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HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper we deal with the quadratic functional equation

$$n^2\binom{n-2}{k-2}f\left(\frac{x_1+\cdots+x_n}{n}\right)+\binom{n-2}{k-1}\sum_{i=1}^{n}f(x_i)$$
$$= k^2\sum_{1\le i_1<\cdots<i_k\le n}f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right),$$

deriving from an inequality of T. Popoviciu for convex functions. We solve this functional equation by proving that its solutions are the polynomials of degree at most two. Likewise, we investigate its stability in the spirit of Hyers, Ulam, and Rassias.

1. Introduction

Let n and k be positive integers such that $2 \le k \le n-1$ (throughout the paper n and k will always have this meaning). T. Popoviciu [12, Théorème 3] proved that if I is a nonempty interval and $f : I \to \mathbf{R}$ is a convex function, then for all $x_1, \ldots, x_n \in I$ it holds that

$$(1.1)\qquad n\binom{n-2}{k-2}f\left(\frac{x_1+\cdots+x_n}{n}\right)+\binom{n-2}{k-1}\sum_{i=1}^{n}f(x_i)$$
$$\ge k\sum_{1\le i_1<\cdots<i_k\le n}f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right).$$

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Starting from (1.1), in [17] we considered the corresponding functional equation

$$n\binom{n-2}{k-2} f\left(\frac{x_1+\cdots+x_n}{n}\right)+\binom{n-2}{k-1}\sum_{i=1}^{n} f(x_i) = k\sum_{1\le i_1<\cdots<i_k\le n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right). \tag{1.2}$$

We proved that a function f between two real linear spaces X and Y satisfies (1.2) for all $x_1,\dots,x_n \in X$ if and only if there exists an additive mapping $A : X \to Y$ such that $f(x) = A(x) + f(0)$ for all $x \in X$. Likewise, we investigated the Hyers–Ulam–Rassias stability of the equation (1.2).

In the special case $n = 3$, $k = 2$ the equation (1.2) reduces to

$$3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z) = 2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]. \tag{1.3}$$

The stability of the functional equation (1.3) has been investigated in [15, 16]. In connection with (1.3), Y. W. Lee [11] had the clever idea to consider the equation

$$9f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z) = 4\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]. \tag{1.4}$$

He proved that a function f between two real linear spaces X and Y satisfies (1.4) for all $x, y, z \in X$ if and only if there exist an additive mapping $A : X \to Y$ as well as a quadratic mapping $Q : X \to Y$, i.e. a mapping satisfying

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y) \qquad \text{for all } x,y \in X,$$

such that $f(x) = Q(x) + A(x) + f(0)$ for all $x \in X$. Besides, he investigated the stability of the functional equation (1.4).

It is the main purpose of the present paper to generalize the above mentioned results of Y. W. Lee for the n variables version of (1.4):

$$(1.5)\qquad n^2\binom{n-2}{k-2} f\left(\frac{x_1+\cdots+x_n}{n}\right)+\binom{n-2}{k-1}\sum_{i=1}^{n} f(x_i) = k^2 \sum_{1\le i_1<\cdots<i_k\le n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right).$$

In Section 2 of this paper we solve the functional equation (1.5), by proving that its solutions are of the same form as those of (1.4). In Section 3, using ideas from the papers of Th. M. Rassias [13] and Z. Gajda [5], we establish the Hyers–Ulam–Rassias stability of Eq. (1.5).

2. Solutions of equation (1.5)

THEOREM 2.1. *Let X and Y be real linear spaces. A function $f : X \to Y$ satisfies (1.5) for all $x_1, \ldots, x_n \in X$ if and only if there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = Q(x) + A(x) + f(0)$ for all $x \in X$.*

Proof. Necessity. Let $A : X \to Y$ and $Q : X \to Y$ be the functions defined by $A(x) := \frac{1}{2}[f(x) - f(-x)]$ and $Q(x) := \frac{1}{2}[f(x) + f(-x)] - f(0)$, respectively. Obviously we have $f(x) = Q(x) + A(x) + f(0)$ for all $x \in X$. We claim that A is additive and Q is quadratic.

