ZEROS OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS OF SMALL LOWER GROWTH

WANG SHENG

Abstract. It is proved that the product of any two linearly independent meromorphic solutions of second order linear differential equations with coefficients of small lower growth must have infinite exponent of convergence of its zero-sequences, under some suitable conditions.

1. Introduction

Let \( f(z) \neq 0 \) be an entire solution of equation (1)

\[
f'' + A(z)f = 0,
\]

where \( A(z) \) is a transcendental entire function of finite order. The upper and lower growth orders of \( f(z) \) denote by respectively,

\[
\sigma(f) = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r},
\]

\[
\mu(f) = \liminf_{r \to \infty} \frac{\log^+ T(r, f)}{\log r},
\]

where \( T(r, f) \) is the characteristic function of \( f(z) \) in the sense of Nevanlinna theory, see [4]. When \( \sigma(A) < \frac{1}{2} \), Bank and Laine in [2] applied Wiman-Valiron theory to proving that the product of any two linearly independent solutions of equation (1) must have infinite exponent of convergence of its zero-sequences. When \( \sigma(A) = \frac{1}{2} \) and \( \sigma(A) \) is not an integer, Shen in [10] and Rossi in [9] independently concluded the same result. When \( \sigma(A) > \frac{1}{2} \) and \( \sigma(A) \) is not an integer, Bank and Laine
conjectured that the same result is valid. But, it still remains an open question.

In this note, we consider that the coefficients of second order linear differential equations are meromorphic of lower order less than $\frac{1}{2}$ and of finite order, and prove the same result, under some suitable conditions. The exponent of convergence of zeros of $f(z)$ is defined by

$$\lambda(f) = \lim_{r \to \infty} \sup \frac{\log^+ N(r, \frac{1}{r})}{\log r},$$

where $N(r, \frac{1}{r})$ is the counting function of zeros of $f(z)$ in $\{|z| < r\}$. $N(r, f)$ is the counting function of poles of $f(z)$ in $\{|z| < r\}, m(r, f)$ is the proximity function of $f(z)$, the Nevanlinna deficiency of $f(z)$ at infinity is expressed as

$$\delta(\infty, f) = \lim_{r \to \infty} \inf \frac{m(r, f)}{T(r, f)} = 1 - \lim_{r \to \infty} \frac{N(r, f)}{T(r, f)},$$

see [4]. Our main result is stated as follows.

**Theorem.** Let $B(z)$ be a transcendental meromorphic function with $\sigma(B) < \infty$, and $\lambda(\frac{1}{B}) < \mu(B) < \alpha < \frac{1}{2}$, for some constant $\alpha$. If $f_1, f_2$ are two linearly independent meromorphic solutions of equation (2) with $N(r, f_1 f_2) \leq O(N(r, B)), r \to \infty$,

$$f'' + B(z) f = 0,$$

then $\max\{\lambda(f_1), \lambda(f_2)\} = \infty$.

By our Theorem, we immediately deduce the following result.

**Corollary.** If $\mu(A) < \frac{1}{2}$, then the product of any two linearly independent solutions of equation (1) must have infinite exponent of convergence of its zero-sequences.

Let $y(z)$ be entire in the complex plane $\mathbb{C}$. For an unbounded subset $U$ of $\mathbb{C}$, if

$$\log |y(z)| \neq O\{\log |z|\},$$

$z \in U$, as $|z| \to \infty$, then we say that $y(z)$ grows transcendentially on $U$. The minimum and maximum moduluses of $y(z)$ are defined respectively by

$$L(r, y) = \inf_{|z|=r} \{|y(z)|\},$$

and

$$M(r, y) = \sup_{|z|=r} \{|y(z)|\}.$$
\( R\)-set\(^{[3,5]} \) denotes a countable union of disc sequences \( \{|z - z_n| < r_n\} \), where \( z_n \to \infty \) as \( n \to \infty \) and \( \sum_{n=1}^{\infty} r_n < \infty \).

Let \( y(z) \neq 0 \) be meromorphic of finite order in the complex plane. Then, it is known that there exists a constant \( q > 0 \) such that
\[
\left| \frac{y'(z)}{y(z)} \right| = O(|z|^q)
\]
for all \( z \) outside a \( R\)-set\(^{[6]} \).

The idea of the proof of our Theorem is from [6].

2. Proof of theorem

In order to prove our theorem, we need the following results.

From the similar proof of the Lemma A in [1], and noting that every pole of any meromorphic solution of equation (2) must be one of \( B(z) \)'s, we can deduce the result as follows, with completed details omitted.

**Lemma 1.** Let \( C(z) \) be a transcendental meromorphic function of finite order, and \( w_1(z), w_2(z) \) be two linearly independent meromorphic solutions of the following equation
\[
w'' + C(z)w = 0.
\]
If \( \lambda(w_1w_2) < \infty \), then \( w_1(z), w_2(z) \) can be expressed as
\[
w_1(z) = G_1(z)e^{g(z)}, \quad w_2(z) = G_2(z)e^{-g(z)},
\]
where \( G_1(z), G_2(z) \) are non-zero meromorphic functions of finite order, and \( g(z) \) is a transcendental entire function.

