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CONFORMAL CHANGES OF A RIZZA MANIFOLD WITH A GENERALIZED FINSLER STRUCTURE

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Abstract. We are devoted to dealing with the conformal theory of a Rizza manifold with a generalized Finsler metric $G_{ij}(x,y)$ and a new conformal invariant non-linear connection $M^i{}_j(x,y)$ constructed from the generalized Chern's non-linear connection $N^i{}_j(x, y)$ and almost complex structure $f^i{}_j(x)$. First, we find a conformal invariant connection $(M_j{}^i{}_k, M^i{}_j, C_j{}^i{}_k)$ and conformal invariant tensors. Next, the nearly Kaehlerian (G, M)-structures under conformal change in a Rizza manifold are investigated.

1. Introduction

Let M^{2n} be a $2n$-dimensional Finsler manifold whose fundamental function is given by $L(x,y)$. The metric tensor $g_{ij}(x,y)$ is introduced by $g_{ij}(x,y) = \dot{\partial}_j\dot{\partial}_i L^2/2$. Moreover, M^{2n} admits an almost complex structure $f^i{}_j(x)$. If L satisfies a Rizza condition ([4]), then the Finsler manifold is called a *Rizza manifold* and the structure $(f^i{}_j(x), g_{ij}(x,y))$ is called a *Rizza structure*. It is known that the Rizza manifold has been studied by many authors ([4], [5], [9], [11], [12]).

In the present paper, we consider the conformal theory of a Rizza manifold with a generalized Finsler metric $G_{ij}(x,y)$ ([3], [7]). A new conformal invariant non-linear connection $M^i{}_j(x,y)$ constructed from a generalized Chern's non-linear connection $N^i{}_j(x,y)$ ([6]) and an almost complex structure $f^i{}_j(x,y)$. First, we find a conformal invariant Finsler connection $(M_j{}^i{}_k, M^i{}_j, C_j{}^i{}_k)$ from a special tensor F_{ijk} in a Rizza manifold. Next, we introduce the invariant tensors under conformal changes

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of the above Finsler connection. Finally, the nearly Kaehlerian (G, M)-structures under conformal change in a Rizza manifold are investigated.

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2. Preliminaries

Let M^{2n} be a $2n$-dimensional Finsler manifold with an almost complex structure $f^i{}_j(x)$ which depends on a point p of M and the Finsler fundamental function $L(x, y)$ satisfying

$$L(x, \tilde{c}y) = |\tilde{c}|L(x, y) \tag{2.1}$$

for any complex number $\tilde{c} = a + ib$.

We define $\tilde{c}y = (ay^i + bf^i{}_m(x)y^m)(\partial/\partial x^i)_p$ for any tangent vector $y = y^i(\partial/\partial x^i)_p$ in tangent space $T_p(M)$. Since $L(x, ky) = kL(x, y)$ for any positive k, we can rewrite (2.1) as

$$L(x, \phi_\theta y) = L(x, y) \tag{2.2}$$

for any θ, where we put $\phi_\theta{}^i{}_j = \cos\theta\delta^i{}_j + \sin\theta f^i{}_j(x)$. It is known ([4]) that the condition (2.2) is equivalent to

$$\{g_{ij}(x, y) - g_{pq}(x, y)f^p{}_i f^q{}_j\}y^j = 0. \tag{2.3}$$

If a manifold admits a Finsler metric $g_{ij}(x, y)$ and an almost complex structure $f^i{}_j(x)$ satisfying the condition (2.1) (or equivalently (2.2) or (2.3)), then the manifold is called a *Rizza manifold* and the structure $(f^i{}_j(x), g_{ij}(x, y))$ is called a *Rizza structure* ([4], [11], [12]).

In a Rizza manifold, If we put

$$G_{ij}(x, y) = \frac{1}{2}(g_{ij}(x, y) + g_{pq}(x, y)f^p{}_i f^q{}_j), \tag{2.4}$$

then it is seen that $G_{ij}(x, y) = G_{ji}(x, y)$, $G_{ij}(x, y)$ is positive homogeneities for y^i in degree zero and $G_{ij}(x, y)\ \xi^i\xi^j$ is positively definite. That is, G_{ij} is a *generalized Finsler metric* ([4]). From (2.4) it is known that

$$\begin{aligned} &G_{pq}(x, y)f^p{}_i(x)f^q{}_j(x) = G_{ij}(x, y), \quad y^r\dot{\partial}_r G_{ij}(x, y) = 0, \\ &y^m f^r{}_m(x)\dot{\partial}_r G_{ij}(x, y) = 0, G_{ij}(x, y) = G_{pq}(x, y)f^p{}_i(x)f^q{}_j(x). \end{aligned} \tag{2.5}$$

In the Rizza manifold, the *generalized Chern's non-linear connection* is given as follows ([6]):

$$
\begin{aligned}
N^i{}_j =& \frac{1}{2}(G^{ih}\dot{\partial}_j G_{hs} - f^i{}_h G^{hr} f^t{}_j \partial_t G_{rs} + S^i{}_{sj} - G^{ih} G_{ms} S^m{}_{jh} \\
& - G^{ih}\dot{\partial}_r G_{hm} y^m S^r{}_{sj} + G^{ih} f^m{}_h G_{rs} f^r{}_t S^t{}_{mj}) y^s,
\end{aligned} \tag{2.6}
$$

where $S^i{}_{kj} = (\partial_k f^i{}_r) f^r{}_j$.