Indeed, since f satisfies (1.5) for all $x_1, \ldots, x_n \in X$, it is immediately seen that

$$(2.1)\qquad n^2\binom{n-2}{k-2} A\left(\frac{x_1+\cdots+x_n}{n}\right)+\binom{n-2}{k-1}\sum_{i=1}^{n} A(x_i) = k^2 \sum_{1\le i_1<\cdots<i_k\le n} A\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)$$

and

$$(2.2)\qquad n^2\binom{n-2}{k-2} Q\left(\frac{x_1+\cdots+x_n}{n}\right)+\binom{n-2}{k-1}\sum_{i=1}^{n} Q(x_i) = k^2 \sum_{1\le i_1<\cdots<i_k\le n} Q\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)$$

for all $x_1, \ldots, x_n \in X$. In addition, remark that $A(0) = Q(0) = 0$,

$$A(-x) = -A(x), \qquad \text{and} \qquad Q(-x) = Q(x) \qquad \text{for all } x \in X.$$

Putting $x_1 = x$ and $x_2 = \cdots = x_n = 0$ in (2.1) yields

(2.3)
$$n^2\binom{n-2}{k-2}A\left(\frac{x}{n}\right) = k^2\binom{n-1}{k-1}A\left(\frac{x}{k}\right) - \binom{n-2}{k-1}A(x) \quad \text{for all } x \in X.$$

Letting $x_1 = x$, $x_2 = y$, and $x_3 = \cdots = x_n = 0$ in (2.1), we get

(2.4)
$$\begin{aligned} & n^2\binom{n-2}{k-2}A\left(\frac{x+y}{n}\right) + \binom{n-2}{k-1}[A(x)+A(y)] \\ = \; & k^2\binom{n-2}{k-2}A\left(\frac{x+y}{k}\right) + k^2\binom{n-2}{k-1}\left[A\left(\frac{x}{k}\right) + A\left(\frac{y}{k}\right)\right] \end{aligned}$$

for all $x, y \in X$. From (2.3) and (2.4) it follows that

(2.5)
$$A(x+y) - A(x) - A(y) = k^2\left[A\left(\frac{x+y}{k}\right) - A\left(\frac{x}{k}\right) - A\left(\frac{y}{k}\right)\right]$$

for all $x, y \in X$. Putting now $x_1 = x$, $x_2 = y$, $x_3 = -x-y$, and $x_4 = \cdots = x_n = 0$ in (2.1) and taking into account that A is odd, we find that

(2.6)
$$\begin{aligned} & \binom{n-2}{k-1}[A(x)+A(y)-A(x+y)] \\ = \; & k^2\left[\binom{n-3}{k-2} - \binom{n-3}{k-1}\right]\left[A\left(\frac{x+y}{k}\right) - A\left(\frac{x}{k}\right) - A\left(\frac{y}{k}\right)\right] \end{aligned}$$

for all $x, y \in X$. From (2.5) and (2.6) we deduce that

$$\left[\binom{n-2}{k-1} + \binom{n-3}{k-2} - \binom{n-3}{k-1}\right][A(x)+A(y)-A(x+y)] = 0$$

for all $x, y \in X$. Since

$$\binom{n-2}{k-1} + \binom{n-3}{k-2} - \binom{n-3}{k-1} = 2\binom{n-3}{k-2} > 0,$$

we conclude that

$$A(x+y) = A(x) + A(y) \qquad \text{for all } x, y \in X.$$

Consequently, A is additive as claimed.

