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MOD p HOMOLOGY OF THE DOUBLE LOOP SPACE OF THE HOMOGENEOUS SPACE $SO(2n)/U(n)$

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ABSTRACT. We compute the mod p homology of the double loop space of $SO(2n)/U(n)$ by the Serre spectral sequence of Hopf algebras. We also obtain the torsion information of the integral homology.

1. Introduction

Let $SO(n)$ be the group of $n \times n$ orthogonal matrices of determinant 1 and $U(n)$ the group of $n \times n$ unitary matrices. Let $\Omega^k M$ be the k–fold loop space of M, that is, the space of all the base point preserving continuous maps from S^k to M. In this paper we study the mod p homology of the double loop space of the homogeneous space $SO(2n)/U(n)$.

There is a natural energy functional on $\Omega^2 SO(2n)/U(n)$ given by $E(\phi) = \frac{1}{2}\int_{S^2} |d\phi(x)|^2 dx$ where $\phi : S^2 \to SO(2n)/U(n)$ is a map between Riemannian manifolds. The absolute minima of this energy functional are precisely the space $Hol^*(S^2, SO(2n)/U(n))$ of all the base preserving holomorphic maps from the Riemannian sphere $S^2 = C \bigcup \infty$ to the homogeneous space $SO(2n)/U(n)$ [4]. Then forgetting the complex structure, we have the natural inclusion $Hol^*_k(S^2, SO(2n)/U(n)) \to \Omega^2_k SO(2n)/U(n)$ where $k \in \pi_0(\Omega^2 Sp(n)/U(n)) = Z$. By exploiting the inclusion map, we can obtain the homological information of the space $Hol^*_k(S^2, SO(2n)/U(n))$ from the homology of $\Omega^2_k SO(2n)/U(n)$. We compute the homology of the double loop space of $SO(2n)/U(n)$ from this point view. Main tool of the computation is the Serre spectral sequence of Hopf algebras.

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2. Homology of $SO(2n)/U(n)$

Throughout this paper p always stands for odd primes and the subscript of an element means the degree of the element, that is, $\deg(x_i) = i$. There are homology Dyer–Lashof operations, $Q_{i(p-1)}$ on the $(n+1)$-loop space $\Omega^{n+1}X$

$$Q_{i(p-1)} : H_q(\Omega^{n+1}X; \mathbb{F}_p) \to H_{pq+i(p-1)}(\Omega^{n+1}X; \mathbb{F}_p)$$

for $0 \le i \le n$ when $p = 2$, and for $0 \le i \le n$ and $i + q$ even when $p > 2$, and they are natural with respect to $(n+1)$-loop maps [3].

The following is well -known. We refer Theorem 6.11 of chapter 3 in [5] for more detail explanation.

THEOREM 2.1. *As an algebra, we have*

$$H^*(SO(2n)/U(n); Z) = Z[e_2, e_4, \cdots, e_{2n-2}]/(e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i} e_{4k-2i}).$$

From above theorem, we get the following.

COROLLARY 2.2. *As an algebra we have*

$$H^*(SO(2n)/U(n); \mathbb{F}_2) = \Delta\ (c_2, \dots, c_{2n-2}),$$

where $\Delta\ (c_2, \dots, c_{2n-2})$ *denotes a graded algebra over* Z_2 *with a basis*

$$\{c_{2i_1} \cdots c_{2i_r} : 1 \le i_1 < i_2 \cdots < i_r \le n-1\}.$$

For odd primes p, as an algebra we have

$$H^*(SO(2n)/U(n); \mathbb{F}_p) = \mathbb{F}_p[c_1, \cdots, c_{n-1}]/(\sum_{i+j=2k, k\ge 1} (-1)^i c_i c_j).$$

Since $\pi_2(SO(2n)/U(n)) = Z$, $\pi_0(\Omega^2 SO(2n)/U(n)) = Z$. So components of the space $\Omega^2 SO(2n)/U(n)$ are labelled by the integer $k \in Z$; we denote the k component by $\Omega^2_k SO(2n)/U(n)$. Since each component is homotopy equivalent to each other, it is enough to compute the homology of any component to get the homology of $\Omega^2 SO(2n)/U(n)$.