It is known that if the given almost complex structure $f^i{}_j$ in the Rizza manifold is integrable, then $N^i{}_j$ defined by (2.6) coincides with the Chern's non-linear connection ([7]). With respect to the generalized Chern's non-linear connection $N^i{}_j(x,y)$ and a generalized Finsler metric $G_{ij}(x,y)$ defined by (2.6) and (2.4) respectively, we introduce a symmetric Finsler connection $F\Gamma = (\Gamma_j{}^i{}_k, N_j, C_j{}^i{}_k)$ as follows ([2], [8]):

$$
\begin{aligned}
\Gamma_j{}^i{}_k &= G^{im}(X_j G_{mk} + X_k G_{mj} - X_m G_{jk})/2, \\
C_j{}^i{}_k(x,y) &= G^{im}(\dot{\partial}_j G_{mk} + \dot{\partial}_k G_{mj} - \dot{\partial}_m G_{jk})/2,
\end{aligned} \tag{2.7}
$$

where $X_j = \partial_j - N^m{}_j \dot{\partial}_m$, $\partial_j = \partial/\partial x^j$, $\dot{\partial}_j = \partial/\partial y^j$. Denoting the h-covariant, v-covariant derivative with respect to $F\Gamma$ by∇ and $\dot{\nabla}$ respectively, we have $\nabla_k G_{ij} = 0$ and $\dot{\nabla}_k G_{ij} = 0$ directly.

Now the pair $(G_{ij}, N^i{}_j)$ defined by (2.4) and (2.6) is called (G,N)-*structure in a Rizza manifold*, and the above Finsler connection $F\Gamma = (\Gamma_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ is said to be the *Finsler connection associated a* (G,N)-*structure*. We write the h-torsion and hv-torsion tensor of the connection $F\Gamma = (\Gamma_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ as

$$
R^i{}_{jk} = X_k N^i{}_j - X_j N^i{}_k, \quad P^i{}_{jk} = \dot{\partial}_k N^i{}_j - \Gamma_j{}^i{}_k, \tag{2.8}
$$

and the curvature tensors as

$$
R_h{}^i{}_{jk} = K_h{}^i{}_{jk} + C_h{}^i{}_m R^m{}_{jk}, \quad P_h{}^i{}_{jk} = \dot{\partial}_k \Gamma_h{}^i{}_j - Q_h{}^i{}_{jk}, \tag{2.9}
$$

where we put

$$
\begin{aligned}
K_h{}^i{}_{jk} &= X_k \Gamma_h{}^i{}_j - X_j \Gamma_h{}^i{}_k + \Gamma_m{}^i{}_k \Gamma_h{}^m{}_j - \Gamma_m{}^i{}_j \Gamma_h{}^m{}_k, \\
Q_h{}^i{}_{jk} &= \nabla_j C_h{}^i{}_k - C_h{}^i{}_m P^m{}_{jk}.
\end{aligned} \tag{2.10}
$$

3. Conformal invariant connection and tensors

In a Rizza manifold M^{2n} with (G, N)-structure, let us consider the conformal changes as follows:

$$\overline{G}_{ij}(x,y) = e^{2\sigma(x)}G_{ij}(x,y), \tag{3.1}$$

where $\sigma(x)$ is any scalar. Then we can see easily

$$\begin{gathered}\overline{G}^{ij} = e^{-2\sigma(x)}G^{ij}, \quad \partial_k\overline{G}_{ij} = e^{2\sigma(x)}\partial_k G_{ij} + 2\sigma_k e^{2\sigma}G_{ij}, \\ \dot{\partial}_k\overline{G}_{ij} = e^{2\sigma(x)}\dot{\partial}_k G_{ij}.\end{gathered} \tag{3.2}$$

The conformal change of the generalized Chern's non-linear connection $N^i{}_j$ defined by (2.6) is given as follows ([6]):

$$\overline{N}^i{}_j = N^i{}_j + y^i\sigma_j - f^i{}_h y^h f^r{}_j\sigma_r, \tag{3.3}$$

where $\sigma(k) = \partial_k\sigma(x)$, and we have

$$\overline{\Gamma}_j{}^i{}_k = \Gamma_j{}^i{}_k + \delta^i_j\sigma_k + \delta^i_k\sigma_j - G_{jk}G^{im}\sigma_m. \tag{3.4}$$

On the other hand, we put $f_{ij}(x,y) = G_{im}(x,y)f^m{}_j(x)$. Then, from (2.5), we find $f_{ij}(x,y) = -f_{ji}(x,y)$ and

$$\bar{f}_{ij} = e^{2\sigma}f_{ij}. \tag{3.5}$$

Here, we put

$$F_{ijk} = X_i f_{jk} + X_j f_{ki} + X_k f_{ij}. \tag{3.6}$$

Since $\Gamma_j{}^i{}_k = \Gamma_k{}^i{}_j$, it follows that

$$F_{ijk} = \nabla_i f_{jk} + \nabla_j f_{ki} + \nabla_k f_{ij}, \tag{3.7}$$

that is, F_{ijk} is a tensor field on a Rizza manifold. Using (3.2) and (3.5), (3.6) leads us to

$$\overline{F}_{ijk} = e^{2\sigma}\{F_{ijk} + 2(\sigma_i f_{jk} + \sigma_j f_{ki} + \sigma_k f_{ij})\}. \tag{3.8}$$