In order to prove that Q is quadratic, we set $x_1 = x$, $x_2 = -x$, and $x_3 = \cdots = x_n = 0$ in (2.2). Taking into account that Q is even and $Q(0) = 0$, we obtain

(2.7)
$$Q(x) = k^2 Q\left(\frac{x}{k}\right) \qquad \text{for all } x \in X.$$

Letting $x_1 = x$ and $x_2 = \cdots = x_n = 0$ in (2.2) and taking account of (2.7) we find that

$$Q(x) = n^2 Q\left(\frac{x}{n}\right) \qquad \text{for all } x \in X. \tag{2.8}$$

From (2.2), (2.7), and (2.8) it follows that

$$\begin{aligned}\binom{n-2}{k-2} Q(x_1 + \cdots + x_n) + \binom{n-2}{k-1} \sum_{i=1}^{n} Q(x_i) \\ = \sum_{1 \le i_1 < \cdots < i_k \le n} Q(x_{i_1} + \cdots + x_{i_k}) \quad \text{for all } x_1, \ldots, x_n \in X.\end{aligned} \tag{2.9}$$

Letting $x_1 = x$, $x_2 = y$, $x_3 = -x$, and $x_4 = \cdots = x_n = 0$ in (2.9), we get

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \qquad \text{for all } x, y \in X.$$

Consequently, Q is quadratic as claimed.

SUFFICIENCY. Assume now that there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = Q(x) + A(x) + f(0)$ for all $x \in X$. By virtue of a well known result, there exists a biadditive symmetric mapping $B : X \times X \to Y$ such that $Q(x) = B(x,x)$ for all $x \in X$. Therefore, we have $f(x) = B(x,x) + A(x) + f(0)$ for all $x \in X$.

Let $x_1, \ldots, x_n$ be any points in X. We have

$$\begin{aligned}
& n^2 \binom{n-2}{k-2} f\left(\frac{x_1 + \cdots + x_n}{n}\right) + \binom{n-2}{k-1} \sum_{i=1}^{n} f(x_i) \\
= \; & \binom{n-2}{k-2} B(x_1 + \cdots + x_n, x_1 + \cdots + x_n) + \binom{n-2}{k-1} \sum_{i=1}^{n} B(x_i, x_i) \\
& + \left[n \binom{n-2}{k-2} + \binom{n-2}{k-1} \right] \sum_{i=1}^{n} A(x_i) \\
& + \left[n^2 \binom{n-2}{k-2} + \binom{n-2}{k-1} \right] f(0) \\
= \; & \binom{n-1}{k-1} \sum_{i=1}^{n} B(x_i, x_i) + 2 \binom{n-2}{k-2} \sum_{1 \le i < j \le n} B(x_i, x_j) \\
& + k \binom{n-1}{k-1} \sum_{i=1}^{n} A(x_i) + k^2 \binom{n}{k} f(0)
\end{aligned}$$

and

$$\begin{aligned} & k^2 \sum_{1\le i_1<\cdots<i_k\le n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \\ = & \sum_{1\le i_1<\cdots<i_k\le n} B(x_{i_1}+\cdots+x_{i_k}, x_{i_1}+\cdots+x_{i_k}) \\ & + k \sum_{1\le i_1<\cdots<i_k\le n} A(x_{i_1}+\cdots+x_{i_k}) + k^2\binom{n}{k} f(0) \\ = & \binom{n-1}{k-1}\sum_{i=1}^{n} B(x_i,x_i) + 2\binom{n-2}{k-2}\sum_{1\le i<j\le n} B(x_i,x_j) \\ & + k\binom{n-1}{k-1}\sum_{i=1}^{n} A(x_i) + k^2\binom{n}{k} f(0). \end{aligned}$$

Therefore, f satisfies (1.5) for all $x_1, \ldots, x_n \in X$. □

3. Hyers–Ulam–Rassias stability of equation (1.4)

Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. Given a function $f : X \to Y$, we set

$$\begin{aligned} Df(x_1,\ldots,x_n) := {} & n^2\binom{n-2}{k-2} f\left(\frac{x_1+\cdots+x_n}{n}\right) + \binom{n-2}{k-1}\sum_{i=1}^{n} f(x_i) \\ & - k^2 \sum_{1\le i_1<\cdots<i_k\le n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \end{aligned}$$

for all $x_1, \ldots, x_n \in X$.

