\partial c} = \frac{\partial L}{\partial \sigma} = 0, \tag{4.4}$$

which can be solved numerically. The unknown parameters appeared in the other two models can be estimated in the same way.

5. NUMERICAL EXAMPLES

5.1 Estimation Results of the Model Parameters

In this section, we analyze actual software fault data to show numerical examples of software reliability measurement for application of our model. A set of fault-detection count data used in this section is obtained from the actual software development project, where we use 19 data pairs given by $(t_j, n_j)(j = 1, 2, \cdots, 19)$ (see Ohba (1984)). So, we have obtained the following maximum-likelihood estimates:

for the case of using the exponential SRGM

$$\hat{a} = 390.305,\ \hat{b} = 0.0966,\ \hat{\sigma} = 0.0561,$$

for the case of using the delayed S-shaped SRGM

$$\hat{a} = 349.449,\ \hat{b} = 0.237,\ \hat{\sigma} = 0.0726,$$

and for the case of using the inflection S-shaped SRGM

$$\hat{a} = 335.927, \quad \hat{b} = 0.360, \quad \hat{c} = 25.867, \quad \hat{\sigma} = 0.0784.$$

5.2 Estimated Expected Number of Detected Faults and MTBF

By using the maximum-likelihood estimates in 5.1, we obtain the estimated expected number of detected faults, the instantaneous MTBF, and the cumulative MTBF.

The estimated expected number of detected faults in equation (3.5), $\hat{\mathrm{E}}[N_i(t)]$, the estimated $\widehat{MTBF}_I^i(t)$ in equation (3.10), and $\widehat{MTBF}_C^i(t)$ in equation (3.14) are shown in Figure 1, Figure 2 and Figure 3, respectively. They show that software reliability grows as the testing procedures go on.

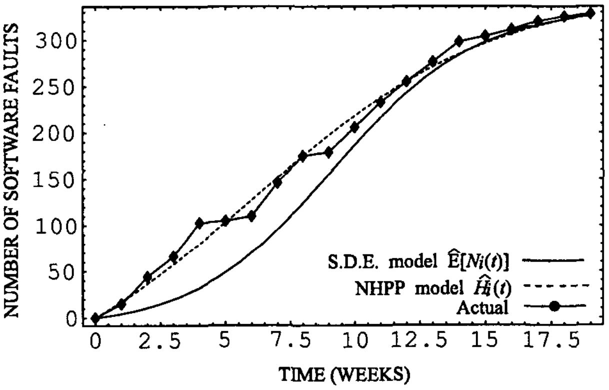

Figure 1. The estimated expected number of detected faults, $\hat{\mathrm{E}}[N_i(t)]$ and $\hat{H}_i(t)$.

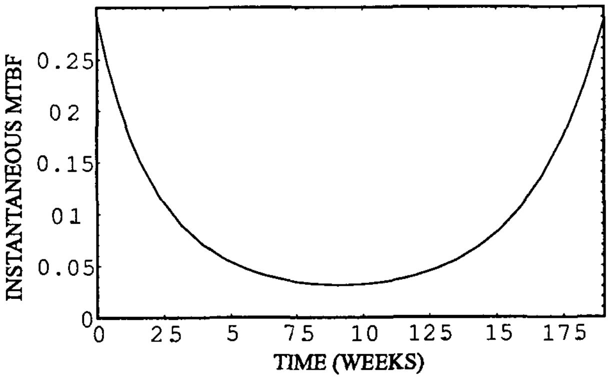

Figure 2. The estimated instantaneous MTBF, $\widehat{MTBF}_I^i(t)$.

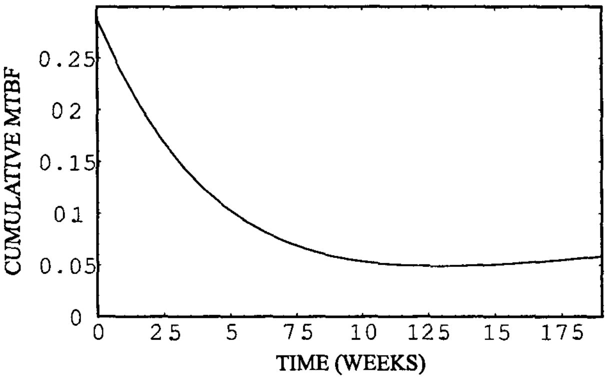

Figure 3. The estimated cumulative MTBF, $\widehat{MTBF}^i_C(t)$.