Now we put $F_k = f^r{}_h G^{ht} F_{rtk}$. Then we get $\overline{F}_k = F_k + 4(n-1)\sigma_k$, that is to say,

$$\sigma_k = \frac{1}{4(n-1)}(\overline{F}_k - F_k), \tag{3.9}$$

where we assumed $n > 1$. Substituting (3.9) into (3.3), we get

$$\overline{N}^i{}_j = N^i{}_j + \frac{1}{4(n-1)}\{(\overline{F}_j - F_j)y^i - f^i{}_h y^h f^r{}_j(\overline{F}_r - F_r)\}.$$

The above equation can be rewritten in the form

$$\begin{aligned}\overline{N}^i{}_j &- \frac{1}{4(n-1)}(\overline{F}_j y^i - f^i{}_h y^h f^r{}_j \overline{F}_r)\\ &= N^i{}_j - \frac{1}{4(n-1)}(F_j y^i - f^i{}_h y^h f^r{}_j F_r).\end{aligned}$$

Putting

$$M^i{}_j = N^i{}_j - \frac{1}{4(n-1)}(F_j y^i - f^i{}_h y^h f^r{}_j F_r), \tag{3.10}$$

we get a conformal invariant non-linear connection $M^i{}_j$, that is, $\overline{M}^i{}_j = M^i{}_j$. Furthermore, substituting (3.9) into (3.4), we get

$$\overline{\Gamma}_j{}^i{}_k = \Gamma_j{}^i{}_k + \frac{1}{4(n-1)}\{\delta^i_j(\overline{F}_k - F_k) + \delta^i_k(\overline{F}_j - F_j) - G_{jk}G^{im}(\overline{F}_m - F_m)\},$$

that is,

$$\begin{aligned}\overline{\Gamma}_j{}^i{}_k &- \frac{1}{4(n-1)}(\delta^i_j \overline{F}_k + \delta^i_k \overline{F}_j - \overline{G}_{jk}\overline{G}^{im}\overline{F}_m)\\ &= \Gamma_j{}^i{}_k - \frac{1}{4(n-1)}(\delta^i_j F_k + \delta^i_k F_j - G_{jk}G^{im}F_m).\end{aligned}$$

Putting

$$M_j{}^i{}_k = \Gamma_j{}^i{}_k - \frac{1}{4(n-1)}(\delta^i_j F_k + \delta^i_k F_j - G_{jk}G^{im}F_m), \tag{3.11}$$

we have a conformally invariant h-connection $(M_j{}^i{}_k, M^i{}_j)$. Thus we have

THEOREM 3.1. *In a Rizza manifold* M^{2n} $(n > 1)$, *we have a conformal invariant Finsler connection* $(M_j{}^i{}_k, M^i{}_j, C_j{}^i{}_k)$ *given by* (3.10), (3.11) *and* (2.8).

With respect to a conformal invariant connection $(M_j{}^i{}_k, M^i{}_j, C_j{}^i{}_k)$, h-torsion and hv-torsion are given by

$$\overset{m}{R^i}{}_{jk} = X_k^* M^i{}_j - X_j^* M^i{}_k, \quad \overset{m}{P^i}{}_{jk} = \dot{\partial}_k M^i{}_j - M_j{}^i{}_k,$$

where $X_k^* = \partial_k - M^r{}_k \dot{\partial}_r$. Therefore $\overset{m}{R^i}{}_{jk}$ and $\overset{m}{P^i}{}_{jk}$ are conformally invariant. Next, the curvature tensors with respect to $(M_j{}^i{}_k, M^i{}_j, C_j{}^i{}_k)$ are given as

$$\overset{m}{R_h}{}^i{}_{jk} = \overset{m}{K_h}{}^i{}_{jk} + C_h{}^i{}_r \overset{m}{R^r}{}_{jk}, \quad \overset{m}{P_h}{}^i{}_{jk} = \dot{\partial}_k M_h{}^i{}_j - \overset{m}{Q_h}{}^i{}_{jk},$$

where we put

$$\begin{aligned} \overset{m}{K_h}{}^i{}_{jk} =& (\partial_k - M^r{}_k \dot{\partial}_r) M_h{}^i{}_j - (\partial_j - M^r{}_j \dot{\partial}_r) M_h{}^i{}_k \\ &+ M_r{}^i{}_k M_h{}^r{}_j - M_r{}^i{}_j M_h{}^r{}_k, \\ \overset{m}{Q_h}{}^i{}_{jk} =& \overset{m}{\nabla}_j C_h{}^i{}_k - C_h{}^i{}_r \overset{m}{P^r}{}_{jk}, \end{aligned}$$

and $\overset{m}{\nabla}_j$ is the h-covariant derivative with respect to $(M_j{}^i{}_k, M^i{}_j, C_j{}^i{}_k)$.