THEOREM 3.1. *Let $\delta, \theta \in [0, \infty[$ and let $p \in]0, 1[$. If a function $f : X \to Y$ satisfies*

$$\|Df(x_1,\ldots,x_n)\| \le \delta + \theta \sum_{i=1}^{n} \|x_i\|^p \tag{3.1}$$

for all $x_1, \ldots, x_n \in X$, then there exist a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$\left\| \frac{f(x)+f(-x)}{2} - Q(x) - f(0) \right\| \le \varepsilon_1(x) \tag{3.2}$$

and

$$\left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| \le \varepsilon_2(x) \tag{3.3}$$

for all $x \in X$, *where*

$$\varepsilon_1(x) = \frac{\delta}{2\binom{n-2}{k-1}(k^2-1)} + \frac{k^p\theta}{\binom{n-2}{k-1}(k^2-k^p)}\|x\|^p,$$
$$\varepsilon_2(x) = \frac{3\delta}{2k^2\binom{n-3}{k-2}} + \frac{(2^p+2)k^{p-2}\theta}{\binom{n-3}{k-2}(2-2^p)}\|x\|^p.$$

In addition, we have

$$\|f(x) - Q(x) - A(x) - f(0)\| \le \varepsilon_1(x) + \varepsilon_2(x) \qquad \textit{for all } x \in X.$$

Proof. Let $f_1 : X \to Y$ be the function defined by $f_1(x) := \frac{1}{2}[f(x) + f(-x)] - f(0)$. Then f_1 is even, $f_1(0) = 0$, and since

$$Df_1(x_1, \dots, x_n) = \frac{1}{2}\left[Df(x_1, \dots, x_n) + Df(-x_1, \dots, -x_n)\right],$$

we have

$$\|Df_1(x_1, \dots, x_n)\| \le \delta + \theta \sum_{i=1}^{n} \|x_i\|^p \tag{3.4}$$

for all $x_1, \dots, x_n \in X$.

Letting $x_1 = kx$, $x_2 = -kx$, and $x_3 = \cdots = x_n = 0$ in (3.4) we get

$$\left\| 2\binom{n-2}{k-1} f_1(kx) - 2k^2\binom{n-2}{k-1} f_1(x) \right\| \le \delta + 2\theta k^p \|x\|^p$$

hence

$$\|f_1(x) - k^{-2} f_1(kx)\| \le \frac{\delta}{2\binom{n-2}{k-1}}\, k^{-2} + \frac{\theta}{\binom{n-2}{k-1}}\, k^{p-2}\|x\|^p \tag{3.5}$$

for all $x \in X$.

Replacing x in (3.5) by $k^{j-1}x$ and then dividing both sides of (3.5) by k^{2j-2} yields

(3.6)
$$\left\| k^{-2(j-1)} f_1(k^{j-1}x) - k^{-2j} f_1(k^j x) \right\| \le \frac{\delta}{2\binom{n-2}{k-1}}\, k^{-2j} + \frac{\theta}{\binom{n-2}{k-1}}\, k^{(p-2)j}\|x\|^p$$

for each positive integer j and all $x \in X$. By virtue of (3.6) we have

$$\begin{aligned}(3.7)\qquad &\left\|k^{-2m}f_1(k^m x)-k^{-2\ell}f_1(k^\ell x)\right\| \\ &\leq \sum_{j=\ell+1}^{m}\left\|k^{-2(j-1)}f_1(k^{j-1}x)-k^{-2j}f_1(k^j x)\right\| \\ &\leq \frac{\delta}{2\binom{n-2}{k-1}}\sum_{j=\ell+1}^{\infty}k^{-2j}+\frac{\theta}{\binom{n-2}{k-1}}\|x\|^p\sum_{j=\ell+1}^{\infty}k^{(p-2)j}\end{aligned}$$

for all nonnegative integers ℓ and m with $\ell < m$ and all $x \in X$. In the special case $\ell = 0$ we find that

$$(3.8)\quad \left\|f_1(x)-k^{-2m}f_1(k^m x)\right\| \leq \frac{\delta}{2\binom{n-2}{k-1}(k^2-1)}+\frac{k^p\theta}{\binom{n-2}{k-1}(k^2-k^p)}\|x\|^p$$

for each positive integer m and all $x \in X$.