3. Mod 2 homology of $\Omega^2 SO(2n)/U(n)$

We have the following identification: $SO(4)/U(2) \cong S^2$. Hence we have

$$\Omega^2 SO(4)/U(2) \cong \Omega^2 S^2 \cong \Omega^2 S^3 \times Z.$$

Therefore we have

$$H_*(\Omega_0^2 SO(4)/U(2); \mathbb{F}_2) = \mathbb{F}_2[Q_1^a z_1 : a \geq 0],$$
$$H_*(\Omega_0^2 SO(4)/U(2); \mathbb{F}_p) = E(Q_{p-1}^a z_1 : a \geq 0) \otimes \mathbb{F}_p[\beta Q_{p-1}^a z_1 : a > 0].$$

First we consider the mod 2 case.

THEOREM 3.1. *$H_*(\Omega_0^2 SO(2n+2)/U(n+1); \mathbb{F}_2), n \geq 2$, is*

$$\mathbb{F}_2[z_{4k} : 0 < 4k \leq n-2]$$
$$\otimes \mathbb{F}_2[Q_1^a w_{2n+8k+5} : a \geq 0, 0 \leq 4k \leq n-4]$$
$$\otimes \mathbb{F}_2[Q_1^a z_{4k} : a \geq 0, n-2 < 4k \leq 2n-2].$$

Proof. It is well-known that

$$H_*(\Omega_0 U; \mathbb{F}_2) = \mathbb{F}_2[e_{2i} : i \geq 1],$$
$$H_*(\Omega_0^2 SO/U; \mathbb{F}_2) = \mathbb{F}_2[y_{4i} : i \geq 1],$$
$$H_*(\Omega^2 SO; \mathbb{F}_2) = E(u_{4i+1} : i \geq 0].$$

In the Serre spectral sequence converging to $H_*(\Omega_0^2 SO/U; \mathbb{F}_2)$ associated to the fibration

$$\Omega^2 SO \longrightarrow \Omega_0^2 SO/U \longrightarrow \Omega_0 U,$$

we have the following differentials for $i \geq 0$, $k \geq 1$:

$$\begin{aligned} &d_{4i+2}(e_{4i+2}) = u_{4i+1}, \\ &d_{(4i+2)2^k}(e_{(4i+2)2^k}) = e_{4i+2} \cdot e_{(4i+2)2} \cdot \dots \cdot e_{(4i+2)2^{k-1}} \cdot u_{4i+1}. \end{aligned} \tag{1}$$

Moreover, e_{2i}^2 survives permanently for each $i \geq 1$.

Now we consider the Serre spectral sequence converging to $H_*(\Omega_0^2 SO(2n+2)/U(n+1); \mathbb{F}_2)$ with

$$E_2 = H_*(\Omega_0 U(n+1); \mathbb{F}_2) \otimes H_*(\Omega^2 SO(2n+2); \mathbb{F}_2).$$

Since the structures of $H_*(\Omega^2 SO(2n);\mathbb{F}_2)$ depend on the congruence of n mod 4 [2], we should consider four cases. Here we will compute only one case because the other cases are followed by the same method.

Consider the following fibration:

$$\begin{array}{ccccc} \Omega^2 SO(8n+2) & \longrightarrow & \Omega_0^2 SO(8n+2)/U(4n+1) & \longrightarrow & \Omega_0 U(4n+1) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^2 SO & \longrightarrow & \Omega_0^2 SO/U & \longrightarrow & \Omega_0 U. \end{array}$$

We recall the following result in [4].