Thus from $\overline{M}_j{}^i{}_k = M_j{}^i{}_k, \overline{M}^i{}_j = M^i{}_j$ and $\overline{C}_j{}^i{}_k = C_j{}^i{}_k$, we have

THEOREM 3.2. *In a Rizza manifold* M^{2n} $(n > 1)$, *the torsion tensors* $\overset{m}{R^i}{}_{jk}$, $\overset{m}{P^i}{}_{jk}$ *and curvature tensors* $\overset{m}{R_h}{}^i{}_{jk}$, $\overset{m}{P_h}{}^i{}_{jk}$, $\overset{m}{K_h}{}^i{}_{jk}$ *and* $\overset{m}{Q_h}{}^i{}_{jk}$ *with respect to* $(M_j{}^i{}_k, M^i{}_j, C_j{}^i{}_k)$ *are conformal invariants.*

4. Conformal change of (G, M)-structure

Let M^{2n} be a Rizza manifold admitting a (G, M)-structure, that is to say, M^{2n} admits a generalized Finsler metric $G_{ij}(x, y)$ and a non-linear $M^i{}_j(x, y)$ defined by (2.4) and (3.10) respectively. We put

$$(4.1) \qquad F_j^{*i}{}_k = \frac{1}{2} G^{im} (X_j^* G_{mk} + X_k^* G_{mj} - X_m^* G_{jk}),$$

then from $(2.5)_{3)}$ and (3.10) $F_j^{*i}{}_k = F_j{}^i{}_k$ and $F^*\Gamma = (F_j^{*i}{}_k, M^i{}_j, C_j{}^i{}_k)$ is a symmetric Finsler connection and it is said to be the *Finsler connection* *associated a (G, M)-structure.* With respect to the connection $F^*\Gamma$, (2.8), (2.9) and (2.10) are expressed as follows:

$$(4.2)\quad \begin{aligned} &R^{*i}{}_{jk} = X_k^* M^i{}_j - X_j^* M^i{}_k = \overset{m}{R}{}^i{}_{jk}, \quad P^{*i}{}_{jk} = \dot{\partial}_k M^i{}_j - F_j{}^i{}_k, \\ &R_h^{*i}{}_{jk} = K_h^{*i}{}_{jk} + C_h{}^i{}_m R^{*m}{}_{jk}, \quad P_h^{*i}{}_{jk} = \dot{\partial}_k F_h{}^i{}_j - Q_h^{*i}{}_{jk}, \end{aligned}$$

where

$$(4.3)\quad \begin{aligned} &K_h^{*i}{}_{jk} = X_k^* F_h{}^i{}_j - X_j^* F_h{}^i{}_k + F_m{}^i{}_k F_h{}^m{}_j - F_m{}^i{}_j F_h{}^m{}_k, \\ &Q_h^{*i}{}_{jk} = \nabla_j^* C_h{}^i{}_k - C_h{}^i{}_m P^{*m}{}_{jk} \end{aligned}$$

and ∇_j^* is the h-covariant derivative with respect to $F^*\Gamma$. The conformal changes of (4.1), (4.2) and (4.3) are induced as follows:

$$(4.4)\quad \begin{aligned} &\overline{F}{}_j^{*i}{}_k = F_j{}^i{}_k + \sigma_j \delta_k^i + \sigma_k \delta_j^i - \sigma^i G_{jk}, \quad \overline{R}{}^{*i}{}_{jk} = R^{*i}{}_{jk}, \\ &\overline{P}{}^{*i}{}_{jk} = P^{*i}{}_{jk} - \sigma_j \delta_k^i - \sigma_k \delta_j^i + \sigma^i G_{jk}, \\ &\overline{K}{}_h^{*i}{}_{jk} = K_h^{*i}{}_{jk} + \delta_j^i \sigma_{kh} - \delta_k^i \sigma_{jh} - G_{hj}\sigma^i{}_k + G_{hk}\sigma^i{}_j, \end{aligned}$$

where

$$\sigma^i = G^{im}\sigma_m, \quad \sigma_{hk} = \nabla_k^* \sigma_h - \sigma_k \sigma_h + \sigma_r \sigma^r G_{hk}/2, \quad \sigma^h{}_k = G^{hm}\sigma_{mk}.$$

Now it is seen from $(4.3)_{3)}$ $\overline{C}_j{}^i{}_k = C_j{}^i{}_k$ that

$$(4.5)\quad \overline{C}_j{}^i{}_m \overline{P}{}^{*m}{}_{k0} = C_j{}^i{}_m P^{*m}{}_{k0} - \sigma_0 C_j{}^i{}_k + C_j{}^i{}_m \sigma^m y_k,$$

from which we get

$$(4.6)\quad \overline{C}_m \overline{P}{}^{*m}{}_{k0} = C_m P^{*m}{}_{k0} - \sigma_0 C_k + C_m \sigma^m y_k,$$

where $\sigma_0 = \sigma_m y^m$, $C_m = C_r{}^r{}_m$, $C^k = G^{rk}C_r$, $y_k = G_{rt}y^k$ and $P^{*m}{}_{k0} = P^{*m}{}_{kr}y^r$.

