Bull. Korean Math. Soc. **40** (2003), No. 2, pp. 235–241

ZEROS OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS OF SMALL LOWER GROWTH

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Abstract. It is proved that the product of any two linearly independent meromorphic solutions of second order linear differential equations with coefficients of small lower growth must have infinite exponent of convergence of its zero-sequences, under some suitable conditions.

1. Introduction

Let $f(z)(\neq 0)$ be an entire solution of equation (1)

$$f'' + A(z)f = 0, \tag{1}$$

where $A(z)$ is a transcendental entire function of finite order. The upper and lower growth orders of $f(z)$ denote by respectively,

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r},$$

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r},$$

where $T(r,f)$ is the characteristic function of $f(z)$ in the sense of Nevanlinna theory, see [4]. When $\sigma(A) < \frac{1}{2}$, Bank and Laine in [2] applied Wiman-Valiron theory to proving that the product of any two linearly independent solutions of equation (1) must have infinite exponent of convergence of its zero-sequences. When $\sigma(A) = \frac{1}{2}$ and $\sigma(A)$ is not an integer, Shen in [10] and Rossi in [9] independently concluded the same result. When $\sigma(A) > \frac{1}{2}$ and $\sigma(A)$ is not an integer, Bank and Laine

Received October 24, 2001.
2000 Mathematics Subject Classification: 30D35, 34M05, 34M10.
Key words and phrases: meromorphic solution, linear differential equation, exponent of convergence of zero-sequences, growth order.

conjectured that the same result is valid. But, it still remains an open question.

In this note, we consider that the coefficients of second order linear differential equations are meromorphic of lower order less than $\frac{1}{2}$ and of finite order, and prove the same result, under some suitable conditions. The exponent of convergence of zeros of $f(z)$ is defined by

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r, \frac{1}{f})}{\log r},$$

where $N(r, \frac{1}{f})$ is the counting function of zeros of $f(z)$ in $\{|z| < r\}$. $N(r, f)$ is the counting function of poles of $f(z)$ in $\{|z| < r\}, m(r, f)$ is the proximity function of $f(z)$, the Nevanlinna deficience of $f(z)$ at infinity is expressed as

$$\delta(\infty, f) = \liminf_{r\to\infty} \frac{m(r,f)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,f)}{T(r,f)},$$

see [4]. Our main result is stated as follows.

THEOREM. *Let $B(z)$ be a transcendental meromorphic function with $\sigma(B) < \infty$, and $\lambda(\frac{1}{B}) < \mu(B) < \alpha < \frac{1}{2}$, for some constant α. If f_1, f_2 are two linearly independent meromorphic solutions of equation* (2) *with $N(r, f_1 f_2) \le O(N(r, B))$, $r \to \infty$,*

$$f'' + B(z)f = 0, \tag{2}$$

then $\max\{\lambda(f_1), \lambda(f_2)\} = \infty$.

By our Theorem, we immediately deduce the following result.

COROLLARY. *If $\mu(A) < \frac{1}{2}$, then the product of any two linearly independent solutions of equation* (1) *must have infinite exponent of convergence of its zero-sequences.*

Let $y(z)$ be entire in the complex plane $\mathbf{C}$. For an unbounded subset U of $\mathbf{C}$, if

$$\log|y(z)| \ne O\{\log|z|\},$$

$z \in U$, as $|z| \to \infty$, then we say that $y(z)$ grows transcendentally on U. The minimum and maximum moduluses of $y(z)$ are defined respectively by

$$L(r, y) = \inf_{|z|=r} \{|y(z)|\},$$

and

$$M(r, y) = \sup_{|z|=r} \{|y(z)|\}.$$

R-set[3,5] denotes a countable union of disc sequences $\{|z - z_n| < r_n\}$, where $z_n \to \infty$ as $n \to \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$.

Let $y(z) \not\equiv 0$ be meromorphic of finite order in the complex plane. Then, it is known that there exists a constant $q > 0$ such that

$$|\frac{y'(z)}{y(z)}| = O(|z|^q) \tag{3}$$

for all z outside a R-set[6].

The idea of the proof of our Theorem is from [6].

2. Proof of theorem

In order to prove our theorem, we need the following results.

From the similar proof of the Lemma A in [1], and noting that every pole of any meromorphic solution of equation (2) must be one of $B(z)$'s, we can deduce the result as follows, with completed details omitted.

LEMMA 1. *Let $C(z)$ be a transcendental meromorphic function of finite order, and $w_1(z), w_2(z)$ be two linearly independent meromorphic solutions of the following equation*

$$w'' + C(z)w = 0.$$

If $\lambda(w_1 w_2) < \infty$, then $w_1(z), w_2(z)$ can be expressed as

$$w_1(z) = G_1(z)e^{g(z)},\ w_2(z) = G_2(z)e^{-g(z)},$$

where $G_1(z), G_2(z)$ are non-zero meromorphic functions of finite order, and $g(z)$ is a transcendental entire function.