$$\begin{aligned} &H_*(\Omega^2 SO(8n+2);\mathbb{F}_2) \\ &= E(u_{4k+1} : 0 \le k \le n-1) \otimes \mathbb{F}_2[v_{8n+8k+6} : 0 \le k \le n-2] \\ &\otimes \mathbb{F}_2[Q_1^a u_{4n+4k+1} : a \ge 0, 0 \le k \le n-1] \\ &\otimes \mathbb{F}_2[Q_1^a w_{8n+2k+1} : a \ge 0, 0 \le k \le 4n-2 \text{ and } k \ne 1 \, mod\, 4] \\ &\otimes E(Q_1^a w_{8n-1} : a \ge 0) \otimes \mathbb{F}_2[Q_2^a v_{16n-2} : a \ge 0]\,. \end{aligned}$$

Recall the following homology in [1].

$$H_*(\Omega_0 U(4n+1);\mathbb{F}_2) = \mathbb{F}_2[e_{2i} : 1 \le i \le 4n]\,.$$

From (1) and the naturality of differentials, we get the following differentials:

$$\begin{aligned} &d(e_{4i+2}) = u_{4i+1}, \quad 0 \le i \le 2n-1, \\ &d(e_{(4i+2)2^k}) = e_{4i+2} \cdot e_{(4i+2)2} \cdots e_{(4i+2)2^{k-1}} \cdot u_{4i+1}, \quad 0 \le i \le n-1, \\ &d(e_{4n+4i+2}^{2^a}) = Q_1^a u_{4n+4i+1}, \quad a \ge 0, 0 \le i \le n-1. \end{aligned}$$

Moreover, if $(4i+2)2^k \le 8n < (4i+2)2^{k+1}$,

$$d(e_{(4i+2)} \cdot e_{(4i+2)2} \cdots e_{(4i+2)2^k} u_{4i+1}) = v_{(4i+2)2^{k+1}-2}, \quad 0 \le i \le n-1.$$

We also have the following differentials.

$$\begin{aligned} &d(e_{4n}^2) = w_{8n-1}, \\ &d(e_{8n}^2) = e_{4n}^2 \cdot w_{8n-1}, \\ &d(e_{4n}^{2^{a+1}} \cdot e_{8n}^{2^{a+1}} \cdot Q_1^a w_{8n-1}) = Q_2^{a+1} v_{16n-2}, \quad a \ge 0, \\ &d(e_{4n+4i}^{2^a}) = Q_1^a w_{8n+8i-1}, \quad a \ge 0, 1 \le i \le n-1. \end{aligned}$$

There is no indecomposable element of degree $4i-1$ for $1 \le i \le 2n-1$ in $H_*(\Omega^2 SO(8n+2); \mathbb{F}_2)$. So e_{2i}^2 survives for each $1 \le i \le 2n-1$, which yields $\mathbb{F}_2[z_{4i} : 1 \le i \le 2n-1]$ in $H_*(\Omega^2 SO(8n+2)/U(4n+1); \mathbb{F}_2)$. By the degree reason, generators, $w_{8n+8i+1}$, $0 \le i \le n-1$, in $H_*(\Omega^2 SO(8n+2); \mathbb{F}_2)$ also survive. Moreover there are choices of the generators with $Q_1(z_{4i}) = w_{8i+1}, n \le i \le 2n-1$, so that we have the following identification:

$$\begin{aligned}\mathbb{F}_2[Q_1^a z_{4i} : a \ge 0, n \le i \le 2n-1] &= \mathbb{F}_2[z_{4i} : n \le i \le 2n-1] \\ &\otimes \mathbb{F}_2[Q_1^a w_{8i+1} : a \ge 0, n \le i \le 2n-1].\end{aligned}$$

By the degree reason, the following terms survive permanently:

$$\mathbb{F}_2[Q_1^a w_{8n+8k+5} : a \ge 0, 0 \le k \le n-1]\,.$$

Hence we get

$$\begin{aligned}&H_*(\Omega_0^2 SO(8n+2)/U(4n+1)); \mathbb{F}_2) \\ &= \mathbb{F}_2[z_{4k} : 1 \le k \le n-1] \\ &\otimes \mathbb{F}_2 Q_1^a w_{8n+8k+5} : a \ge 0, 0 \le k \le n-1] \\ &\otimes \mathbb{F}_2[Q_1^a z_{4k} : a \ge 0, 4n < 4k+2 \le 8n]\,.\end{aligned}$$