On the other hand, we get $\overline{C}^k = e^{-2\sigma}C_k$, $\overline{C}^2 == \overline{C}_r\overline{C}^r = e^{-2\sigma}C^2$. Hence we have

$$\overline{C}_m \overline{P}{}^{*m}{}_{r0}\overline{C}^r = e^{-2\sigma}(P^{*m}{}_{r0}C^r - \sigma_0 C^2).$$

We assume that $C^2 \neq 0$. So we get

$$(4.7)\quad \sigma_0 = B - \overline{B}, \quad \sigma_k = B_k - \overline{B}_k,$$

where

$$B = C_m P^{*m}{}_{r0} C^r / C^2, \quad B_k = \dot{\partial}_k B, \quad B^k = G^{km} B_m.$$

Substituting (4.7) into (4.6), we get

$$\overline{C}_j{}^i{}_m \overline{P}^{*m}{}_{r0} = C_j{}^i{}_m P^{*m}{}_{r0} - (B - \overline{B}) C_j{}^i{}_k + C_j{}^i{}_m G^{mr}(B_r - \overline{B}_r) y_k,$$

that is,

$$\overline{C}_j{}^i{}_m \overline{P}^{*m}{}_{k0} - \overline{B}\, \overline{C}_j{}^i{}_k + \overline{C}_j{}^i{}_m \overline{G}^{mr} \overline{P}_r y_k$$
$$= C_j{}^i{}_m P^{*m}{}_{k0} - B C_j{}^i{}_k + C_j{}^i{}_m G^{mr} B_r y_k.$$

Hence, the quantity defined by

$$Q_j^{*i}{}_k = C_j{}^i{}_m P^{*m}{}_{k0} - B C_j{}^i{}_k + C_j{}^i{}_m G^{mr} P_r y_k \tag{4.8}$$

is invariant under the conformal change of given $F^*\Gamma$-connection in a Rizza manifold, that is, $\overline{Q}_j^{*i}{}_k = Q_j^{*i}{}_k$.

Next, by means of $(4.4)_{1)}$ and (4.7), we have

$$\dot{\partial}_h \overline{F}_j^{*i}{}_k = \dot{\partial}_h F_j{}^i{}_k - (\dot{\partial}_h G^{im})(B_m - \overline{B}_m) G_{jk} - G^{im}(B_m - \overline{B}_m) \dot{\partial}_h G_{jk}.$$

By virtue of $\dot{\partial}_h \overline{G}^{im} \overline{G}_{jk} = \dot{\partial}_h G^{im} G_{jk}$, we have

$$\dot{\partial}_h \overline{F}_j{}^i{}_k + (\dot{\partial}_h \overline{G}^{im}) \overline{B}_m \overline{G}_{jk} - \overline{B}^i \dot{\partial}_h \overline{G}_{jk}$$
$$= \dot{\partial}_h F_j{}^i{}_k + (\dot{\partial}_h G^{im}) B_m G_{jk} - B^i \dot{\partial}_h G_{jk}.$$

If we put

$$F_h^{*i}{}_{jk} = \dot{\partial}_h F_j{}^i{}_k + (\dot{\partial}_h G^{im}) B_m G_{jk} - B^i \dot{\partial}_h G_{jk}, \tag{4.9}$$

then the tensor field $F_h^{*i}{}_{jk}(x, y)$ is also invariant under the conformal change of the given connection $F^*\Gamma$.

For the tensor field $Q_h^{*i}{}_{jk} = \nabla_j^* C_h{}^i{}_k - C_k{}^i{}_m P^{*m}{}_{jk}$, we get the following from $(4.4)_{3)}$,

$$\overline{Q}_h^{*i}{}_{jk} = Q_h^{*i}{}_{jk} + \sigma_m \delta_j^i C_h{}^m{}_k - \sigma_h C_j{}^i{}_k - \sigma^i G_{jm} C_h{}^m{}_k$$
$$+ \sigma^m G_{hj} C_m{}^i{}_k - \sigma^m G_{jk} C_h{}^i{}_m.$$

Using $(4.7)_{2)}$, we see that the tensor $\Psi^{*}{}_{h}{}^{i}{}_{jk}(x,y)$ defined by

$$(4.10)\qquad \begin{aligned}\Psi^{*}{}_{h}{}^{i}{}_{jk} &= Q^{*}{}_{h}{}^{i}{}_{jk} + B_m\delta^i_j C_h{}^m{}_k - B_h C_j{}^i{}_k \\ &\quad - B^i G_{jm} C_h{}^m{}_k + B^m G_{hj} C_m{}^i{}_k - B^m G_{jk} C_h{}^i{}_m\end{aligned}$$

is invariant under the conformal change of given connection $F^*\Gamma$. From $\sigma_i = \sigma_i(x)$, $(4.7)_{2)}$ leads us to $\dot\partial_j \overline{B}_k = \dot\partial_j B_k$, that is, the tensor $\dot\partial_j B_k$ itself is invariant under the conformal change of the given connection $F^*\Gamma$. In addition to the above, we have

$$\begin{aligned}\overline{\nabla}^*_j \overline{B}_k &= \nabla^*_j B_k - \nabla^*_j \sigma_k - \sigma_k B_j - \sigma_j B_k + \sigma^m B_m G_{jk} \\ &\quad + 2\sigma_j\sigma_k - \sigma_m\sigma^m G_{jk},\end{aligned}$$

from which we have

$$(4.11)\qquad \begin{aligned}\nabla^*_j \sigma_k &= \nabla^*_j B_k - \overline{\nabla}^*_j \overline{B}_k - \sigma_k B_j - \sigma_j B_k + \sigma_m B^m G_{jk} \\ &\quad + 2\sigma_j\sigma_k - \sigma_m\sigma^m G_{jk}.\end{aligned}$$