LEMMA 2. [8] *Let m and n be positive integers and let $f_1, \cdots, f_m$ and $g_1, \cdots, g_n$ be meromorphic functions on a domain D. Then the following Wronskian holds on D*

$$\begin{aligned} &W(f_1, \cdots, f_m, g_1, \cdots, g_n)W(f_1, \cdots, f_m)^{n-1} \\ &= W(W(f_1, \cdots, f_m, g_1), \cdots, W(f_1, \cdots, f_m, g_n)). \end{aligned}$$

The following result may be refered to [6].

LEMMA 3. *Let*

$$f_j(z) = G_j(z)e^{g_j(z)}, j = 1, 2, \cdots, n$$

be meromorphic functions, and linearly independent, where $G_j(j = 1, \cdots, n)$ are meromorphic of finite order, $g_j(j = 1, \cdots, n)$ are entire.

Suppose that if for an unbounded subset U of $\mathbf{C}$ such that $g_n' - g_j'$ or $g_j'(j = 1, 2, \cdots, n-1)$ grows transcendentally on U, then

$$\frac{W(f_1, \cdots, f_{n-2}, f_{n-1}, f_n)}{f_n W(f_1, \cdots, f_{n-2}, f_{n-1})}$$

grows transcendentally on $U \backslash R$-set.

Proof. We apply the mathematical induction method to the proof of Lemma 3.

Let $n = 2$, then

$$\frac{W(f_1, f_2)}{f_2 W(f_1)} = \frac{G_2'}{G_2} - \frac{G_1'}{G_1} + g_2' - g_1'. \tag{4}$$

By (3), it follows

$$|\frac{G_j'}{G_j}| = O(|z|^M), j = 1, 2,$$

on $U \backslash D_1$, where D_1 is a R-set in the complex plane, M is a positive number. From (4)

$$|g_2' - g_1'| \leq |\frac{W(f_1, f_2)}{f_2 W(f_1)}| + O(|z|^M),$$

it follows

$$|\frac{W(f_1, f_2)}{f_2 W(f_1)}| \tag{5}$$

grows transcendentally on $U \backslash D_2$, where D_2 is a R-set.

Let $n \geq 3$. Assume that

$$\frac{W(f_1, \cdots, f_{n-2}, f_{n-1})}{f_{n-1} W(f_1, \cdots, f_{n-2})} \tag{6}$$

grows transcendentally on $U \backslash D_3$, where D_3 is a R-set. Then by Lemma 2

$$\begin{aligned} &W(f_1, \cdots, f_{n-2}, f_{n-1}, f_n) W(f_1, \cdots, f_{n-2}) \\ &= W(W(f_1, \cdots, f_{n-2}, f_{n-1}), W(f_1, \cdots, f_{n-2}, f_n)), \end{aligned}$$

it follows

$$\begin{aligned} &\frac{W(f_1, \cdots, f_{n-2}, f_{n-1}, f_n)}{f_n W(f_1, \cdots, f_{n-2}, f_{n-1})} \\ &= \frac{W(W(f_1, \cdots, f_{n-2}, f_{n-1}), W(f_1, \cdots, f_{n-2}, f_n))}{W(f_1, \cdots, f_{n-2}, f_{n-1}) W(f_1, \cdots, f_{n-2}, f_n)} \cdot \frac{W(f_1, \cdots, f_{n-2}, f_n)}{f_n W(f_1, \cdots, f_{n-2})}. \end{aligned}$$

According to the hypotheses of Lemma 3, we may set

$$W(f_1, \cdots, f_{n-2}, f_{n-1}) = H_1 e^{h_1},$$

$$W(f_1, \cdots, f_{n-2}, f_n) = H_2 e^{h_2},$$

where H_1 and H_2 are entire of finite order, and $h_2' - h_1' = g_n' - g_{n-1}'$ or g_{n-1}' grows transcendentally on $U \backslash D_4$, where D_4 is a R-set. Similarly, from the proof of (4), it follows that

$$\frac{W(W(f_1, \cdots, f_{n-2}, f_{n-1}), W(f_1, \cdots, f_{n-2}, f_n))}{W(f_1, \cdots, f_{n-2}, f_{n-1}) W(f_1, \cdots, f_{n-2}, f_n)}$$

grows transcendentally on $U \backslash D_5$, where D_5 is a R-set. By (5) and (6),

$$\frac{W(f_1, \cdots, f_{n-2}, f_n)}{f_n W(f_1, \cdots, f_{n-2})}$$

grows transcendentally on $U \backslash D_6$, where D_6 is a R-set.