**Lemma 2.** [8] Let \( m \) and \( n \) be positive integers and let \( f_1, \cdots, f_m \) and \( g_1, \cdots, g_n \) be meromorphic functions on a domain \( D \). Then the following Wronskian holds on \( D \)
\[
W(f_1, \cdots, f_m, g_1, \cdots, g_n)W(f_1, \cdots, f_m)^{n-1} = W(W(f_1, \cdots, f_m, g_1), \cdots, W(f_1, \cdots, f_m, g_n)).
\]
The following result may be referred to [6].

**Lemma 3.** Let
\[
f_j(z) = G_j(z)e^{g_j(z)}, \quad j = 1, 2, \cdots, n
\]
be meromorphic functions, and linearly independent, where \( G_j(j = 1, \cdots, n) \) are meromorphic of finite order, \( g_j(j = 1, \cdots, n) \) are entire.
Suppose that if for an unbounded subset $U$ of $\mathbb{C}$ such that $g'_n - g'_j$ or $g'_j (j = 1, 2, \cdots, n - 1)$ grows transcendently on $U$, then
\[
\frac{W(f_1, \cdots, f_{n-2}, f_{n-1}, f_n)}{f_n W(f_1, \cdots, f_{n-2}, f_{n-1})}
\]
grows transcendently on $U \setminus R$-set.

**Proof.** We apply the mathematical induction method to the proof of Lemma 3.

Let $n = 2$, then
\[
\frac{W(f_1, f_2)}{f_2 W(f_1)} = \frac{G'_2}{G_2} - \frac{G'_1}{G_1} + g'_2 - g'_1.
\]
By (3), it follows
\[
\left|\frac{G'_2}{G_2}\right| = O(|z|^M), j = 1, 2,
\]
on $U \setminus D_1$, where $D_1$ is a $R$-set in the complex plane, $M$ is a positive number. From (4)
\[
|g'_2 - g'_1| \leq \frac{|W(f_1, f_2)|}{f_2 W(f_1)} + O(|z|^M),
\]
it follows
\[
\frac{|W(f_1, f_2)|}{f_2 W(f_1)}
\]
grows transcendally on $U \setminus D_2$, where $D_2$ is a $R$-set.

Let $n \geq 3$. Assume that
\[
\frac{W(f_1, \cdots, f_{n-2}, f_{n-1})}{f_{n-1} W(f_1, \cdots, f_{n-2})}
\]
grows transcendally on $U \setminus D_3$, where $D_3$ is a $R$-set. Then by Lemma 2
\[
W(f_1, \cdots, f_{n-2}, f_{n-1}) = W(W(f_1, \cdots, f_{n-2}, f_{n-1}), f_1, \cdots, f_{n-2}, f_n),
\]
it follows
\[
\frac{W(f_1, \cdots, f_{n-2}, f_{n-1}, f_n)}{f_n W(f_1, \cdots, f_{n-2}, f_{n-1})}
\]
\[
= \frac{W(W(f_1, \cdots, f_{n-2}, f_{n-1}), W(f_1, \cdots, f_{n-2}, f_n))}{W(f_1, \cdots, f_{n-2}, f_{n-1}) W(f_1, \cdots, f_{n-2}, f_n)} \cdot \frac{W(f_1, \cdots, f_{n-2}, f_n)}{f_n W(f_1, \cdots, f_{n-2})}.
\]

According to the hypotheses of Lemma 3, we may set
\[
W(f_1, \cdots, f_{n-2}, f_{n-1}) = H_1 e^{h_1},
\]
where $H_1$ and $H_2$ are entire of finite order, and $h'_2 = h_1' = g'_n - g'_{n-1}$ or $g'_{n-1}$ grows transcendentally on $U \setminus D_4$, where $D_4$ is a $R$-set. Similarly, from the proof of (4), it follows that

$$W(W(f_1, \cdots, f_{n-2}, f_n), W(f_1, \cdots, f_{n-2}, f_n))$$

$$W(f_1, \cdots, f_{n-2}, f_{n-1})W(f_1, \cdots, f_{n-2}, f_n)$$

grows transcendentally on $U \setminus D_5$, where $D_5$ is a $R$-set. By (5) and (6),

$$W(f_1, \cdots, f_{n-2}, f_n)$$

$$f_n W(f_1, \cdots, f_{n-2})$$

grows transcendentally on $U \setminus D_6$, where $D_6$ is a $R$-set.

Thereafter, we deduce that

$$W(f_1, \cdots, f_{n-2}, f_{n-1}, f_n)$$

$$f_n W(f_1, \cdots, f_{n-2}, f_{n-1})$$

grows transcendentally on $U \setminus D_7$, where $D_7$ is a $R$-set. Lemma 3 follows.