Since $\nabla^*_j\sigma_k = \nabla^*_k\sigma_j$, we have $\overline{\nabla}^*_j\overline{B}_k - \overline{\nabla}^*_k\overline{B}_j = \nabla^*_j B_k - \nabla^*_k B_j$. Namely, the tensor field defined by $\nabla^*_j B_k - \nabla^*_k B_j$ is also invariant under the conformal change of the given connection $F^*\Gamma$. Finally, on account of (4.11) and (4.7), we have

$$\sigma_{kj} = \nabla^*_j B_k - \overline{\nabla}^*_j \overline{B}_k - B_j B_k + \overline{B}_j \overline{B}_k + \frac{1}{2} B_m B^m G_{jk} - \frac{1}{2}\overline{B}_m \overline{B}^m \overline{G}_{jk}.$$

Hence we put

$$B_{kj} = \nabla^*_j B_k - B_j B_k + \frac{1}{2} B_m B^m G_{jk},$$

then we have $\sigma_{kj} = B_{kj} - \overline{B}_{kj}$, from which

$$\begin{aligned}\overline{K}^{*}{}_{h}{}^{i}{}_{jk} &= K^{*}{}_{h}{}^{i}{}_{jk} + \delta^i_j(B_{kh} - \overline{B}_{kh}) - \delta^i_k(B_{jh} - \overline{B}_{jh}) \\ &\quad - G_{hj}G^{im}(B_{mk} - \overline{B}_{mk}) + G_{hk}G^{im}(B_{mj} - \overline{B}_{mj}).\end{aligned}$$

Thus the tensor field defined by

$$(4.12)\quad \Omega^{*}{}_{h}{}^{i}{}_{jk} = K^{*}{}_{h}{}^{i}{}_{jk} + \delta^i_j B_{kh} - \delta^i_k B_{jh} - G_{hj}G^{im}B_{mk} + G_{hk}G^{im}B_{mj}$$

is also invariant under the conformal change of the given connection $F^*\Gamma$.

Consequently we conclude the following

THEOREM 4.1. *In a Rizza manifold M^{2n} $(n > 1)$ satisfying $C^2 \neq 0$, the tensors $Q^{*}{}_{j}{}^{i}{}_{k}$, $F^{*}{}_{h}{}^{i}{}_{jk}$, $\Psi^{*}{}_{h}{}^{i}{}_{jk}$ and $\Omega^{*}{}_{h}{}^{i}{}_{jk}$ which are given respectively* (4.8), (4.9), (4.10) *and* (4.12) *are invariant under the conformal change of a given connection $F^*\Gamma$.*

5. A nearly Kaehlerian Finsler (G, M)-structure

We consider a Rizza manifold admitting a (G, M)-structure, where the non-linear connection $M^i{}_j$ is given by (3.10). If a (G, M)-structure satisfies

$$\nabla^*_k f^i{}_j + \nabla^*_j f^i{}_k = 0, \tag{5.1}$$

then the (G, M)-structure is called a *nearly Kaehlerian* (G, M)*-structure*. On the other hand, the Nejenhuis tensor $N^h{}_{ij}$ of $f^i{}_j(x)$ with respect to $F^*\Gamma$ in a Rizza manifold is written as

$$N^h{}_{ij} = (\nabla^*_r f^h{}_i)f^r{}_j - (\nabla^*_r f^h{}_j)f^r{}_i + f^h{}_r\nabla^*_i f^r{}_j - f^h{}_r\nabla^*_j f^r{}_i.$$

This tensor may be rewritten in the form

$$\begin{aligned} N^h{}_{ij} &= 4f^h{}_r\nabla^*_i f^r{}_j - 2f^h{}_r(\nabla^*_i f^r{}_j + \nabla^*_j f^r{}_i) \\ &\quad - (\nabla^*_j f^h{}_r + \nabla^*_r f^h{}_j)f^r{}_i + (\nabla^*_r f^h{}_i + \nabla^*_i f^h{}_r)f^r{}_j. \end{aligned} \tag{5.2}$$

Let us put $N_{hij} = G_{hm}N^m{}_{ij}$. Then we have

$$N_{hij} = f^r{}_j F^*_{rhi} - f^r{}_i F^*_{rhj} - 2f_{jr}\nabla^*_h f^r{}_i, \tag{5.3}$$

where $F_{hij} = \nabla^*_h f_{ij} + \nabla^*_i f_{jh} + \nabla^*_j f_{hi}$. Therefore

$$N_{hij} + N_{ihj} = -f^r{}_i F^*_{rhj} - f^r{}_h F^*_{rij} - 2f_{jr}(\nabla^*_h f^r{}_i + \nabla^*_i f^r{}_h). \tag{5.4}$$

Hence, if $N_{hij} + N_{ihj} = 0$ and $f^r{}_i F^*_{rhj} + f^r{}_h F^*_{rij} = 0$, then (5.1) holds, that is, the (G, M)-structure is to be a nearly Kaehlerian (G, M)-structure.