Thereafter, we deduce that

$$\frac{W(f_1, \cdots, f_{n-2}, f_{n-1}, f_n)}{f_n W(f_1, \cdots, f_{n-2}, f_{n-1})}$$

grows transcendentally on $U \backslash D_7$, where D_7 is a R-set. Lemma 3 follows. □

Proof of theorem. Assume that f_1, f_2 are two linearly independent solutions of equation (2), with

$$\max_{1 \leq j \leq 2} \{\lambda(f_j)\} < \infty.$$

By Lemma 1, set

$$f_1(z) = G_1(z) e^{g(z)}, \; f_2(z) = G_2(z) e^{-g(z)},$$

where $G_j(z) \not\equiv 0 (j = 1, 2)$ are meromorphic of finite order, $g(z)$ is entire, and set

$$G_1(z) G_2(z) = \frac{H_1(z)}{H_2(z)},$$

where $H_1(z), H_2(z)$ are the canonical products of zeros and poles of $f_1(z) f_2(z)$ respectively. Substituting $f_1(z)$ into equation (2), we have

$$-B(z) = \frac{G_1''(z)}{G_1(z)} + (2\frac{G_1'(z)}{G_1(z)} + \frac{g''(z)}{g'(z)} + g'(z)) g'(z),$$

and then for some $d > 0$,

$$|B(z)| \leq (O\{|z|^d\} + |g'(z)|)|g'(z)|,$$

outside a R-set, say D_1. And then

(7) $\log^+ \log^+ |B(z)| \leq \log^+ \log^+ |g'(z)| + O(\log \log |z|), \; z \notin D_1.$

Since $\lambda(\frac{1}{B}) < \mu(B)$ implies that $\delta(\infty, B) = 1$, from (9) in [7], we have

$$(8) \qquad \log L(r, B) > \frac{\alpha\pi}{\sin\alpha\pi}(\cos\alpha\pi + \delta(\infty, B) - 1)T(r, B), \ r \in E,$$

where

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_{E\cap(0,r)} \frac{dt}{t} \geq 1 - \frac{\mu(B)}{\alpha} > 0.$$

Hence, it follows, from (7) and (8), that $g'(z)$ grows transcendentally on $\{z \in \mathbf{C} : |z| \in E\}\backslash D_1$. By Lemma 3,

$$(9) \qquad \frac{W(f_1, f_2)}{f_1 \cdot f_2}$$

grows transcendentally on $\{z \in \mathbf{C} : |z| \in E\}\backslash D_2$, where D_2 is a R-set. Noting that the Wronskian of $f_1(z)$ and $f_2(z)$,

$$W(f_1, f_2)$$

is identically non-zero constant c, from (9), as $z \notin D_2$ and $|z| \to \infty$ in E, we have

$$(10) \qquad |\frac{c}{G_1(z)G_2(z)}| = |\frac{H_2(z)}{H_1(z)}| \to \infty.$$

If $H_1(z) \equiv const.(\neq 0)$, for some constant $\beta \geq 1$, we get

$$|\frac{H_2(z)}{H_1(z)}| \leq \beta|H_2(z)|.$$

If $H_1(z) \not\equiv const.$, then take a series of points, say, $\{z_n\} \subset \{z \in \mathbf{C} : |z| \in E\}\backslash\{D_1 \cup D_2\}$, $z_n \to \infty$, as $n \to \infty$, such that

$$|H_1(z_n)| \geq 1, \ n = 1, 2, \cdots,$$

and then

$$(11) \qquad |\frac{H_2(z_n)}{H_1(z_n)}| \leq \beta|H_2(z_n)|, \ n = 1, 2, \cdots.$$

From (9) and (10), we have

$$(12) \qquad \frac{G_2'(z)}{G_2(z)} - \frac{G_1'(z)}{G_1(z)} - 2g'(z) = \frac{H_2(z)}{H_1(z)}.$$

Combining (11) and (12), it follows

$$(13) \qquad 2|g'(z_n)| \leq \beta|H_2(z_n)| + O(|z_n|^q), \ n = 1, 2, \cdots$$

for some positive number q. Therefore, from (7), (8) and (13), we obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log^+ T(|z_n|,B)}{\log|z_n|} &\le \lim_{n\to\infty} \frac{\log^+\log^+ L(|z_n|,B)}{\log|z_n|} \\
&\le \lim_{n\to\infty} \frac{\log^+\log^+ |g'(z_n)|}{\log|z_n|} \\
&\le \lim_{n\to\infty} \frac{\log^+\log^+ M(|z_n|,H_2)}{\log|z_n|} \\
&\le \limsup_{r\to\infty} \frac{\log^+ N(r,\frac{1}{H_2})}{\log r} \\
&\le \limsup_{r\to\infty} \frac{\log^+ N(r,B)}{\log r}.
\end{aligned}$$

This contradicts to $\lambda(\frac{1}{B}) < \mu(B)$. The proof is complete. □

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