Proof of theorem. Assume that $f_1, f_2$ are two linearly independent solutions of equation (2), with

$$\max_{1 \leq j \leq 2} \{\lambda_j(f_j)\} < \infty.$$ 

By Lemma 1, set

$$f_1(z) = G_1(z)e^{g(z)}, \quad f_2(z) = G_2(z)e^{-g(z)},$$

where $G_j(z) \not= 0 (j = 1, 2)$ are meromorphic of finite order, $g(z)$ is entire, and set

$$G_1(z)G_2(z) = \frac{H_1(z)}{H_2(z)},$$

where $H_1(z), H_2(z)$ are the canonical products of zeros and poles of $f_1(z)f_2(z)$ respectively. Substituting $f_1(z)$ into equation (2), we have

$$-B(z) = \frac{G''_1(z)}{G_1(z)} + (2 \frac{G'_1(z)}{G_1(z)} + \frac{g''(z)}{g'(z)} + g'(z))g'(z),$$

and then for some $d > 0$,

$$|B(z)| \leq (O(|z|^d) + |g'(z)|)|g'(z)|,$$

outside a $R$-set, say $D_1$. And then

$$\log^+ \log^+ |B(z)| \leq \log^+ \log^+ |g'(z)| + O(\log \log |z|), \quad z \not\in D_1.$$
Since $\lambda(\frac{1}{2}) < \mu(B)$ implies that $\delta(\infty, B) = 1$, from (9) in [7], we have
\begin{align}
\log L_r(B) > \frac{\alpha \pi}{\sin \alpha \pi} (\cos \alpha \pi + \delta(\infty, B) - 1)T(r, B), \ r \in E,
\end{align}
where
\begin{align}
\limsup_{r \to \infty} \frac{1}{\log r} \int_{E \cap (0, r)} \frac{dt}{t} \geq 1 - \frac{\mu(B)}{\alpha} > 0.
\end{align}
Hence, it follows, from (7) and (8), that $g'(z)$ grows transcendentally on \{z $\in$ C : $|z| \in E\}$\ $\setminus$ $D_1$. By Lemma 3,
\begin{align}
\frac{W(f_1, f_2)}{f_1 \cdot f_2}
\end{align}
grows transcendentally on \{z $\in$ C : $|z| \in E\}$\ $\setminus$ $D_2$, where $D_2$ is a R-set. Noting that the Wronskian of $f_1(z)$ and $f_2(z)$,
\begin{align}
W(f_1, f_2)
\end{align}
is identically non-zero constant $c$, from (9), as $z \notin D_2$ and $|z| \to \infty$ in $E$, we have
\begin{align}
|\frac{c}{G_1(z)G_2(z)}| = \left| \frac{H_2(z)}{H_1(z)} \right| \to \infty.
\end{align}
If $H_1(z) \equiv const.(\neq 0)$, for some constant $\beta \geq 1$, we get
\begin{align}
\frac{H_2(z)}{H_1(z)} \leq \beta |H_2(z)|.
\end{align}
If $H_1(z) \neq const.$, then take a series of points, say, \{z$_n$\} $\subset$ \{z $\in$ C : $|z| \in E\}$\ $\setminus$ \{D$_1$ $\cup$ D$_2$\}, $z_n \to \infty$, as $n \to \infty$, such that
\begin{align}
|H_1(z$_n$)| \geq 1, \ n = 1, 2, \ldots,
\end{align}
and then
\begin{align}
\frac{|H_2(z_n)|}{H_1(z_n)} \leq \beta |H_2(z_n)|, \ n = 1, 2, \ldots.
\end{align}
From (9) and (10), we have
\begin{align}
\frac{G_2'(z)}{G_2(z)} - \frac{G_1'(z)}{G_1(z)} - 2g'(z) = \frac{H_2(z)}{H_1(z)}.
\end{align}
Combining (11) and (12), it follows
\begin{align}
2|g'(z_n)| \leq \beta |H_2(z_n)| + O(|z_n|^q), \ n = 1, 2, \ldots
\end{align}
for some positive number \( q \). Therefore, from (7), (8) and (13), we obtain
\[
\lim_{n \to \infty} \frac{\log^+ T(|z_n|, B)}{\log |z_n|} \leq \lim_{n \to \infty} \frac{\log^+ \log^+ L(|z_n|, B)}{\log |z_n|} \\
\leq \lim_{n \to \infty} \frac{\log^+ \log^+ \log^+ |g'(z_n)|}{\log |z_n|} \\
\leq \lim_{n \to \infty} \frac{\log^+ \log^+ M(|z_n|, H_2)}{\log |z_n|} \\
\leq \lim \sup_{r \to \infty} \frac{\log^+ N(r, h_2)}{\log^+ N(r, B)}.
\]
This contradicts to \( \lambda \left( \frac{1}{B} \right) < \mu(B) \). The proof is complete. \( \Box \)

References


DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084
P. R. CHINA
E-mail: swang@math.tsinghua.edu.cn
wangsh@pub.zaoqing.gd.cn