Conversely, if the (G, M)-structure is nearly Kaehlerian, that is, (5.1) is satisfied, then from (5.2), we have $N_{hij} = 4f_{hr}\nabla^*_i f^r{}_j$. Therefore

$$\begin{aligned} N_{hij} + N_{ihj} &= 4(f_{hr}\nabla^*_i f^r{}_j + f_{ir}\nabla^*_h f^r{}_j) \\ &= 4f_{mj}(\nabla^*_i f^m{}_h + \nabla^*_h f^m{}_i) = 0, \end{aligned} \tag{5.5}$$

$$\begin{aligned} f^r{}_i F^*_{rhj} + f^r{}_h F^*_{rij} & \\ &= f^r{}_i\nabla^*_h f_{jr} + f^r{}_h\nabla^*_i f_{jr} + 2(f^r{}_i\nabla^*_j f_{rh} + f^r{}_h\nabla^*_j f_{ri}) \\ &= -f_{jr}(\nabla^*_h f^r{}_i + \nabla^*_i f^r{}_h) = 0 \end{aligned} \tag{5.6}$$

hold good by virtue of (5.1). Since $F^*_{ijk} = -F^*_{jik}$, the equation (5.6) may be written as

$$F^*_{ijk} + F^*_{stk}f^s{}_i f^t{}_j = 0. \tag{5.7}$$

Using the operators as follows: $O^{st}_{ij} = (\delta^s_i\delta^t_j - f^s{}_i f^t{}_j)/2$, ${}^*O^{st}_{ij} = (\delta^s_i\delta^t_j + f^s{}_i f^t{}_j)/2$ ([13]), the equation (5.7) is written as ${}^*O^{st}_{ij}F^*_{stk} = 0$, that is, F^*_{ijk} is pure in i and j. Consequently we have

THEOREM 5.1. *In a Rizza manifold with a (G, M)-structure, the necessary and sufficient condition for a (G, M)-structure to be a nearly Kaehlerian (G, M)-structure is that N_{hij} is skew-symmetric in i and h, and F^*_{rhj} is pure in i and j respectively.*

From the conformal change (3.1), we have $\overline{N}_{hij} = e^{2\sigma} N_{hij}$. Therefore

$$\overline{N}_{hij} + \overline{N}_{ihj} = e^{2\sigma}(N_{hij} + N_{ihj}) \tag{5.8}$$

and

$$\begin{aligned} f^r{}_i \overline{F}^*_{rhj} + f^r{}_h \overline{F}^*_{rij} = e^{2\sigma}\{&(f^r{}_i F^*_{rhj} + f^r{}_h F^*_{rij}) \\ &+ 2\sigma_r (f^r{}_i f_{hj} + f^r{}_h f_{ij}) - 2(\sigma_h G_{ji} + \sigma_i G_{jh})\}. \end{aligned} \tag{5.9}$$

In a Rizza manifold, if the (G, M)-structure is conformal to a nearly Kaehlerian Finsler structure, then from Theorem 5.1, (5.8) and (5.9), we have $N_{hij} + N_{ihj} = 0$ and

$$ {}^*O^{ri}_{hk} F^*_{rij} = 2e^{-2\sigma} \sigma_r \{(\delta^r_k f_{hj} - \delta^r_h f_{kj}) + (f^r{}_k G_{hj} - f^r{}_h G_{kj})\}. \tag{5.10}$$

Thus we have the following

THEOREM 5.2. *In a Rizza manifold with a non-linear connection $M^i{}_j$ given by (3.10), if the conformal change of a (G, M)-structure is a nearly Kaehlerian structure, then a (G, M)-structure is also a nearly Kaehlerian Finsler structure provided $\delta^r_k f_{hj} - \delta^r_h f_{kj} + f^r_k G_{hj} - f^r_h G_{kj} = 0$.*

Next, according to [10], if there exists a coordinate neighborhood (U, x^i) containing any point $p \in M^{2n}$ such that $\partial_k G_{ij} = 0$ and $M^m{}_k \dot{\partial}_m G_{ij}$ $(= N^m{}_k \dot{\partial}_m G_{ij}) = 0$ hold on U, then the (G, M)-structure to be *flat.* Let a $(\overline{G}, \overline{M})$-structure be the locally conformal change of a nearly Kaehlerian (G, M)-structure. If a $(\overline{G}, \overline{M})$-structure is flat, then it follows that $\partial_k \overline{G}_{il} = 0$ and $\overline{M}^m{}_k \dot{\partial}_m \overline{G}_{ij} = 0$ locally and naturally, $\overline{X}^*_h \overline{G}_{ij} = 0$, from which $\overline{F}^*_j{}^i{}_k = 0$. Also $\partial_k \overline{G}_{ij} = \partial_k(e^{2\sigma} G_{ij}) = 0$ leads us to

$$\partial_k G_{ij} = -2\sigma_k G_{ij}. \tag{5.11}$$

Hence we see that

$$\partial_k G^{ij} = 2\sigma_k G^{ij}. \tag{5.12}$$

And, since $\overline{M}^i{}_j = M^i{}_j$ and $\overline{M}^m{}_k\dot{\partial}_m\overline{G}_{ij} = 0$, we have

$$M^m{}_k\dot{\partial}_m G_{ij} = 0. \tag{5.13}$$

On the other hand, applying the Ricci identity to the tensor $f^i{}_j$ with respect to $F^*\Gamma$, we find

$$\nabla^*_k\nabla^*_h f^i{}_j - \nabla^*_h\nabla^*_k f^i{}_j = f^m{}_j K^*_m{}^i{}_{hk} - f^i{}_m K^*_j{}^m{}_{hk}.$$