Since

$$\lim_{\ell\to\infty}\sum_{j=\ell+1}^{\infty}k^{-2j}=\lim_{\ell\to\infty}\sum_{j=\ell+1}^{\infty}k^{(p-2)j}=0,$$

from (3.7) we conclude that $\left(k^{-2j}f_1(k^j x)\right)_{j\in\mathbf{N}}$ is a Cauchy sequence for all $x \in X$. Therefore, we can define the mapping $Q : X \to Y$ by $Q(x) := \lim_{j\to\infty} k^{-2j} f_1(k^j x)$.

Let $x_1, \ldots, x_n$ be any points in X. By virtue of (3.4) we have

$$\begin{aligned}\|DQ(x_1,\ldots,x_n)\| &= \lim_{j\to\infty}k^{-2j}\|Df_1(k^j x_1,\ldots,k^j x_n)\| \\ &\leq \lim_{j\to\infty}\left(\delta k^{-2j}+\theta k^{(p-2)j}\sum_{i=1}^{n}\|x_i\|^p\right) \\ &= 0.\end{aligned}$$

Hence $DQ(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in X$. Since Q is even and $Q(0) = 0$, Theorem 2.1 ensures that Q is quadratic. Moreover, by passing to the limit in (3.8) when $m \to \infty$, it follows that (3.2) holds true for all $x \in X$.

Now, let $\tilde{Q} : X \to Y$ be another quadratic mapping satisfying

$$\left\|\frac{f(x)+f(-x)}{2}-\tilde{Q}(x)-f(0)\right\| \leq \frac{\delta}{2\binom{n-2}{k-1}(k^2-1)}+\frac{k^p\theta}{\binom{n-2}{k-1}(k^2-k^p)}\|x\|^p$$

for all $x \in X$. Then we have

$$\begin{aligned}
\|Q(x)-\tilde{Q}(x)\| &= k^{-2m}\|Q(k^m x)-\tilde{Q}(k^m x)\| \\
&\le k^{-2m}\left(\left\|Q(k^m x)+f(0)-\frac{f(k^m x)+f(-k^m x)}{2}\right\|\right. \\
&\quad \left.+\left\|\frac{f(k^m x)+f(-k^m x)}{2}-\tilde{Q}(k^m x)-f(0)\right\|\right) \\
&\le k^{-2m}\left(\frac{\delta}{\binom{n-2}{k-1}(k^2-1)}+\frac{2k^{(m+1)p}\theta}{\binom{n-2}{k-1}(k^2-k^p)}\|x\|^p\right)
\end{aligned}$$

for each positive integer m and all $x \in X$. Since

$$\lim_{m\to\infty} k^{-2m}\left(\frac{\delta}{\binom{n-2}{k-1}(k^2-1)}+\frac{2k^{(m+1)p}\theta}{\binom{n-2}{k-1}(k^2-k^p)}\|x\|^p\right)=0,$$

we can conclude that $Q(x)=\tilde{Q}(x)$ for all $x \in X$. This proves the uniqueness of Q.

Next, let $f_2 : X \to Y$ be the function defined by $f_2(x) := \frac{1}{2}[f(x) -f(-x)]$. Then f_2 is odd, $f_2(0)=0$, and since

$$Df_2(x_1,\ldots,x_n)=\frac{1}{2}\left[Df(x_1,\ldots,x_n)-Df(-x_1,\ldots,-x_n)\right],$$

we have

$$\|Df_2(x_1,\ldots,x_n)\| \le \delta+\theta\sum_{i=1}^{n}\|x_i\|^p \tag{3.9}$$

for all $x_1,\ldots,x_n \in X$.

Letting $x_1=2x$ and $x_2=\cdots=x_n=0$ in (3.9) yields

$$\begin{aligned}
&\left\|n^2\binom{n-2}{k-2}f_2\left(\frac{2x}{n}\right)+\binom{n-2}{k-1}f_2(2x)-k^2\binom{n-1}{k-1}f_2\left(\frac{2x}{k}\right)\right\| \\
&\le \delta+2^p\theta\|x\|^p \qquad \text{for all } x\in X.
\end{aligned} \tag{3.10}$$