5.3 Performance Comparison of the Models

We compare our S.D.E. (stochastic differential equation) models discussed in this paper with three NHPP models which have the mean value functions $H_e(t)$, $H_d(t)$, and $H_i(t)$, in terms of goodness-of-fit. As a comparison criterion of goodness-of-fit, we adopt the values of Akaike's Information Criterion (AIC) (see Akaike (1974)). Table 1 shows the comparison results among the estimated our S.D.E. models and NHPP ones. We conclude that our S.D.E. model using the inflection S-shaped SRGM fits best among the models.

Table 1. Comparison results of goodness-of-fit based on the actual data.

Compared model	AIC
S.D.E. model $\hat{E}[N_e(t)]$	143.61
$\hat{E}[N_d(t)]$	138.04
$\hat{E}[N_i(t)]$	133.72
NHPP model $\hat{H}_e(t)$	220.76
$\hat{H}_d(t)$	222.38
$\hat{H}_i(t)$	205.01

6. CONCLUDING REMARKS

In this paper, we have treated the event of fault-detection during the software testing phase as the stochastic process with continuous state space. We have introduced stochastic differential equations in order to construct new software reliability growth models. And we can derive the probability distribution of a software fault-detection process by using the mean value function of NHPP models. We can also estimate the unknown parameters in our models by using actual fault detection data sets. As software reliability assessment measures, we have derived the instantaneous MTBF and the cumulative MTBF. Comparing our S.D.E. models with the NHPP models, we show that our S.D.E. model using the inflection S-shaped SRGM fits best. However, we need to clarify the performance of models by applying another data sets in the future study.

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BIOGRAPHY OF AUTHOR

Shigeru Yamada was born in Hiroshima Prefecture, Japan, on July 6, 1952. He received the B.S.E., M.S., and Ph.D. degrees from Hiroshima University in 1975, 1977, and 1985, respectively. From 1977 to 1980, he worked at the Quality Assurance Department of Nippondenso Co., Aichi Prefecture, Japan. From 1983 to 1988, he was an assistant professor of Okayama University of Science, Okayama, Japan. From 1988 to 1993, he was an associate professor at the Faculty of Engineering of Hiroshima University, Japan. Since 1993, he has been working as a professor at the

Faculty of Engineering of Tottori University, Tottori-shi, Japan. He has published numerous technical papers in the area of software reliability models, project management, reliability engineering, and quality control. He has authored several books entitled *Software Reliability Assessment Technology* (HBJ Japan, Tokyo, 1989), *Software Reliability: Theory and Practical Application* (Soft Research Center, Tokyo, 1990), *Introduction to Software Management Model* (Kyoritsu Shuppan, 1993), *Software Reliability Models: Fundamentals and Applications* (JUSE, Tokyo, 1994), and *Statistical Quality Control for TQM* (Corona Publishing, Tokyo, 1998). Dr. Yamada received the Best Author Award from the Information Processing Society of Japan in 1992, the TELECOM System Technology Award from the Telecommunications Advancement Foundation in 1993, and the Paper Award from the Reliability Engineering Association of Japan in 1999. He is a regular member of the Information Processing Society of Japan, the Institute of Electronics, Information and Communication Engineers, the Operations Research Society of Japan, the Japan SIAM, the Reliability Engineering Association of Japan, Japan Industrial Management Association, the Japanese Society for Quality Control, and the IEEE.

Akio Nishigaki was born in Tottori Prefecture, Japan, on December 4, 1978. He received the B.S. degree from Tottori University, Tottori, Japan in 2001. Currently he is a graduate student in the Department of Social Systems Engineering of Tottori University, Japan. His research interests include software reliability modeling and assessment. He is a student member of the Institute of Electronics, Information and Communication Engineers.

Mitsuhiro Kimura was born in Hiroshima Prefecture, Japan, on October 1, 1964. He received the B.S.E., M.S., and Ph.D. degrees from Hiroshima University in 1989, 1991, and 1994, respectively. From 1994 to 1995, he was an assistant professor at Tottori University, Tottori, Japan. From 1995 to 2001, he was an associate professor at Tottori University. Since 2001, he has been working as an associate professor at the Faculty of Engineering of Hosei University, Japan. He has authored a book entitled *Statistical Quality Control for TQM* (Corona Publishing, Tokyo, 1998). He is now mainly interested in software reliability assessment models and quality control. Dr. Kimura received the Paper Award from the Reliability Engineering Association of Japan in 1999. He is a regular member of the Institute of Electronics, Information and Communication Engineers, the Operations Research Society of Japan, the Information Processing Society of Japan, and the IEEE computer society.