The other three cases follow by the same method. □

For example, we have

$$\begin{aligned}H_*(\Omega_0^2 SO(18)/U(9)); \mathbb{F}_2) = \mathbb{F}_2[z_4] &\otimes \mathbb{F}_2[Q_1^a z_{21}, Q_1^a z_{29} : a \ge 0] \\ &\otimes \mathbb{F}_2[Q_1^a z_8, Q_1^a z_{12} : a \ge 0].\end{aligned}$$

COROLLARY 3.2. *2 annihilates all the 2–torsions in* $H_*(\Omega^2 SO(2n)/U(n); Z)$.

Proof. Consider the Bockstein spectral sequence. Then

$$E_1 = H_*(\Omega_0^2 SO(2n)/U(n)); \mathbb{F}_2)\,.$$

By Nishida relation, we have $\beta Q_1^{a+1} w_{2n+8k+3} = (Q_1^a w_{2n+8k+3})^2$ for $a \ge 0, 0 \le 4k \le n-5$ and $Q_1^{a+1} z_{4k} = (Q_1^a z_{4k})^2$ for $a \ge 0, n-3 < 4k \le 2n-4$. Since this Bockstein spectral sequence is a spectral sequence of an Hopf algebra, we have the following E_2–term:

$$\begin{aligned}\mathbb{F}_2[z_{4k} : 1 \le 4k \le n-3] &\otimes E(w_{2n+8k+5} : 0 \le 4k \le n-5) \\ &\otimes E(z_{4k} : n-3 < 4k \le 2n-4)\,.\end{aligned}$$

Hence there is no higher differential and $E_2 = E_\infty$. So the 2–torsions of $H_*(\Omega^2 SO(2n)/U(n); Z)$ are all of order 2. □

COROLLARY 3.3. *The rational homology of $\Omega^2SO(2n)/U(n)$ is as follows.*

$$\begin{aligned}
&H_*(\Omega^2SO(2n)/U(n);\mathbb{Q})\\
&= \mathbb{Q}[z_{4k} : 1 \le 4k \le n-3] \otimes E(w_{2n+8k+5} : 0 \le 4k \le n-5)\\
&\otimes E(z_{4k} : n-3 < 4k \le 2n-4)\,.
\end{aligned}$$

4. Mod p homology of $\Omega^2SO(2n)/U(n)$

We compute odd prime cases. From now on we denote $H_*(\Omega^2S^n;\mathbb{F}_p)$ by $\Omega_2(n)$ and $\otimes_{k=1}^r H_*(\Omega^2S^{n_k};\mathbb{F}_p)$ by $\Omega_2(n_1,\cdots,n_r)$.

THEOREM 4.1. *For odd primes p, we have*

$$\begin{aligned}
H_*(\Omega^2SO(4n)/U(2n);\mathbb{F}_p) &= \mathbb{F}_p[x_{4i} : 0 \le i \le n-1]\\
&\otimes \Omega_2(n_{4i+3} : n-1 \le i \le 2n-2),\\
H_*(\Omega^2SO(4n+2)/U(2n+1);\mathbb{F}_p) &= \mathbb{F}_p[x_{4i} : 0 \le i \le n-1]\\
&\otimes \Omega_2(n_{4i+3} : n \le i \le 2n-1)\,.
\end{aligned}$$

Proof. We will prove this by induction. For $n=1$, we have that

$$H_*(\Omega^2SO(4)/U(2);\mathbb{F}_p) = H_*(\Omega^2S^2;\mathbb{F}_p) = \mathbb{F}_p[x_0] \otimes H_*(\Omega^2S^3;\mathbb{F}_p)\,.$$

By induction, we assume that

$$\begin{aligned}
H_*(\Omega^2SO(4n-2)/U(2n-1);\mathbb{F}_p) &= \mathbb{F}_p[x_{4i} : 0 \le i \le n-2]\\
&\otimes\Omega_2(n_{4i+3} : n-1 \le i \le 2n-3)\,.
\end{aligned}$$