Putting $x_1=x_2=x$ and $x_3=\cdots=x_n=0$ in (3.9) we get

$$\begin{aligned}
&\left\|n^2\binom{n-2}{k-2}f_2\left(\frac{2x}{n}\right)+2\binom{n-2}{k-1}f_2(x)-k^2\binom{n-2}{k-2}f_2\left(\frac{2x}{k}\right)\right. \\
&\quad\left.-2k^2\binom{n-2}{k-1}f_2\left(\frac{x}{k}\right)\right\| \le \delta+2\theta\|x\|^p \qquad \text{for all } x\in X.
\end{aligned} \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$(3.12)\qquad \binom{n-2}{k-1}\left\|f_2(2x)-2f_2(x)-k^2\left[f_2\left(\frac{2x}{k}\right)-2f_2\left(\frac{x}{k}\right)\right]\right\| \leq 2\delta+(2^p+2)\theta\|x\|^p \qquad \text{for all } x\in X.$$

Letting now $x_1 = x_2 = x$, $x_3 = -2x$, and $x_4 = \cdots = x_n = 0$ in (3.9) and taking into account that f_2 is odd, we find that

$$(3.13)\qquad \left\|\binom{n-2}{k-1}[2f_2(x)-f_2(2x)]-k^2\left[\binom{n-3}{k-2}-\binom{n-3}{k-1}\right]\left[f_2\left(\frac{2x}{k}\right)-2f_2\left(\frac{x}{k}\right)\right]\right\| \leq \delta+(2^p+2)\theta\|x\|^p \qquad \text{for all } x\in X.$$

From (3.12) and (3.13) we deduce that

$$\left\|f_2\left(\frac{2x}{k}\right)-2f_2\left(\frac{x}{k}\right)\right\| \leq \frac{3\delta}{2k^2\binom{n-3}{k-2}}+\frac{(2^p+2)\theta}{k^2\binom{n-3}{k-2}}\|x\|^p$$

for all $x \in X$. Replacing x by kx, we get

$$(3.14)\qquad \|f_2(2x)-2f_2(x)\| \leq \frac{3\delta}{2k^2\binom{n-3}{k-2}}+\frac{(2^p+2)k^{p-2}\theta}{\binom{n-3}{k-2}}\|x\|^p$$

for all $x \in X$.

Replacing x in (3.14) by $2^{j-1}x$ and then dividing both sides of (3.14) by 2^j yields

$$(3.15)\qquad \left\|2^{-j+1}f_2(2^{j-1}x)-2^{-j}f_2(2^jx)\right\| \leq \frac{3\delta}{2k^2\binom{n-3}{k-2}}2^{-j}+\frac{(2^p+2)k^{p-2}\theta}{2^p\binom{n-3}{k-2}}2^{(p-1)j}\|x\|^p$$

for each positive integer j and all $x \in X$. By virtue of (3.15) we have

$$(3.16)\qquad \left\|2^{-m}f_2(2^mx)-2^{-\ell}f_2(2^\ell x)\right\| \leq \sum_{j=\ell+1}^{m}\left\|2^{-j+1}f_2(2^{j-1}x)-2^{-j}f_2(2^jx)\right\| \leq \frac{3\delta}{2k^2\binom{n-3}{k-2}}\sum_{j=\ell+1}^{\infty}2^{-j}+\frac{(2^p+2)k^{p-2}\theta}{2^p\binom{n-3}{k-2}}\|x\|^p\sum_{j=\ell+1}^{\infty}2^{(p-1)j}$$

for all nonnegative integers ℓ and m with $\ell < m$ and all $x \in X$. In the special case $\ell = 0$ we find that

$$(3.17) \qquad \left\|f_2(x) - 2^{-m}f_2(2^m x)\right\| \leq \frac{3\delta}{2k^2\binom{n-3}{k-2}} + \frac{(2^p+2)k^{p-2}\theta}{\binom{n-3}{k-2}(2-2^p)}\|x\|^p$$

for each positive integer m and all $x \in X$.