Contraction with respect to i and k in this equation gives

$$\nabla^*_r\nabla^*_h f^r{}_j = f^m{}_j K^*_{mh} - f^r{}_m K^*_j{}^m{}_{hr},$$

where $K^*_{hj} = K^*_h{}^m{}_{jm}$, and similarly we have

$$\nabla^*_r\nabla^*_j f^r{}_h = f^m{}_h K^*_{mj} - f^r{}_m K^*_h{}^m{}_{jr},$$

From above two equations, we have

$$\nabla^*_r(\nabla^*_h f^r{}_j + \nabla^*_j f^r{}_h) = f^m{}_j K^*_{mh} + f^m{}_h K^*_{mj},$$

from which

$$f^m{}_j K^*_{mh} + f^m{}_h K^*_{mj} = 0 \tag{5.14}$$

by virtue of (5.1). Since $\overline{F}^*_j{}^i{}_k = 0$, $(4.3)_{1)}$ shows that

$$F^*_j{}^i{}_k = -\delta^i_j\sigma_k - \delta^i_k\sigma_j + G_{jk}G^{im}\sigma_m. \tag{5.15}$$

From $M^m{}_k\dot{\partial}_m(G^{ir}G_{rj}) = M^m{}_k\{(\dot{\partial}_m G^{ir})G_{rj} + G^{ir}\dot{\partial}_m G_{rj}\} = 0$ and (5.13), we have $M^m{}_k(\dot{\partial}_m G^{ir})G_{rj} = 0$, from which $M^m{}_k\dot{\partial}_m G^{ir} = 0$. Using these results, we have

$$\begin{aligned}
K^*_h{}^i{}_{jk} &= X^*_k(-\delta^i_h\sigma_j - \delta^i_j\sigma_h + G_{hj}G^{im}\sigma_m)\\
&\quad - X^*_j(-\delta^i_h\sigma_k - \delta^i_k\sigma_h + G_{kh}G^{im}\sigma_m)\\
&\quad + (\delta^i_m\sigma_k + \delta^i_k\sigma_m - G_{mk}G^{ir}\sigma_r)(\delta^m_h\sigma_j + \delta^m_j\sigma_h - G_{hj}G^{ms}\sigma_s)\\
&\quad - (\delta^i_m\sigma_j + \delta^i_j\sigma_m - G_{mj}G^{ir}\sigma_r)(\delta^m_h\sigma_k + \delta^m_k\sigma_h - G_{hk}G^{ms}\sigma_s)\\
&= \delta^i_k(\partial_j\sigma_h + \sigma_j\sigma_h - G_{hj}G^{rm}\sigma_r\sigma_m)\\
&\quad - \delta^i_j(\partial_k\sigma_h + \sigma_k\sigma_h - G_{hk}G^{rm}\sigma_r\sigma_m)\\
&\quad + G_{hj}G^{im}(\partial_k\sigma_m + \sigma_k\sigma_m) - G_{hk}G^{im}(\partial_j\sigma_m + \sigma_j\sigma_m).
\end{aligned}$$

Putting $\rho_{jh} = \partial_j\sigma_h + \sigma_j\sigma_h - G_{jh}\sigma^m\sigma_m/2$, we have $\rho_{jh} = \rho_{hj}$. And we find

$$K^{*i}_{h\ jk} = \delta^i_k\rho_{jh} - \delta^i_j\rho_{kh} + G_{hj}G^{im}\rho_{km} - G_{hk}G^{im}\rho_{jm}.$$

Contraction with respect to i and k in this equation gives

$$K^*_{hj} = 2(n-1)\rho_{jh} + G_{hj}\rho^m{}_m, \tag{5.16}$$

where $\rho^m{}_m = G^{rm}\rho_{rm}$. Substituting (5.16) into (5.14), we have

$$2(n-1)(f^m{}_j\rho_{mh} + f^m{}_h\rho_{mj}) + (f^m{}_jG_{mh} + f^m{}_hG_{mj})\rho^r{}_r = 0,$$

which implies

$$f^m{}_j\rho_{mh} + f^m{}_h\rho_{mj} = 0 \tag{5.17}$$

by virtue of $f_{hj} + f_{jh} = 0$. The equation (5.17) may be written in the form $\rho_{hr} - \rho_{mj}f^m{}_hf^j{}_r = 0$, that is,

$$O^{hk}_{ij}\rho_{hk} = 0. \tag{5.18}$$

By virtue of $(2.5)_{4)}$, (5.18) is rewritten as

$$O^{hk}_{ij}(\partial_h\sigma_k + \sigma_h\sigma_k) = 0.$$

Thus we have the following

THEOREM 5.3. *Let M^{2n} be a Rizza manifold with a nearly Kaehlerian Finsler structure (G, M). If the locally conformal change of a (G, M)-structure is flat, then $\partial_h\sigma_k + \sigma_h\sigma_k$ is hybrid in M^{2n}.*

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