Consider the Serre spectral sequence associated to the following fibration:

$$\Omega^2SO(4n-2)/U(2n-1) \longrightarrow \Omega^2SO(4n)/U(2n) \longrightarrow \Omega^2S^{4n-2}.$$

Since this spectral sequence is a spectral sequence of Hopf algebras, the source of the first non zero differential is a generator and the target is a primitive element. For odd prime p, we have that

$$(\Omega^2S^{2n})_{(p)} \simeq (\Omega S^{2n-1})_{(p)} \times (\Omega^2S^{4n-1})_{(p)}\,.$$

Then we have that

$$H_*(\Omega^2 S^{4n-2};\mathbb{F}_p)=\mathbb{F}_p[z_{4n-4}]\otimes E(Q^a_{p-1}z_{8n-7}:a\geq 0)$$
$$\otimes\,\mathbb{F}_p[\beta Q^a_{p-1}z_{8n-7}:a>0]\,.$$

Hence there is no $4n-5$, $8n-8$ dimensional primitive element in the part $\Omega_2(n_{4i+3}:n-1\leq i\leq 2n-3)$ of $H_*(\Omega^2 SO(4n-2)/U(2n-1);\mathbb{F}_p)$ and there is no $4n-5$ dimensional primitive element in $\mathbb{F}_p[x_{4i}:0\leq i\leq n-2]$. Moreover if there exists nontrivial differential from $8n-7$ dimensional generator z_{8n-7} to $8n-8$ dimensional primitive element in $\mathbb{F}_p[x_{4i}:0\leq i\leq n-2]$, then it leads to contradiction to the fact that $H_*(\Omega^2_0 SO/U;\mathbb{F}_p)=\mathbb{F}_p[x_{4i}:i\geq 1]$. So the Serre spectral sequence collapses at the E_2-term and we get

$$H_*(\Omega^2 SO(4n)/U(2n);\mathbb{F}_p)=\mathbb{F}_p[x_{4i}:0\leq i\leq n-1]$$
$$\otimes\,\Omega_2(n_{4i+3}:n-1\leq i\leq 2n-2)\,.$$

Next we consider the Serre spectral sequence associated to the following fibration:

$$\Omega^2 SO(4n)/U(2n)\longrightarrow\Omega^2 SO(4n+2)/U(2n+1)\longrightarrow\Omega^2 S^{4n}.$$

Now we have that

$$H_*(\Omega^2 S^{4n};\mathbb{F}_p)==\mathbb{F}_p[z_{4n-2}]\otimes E(Q^a_{p-1}z_{8n-3}:a\geq 0)$$
$$\otimes\,\mathbb{F}_p[\beta Q^a_{p-1}z_{8n-3}:a>0]\,.$$

Then there should be nontrivial differential from $4n-2$ dimensional generator because $H_*(\Omega^2_0 SO/U;\mathbb{F}_p)$ does not contain a generator of dimension $4n-2$.

Since The elements $(z_{4n-2})^{p^k}$ for $k\geq 0$ in $H_*(\Omega S^{4n-1});\mathbb{F}_p)$ hits all generators in $H_*(\Omega^2 S^{4n-1});\mathbb{F}_p)$, there is no $8n-4$ dimensional primitive element in $H_*(\Omega^2 SO(4n)/U(2n);\mathbb{F}_p)$. Therefore the part $H_*(\Omega^2 S^{8n-1};\mathbb{F}_p)$ of $H_*(\Omega^2 SO(4n)/U(2n);\mathbb{F}_p)$ survives permanently and we get

$$H_*(\Omega^2 SO(4n+2)/U(2n+1);\mathbb{F}_p)=\mathbb{F}_p[x_{4i}:0\leq i\leq n-1]$$
$$\otimes\,\Omega_2(n_{4i+3}:n\leq i\leq 2n-1).$$

□

By the same method as Corollary 3.2, we get the following result for odd primes p.

COROLLARY 4.2. *p annihilates all the p-torsions in $H_*(\Omega^2 SO(2n)/U(n);Z)$ for odd primes p.*

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