Since

$$\lim_{\ell\to\infty}\sum_{j=\ell+1}^{\infty} 2^{-j} = \lim_{\ell\to\infty}\sum_{j=\ell+1}^{\infty} 2^{(p-1)j} = 0,$$

from (3.16) we conclude that $\left(2^{-j}f_2(2^j x)\right)_{j\in\mathbf{N}}$ is a Cauchy sequence for all $x \in X$. Therefore, we can define the mapping $A : X \to Y$ by $A(x) := \lim_{j\to\infty} 2^{-j}f_2(2^j x)$.

Let $x_1, \ldots, x_n$ be any points in X. By virtue of (3.9) we have

$$\begin{aligned}
\|DA(x_1,\ldots,x_n)\| &= \lim_{j\to\infty} 2^{-j}\|Df_2(2^j x_1,\ldots,2^j x_n)\| \\
&\leq \lim_{j\to\infty}\left(\delta 2^{-j} + \theta 2^{(p-1)j}\sum_{i=1}^{n}\|x_i\|^p\right) \\
&= 0.
\end{aligned}$$

Hence $DA(x_1,\ldots,x_n) = 0$ for all $x_1,\ldots,x_n \in X$. Since A is odd and $A(0) = 0$, Theorem 2.1 ensures that A is additive. Moreover, by passing to the limit in (3.17) when $m \to \infty$, we see that (3.3) holds true for all $x \in X$.

Now let $\tilde{A} : X \to Y$ be another additive mapping satisfying

$$\left\|\frac{f(x)-f(-x)}{2} - \tilde{A}(x)\right\| \leq \frac{3\delta}{2k^2\binom{n-3}{k-2}} + \frac{(2^p+2)k^{p-2}\theta}{\binom{n-3}{k-2}(2-2^p)}\|x\|^p$$

for all $x \in X$. Then we have

$$\begin{aligned}
\|A(x) - \tilde{A}(x)\| &= 2^{-m}\|A(2^m x) - \tilde{A}(2^m x)\| \\
&\leq 2^{-m}\left(\left\|A(2^m x) - \frac{f(2^m x) - f(-2^m x)}{2}\right\|\right. \\
&\quad \left. + \left\|\frac{f(2^m x) - f(-2^m x)}{2} - \tilde{A}(2^m x)\right\|\right) \\
&\leq 2^{-m}\left(\frac{3\delta}{k^2\binom{n-3}{k-2}} + 2^{mp}\frac{2(2^p+2)k^{p-2}\theta}{\binom{n-3}{k-2}(2-2^p)}\|x\|^p\right)
\end{aligned}$$

for each positive integer m and all $x \in X$. Since

$$\lim_{m\to\infty} 2^{-m}\left(\frac{3\delta}{k^2\binom{n-3}{k-2}} + 2^{mp}\frac{2(2^p+2)k^{p-2}\theta}{\binom{n-3}{k-2}(2-2^p)}\|x\|^p\right) = 0,$$

we conclude that $A(x) = \tilde{A}(x)$ for all $x \in X$. This proves the uniqueness of A, completing also the proof. □

The proof of the next theorem (containing the case $p = 0$), being similar to that of Theorem 3.1, is omitted.

THEOREM 3.2. *Let $\delta \in [0, \infty[$ and let $f : X \to Y$ be a function satisfying*

$$\|Df(x_1, \dots, x_n)\| \le \delta \qquad \text{for all } x_1, \dots, x_n \in X.$$

Then there exist a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$\left\|\frac{f(x)+f(-x)}{2} - Q(x) - f(0)\right\| \le \frac{\delta}{2\binom{n-2}{k-1}(k^2-1)}$$

and

$$\left\|\frac{f(x)-f(-x)}{2} - A(x)\right\| \le \frac{3\delta}{2k^2\binom{n-3}{k-2}}$$

for all $x \in X$. In addition, we have

$$\|f(x)-Q(x)-A(x)-f(0)\| \le \frac{\delta}{2\binom{n-2}{k-1}(k^2-1)} + \frac{3\delta}{2k^2\binom{n-3}{k-2}} \qquad \text{for all } x \in X.$$

THEOREM 3.3. *Let $\theta \in [0, \infty[$ and let $p \in \,]1, \infty[\, \setminus \{2\}$. If a function $f : X \to Y$ satisfies*

$$\|Df(x_1, \dots, x_n)\| \le \theta \sum_{i=1}^{n} \|x_i\|^p \qquad \text{for all } x_1, \dots, x_n \in X,$$

then there exist a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$\left\|\frac{f(x)+f(-x)}{2} - Q(x) - f(0)\right\| \le \varepsilon_1(x) \tag{3.18}$$

and

$$\left\|\frac{f(x)-f(-x)}{2} - A(x)\right\| \le \varepsilon_2(x) \tag{3.19}$$

for all $x \in X$, *where*

$$\varepsilon_1(x) = \frac{k^p\theta}{\binom{n-2}{k-1}|k^2 - k^p|}\|x\|^p \qquad \textit{and} \qquad \varepsilon_2(x) = \frac{(2^p+2)k^{p-2}\theta}{\binom{n-3}{k-2}(2^p-2)}\|x\|^p.$$

In addition, we have

$$\|f(x) - Q(x) - A(x) - f(0)\| \le \varepsilon_1(x) + \varepsilon_2(x) \qquad \textit{for all } x \in X.$$

Proof. Let $f_1 : X \to Y$ and $f_2 : X \to Y$ be the functions defined as in the proof of Theorem 3.1. By proceeding as in the proof of Theorem 3.1, we find that f_1 and f_2 satisfy

$$\left\|f_1(x) - k^{-2}f_1(kx)\right\| \le \frac{\theta}{\binom{n-2}{k-1}}\, k^{p-2}\|x\|^p \tag{3.20}$$

and

$$\|f_2(2x) - 2f_2(x)\| \le \frac{(2^p+2)k^{p-2}\theta}{\binom{n-3}{k-2}}\|x\|^p \tag{3.21}$$

for all $x \in X$.

If $1 < p < 2$, then as in the proof of Theorem 3.1 it can be proved that $\left(k^{-2j}f_1(k^jx)\right)_{j\in\mathbf{N}}$ is a Cauchy sequence for all $x \in X$ and that the mapping $Q : X \to Y$, defined by $Q(x) := \lim_{j\to\infty} k^{-2j}f_1(k^jx)$, is the unique quadratic mapping satisfying (3.18) for all $x \in X$.

Assume now that $p > 2$. Replacing x in (3.20) by $k^{-j}x$ and then multiplying both sides of (3.20) by k^{2j}, we get

$$\left\|k^{2(j-1)}f_1(k^{-j+1}x) - k^{2j}f_1(k^{-j}x)\right\| \le \frac{k^{p-2}\theta}{\binom{n-2}{k-1}}\, k^{(2-p)j}\|x\|^p \tag{3.22}$$

for each positive integer j and all $x \in X$.

Starting from (3.22), as in Theorem 3.1, it is easy to prove that $\left(k^{2j}f_1(k^{-j}x)\right)_{j\in\mathbf{N}}$ is a Cauchy sequence for all $x \in X$ and that the mapping $Q : X \to Y$, defined by $Q(x) := \lim_{j\to\infty} k^{2j}f_1(k^{-j}x)$, is the unique quadratic mapping satisfying (3.18) for all $x \in X$.

On the other hand, replacing x in (3.21) by $2^{-j}x$ and then multiplying both sides of (3.21) by 2^{j-1}, we find that

$$\left\|2^{j-1}f_2(2^{-j+1}x) - 2^jf_2(2^{-j}x)\right\| \le \frac{(2^p+2)k^{p-2}\theta}{2\binom{n-3}{k-2}}\, 2^{(1-p)j}\|x\|^p \tag{3.23}$$

for each positive integer j and all $x \in X$.

Starting from (3.23) and proceeding as in Theorem 3.1, it is easily seen that $\left(2^jf_2(2^{-j}x)\right)_{j\in\mathbf{N}}$ is a Cauchy sequence for all $x \in X$ and that the mapping $A : X \to Y$, defined by $A(x) := \lim_{j\to\infty} 2^j f_2(2^{-j}x)$, is the unique additive mapping satisfying (3.19) for all $x \in X$. We omit the details. □

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