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AN EMPIRICAL BAYESIAN ESTIMATION OF MONTHLY LEVEL AND CHANGE IN TWO-WAY BALANCED ROTATION SAMPLING

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ABSTRACT

An empirical Bayesian approach is discussed for estimation of characteristics from the two-way balanced rotation sampling design which includes U.S. Current Population Survey and Canadian Labor Force Survey as special cases. An empirical Bayesian estimator is derived for monthly effect under presence of two types of biases and correlations. It is shown that the marginal distribution of observation provides more general correlation structure than that frequentist has assumed. Consistent estimators are derived for hyper-parameters in Normal priors.

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1. INTRODUCTION

Rotation sampling design has been used for an effective estimation of characteristic in a panel survey by partially replacing sample units as time advances. Sample units are partitioned into a finite number of groups, called as rotation groups, so that sample units in a rotation group are homogeneous. To reduce or control biases arising from different interview times and rotation groups in rotation design in a systematic way, Park, Kim and Choi (2001) introduced the rotation design balanced in two-ways and they called it two-way balanced rotation design. The two-way balanced rotation design by balancing monthly sample on rotation groups and interview time is shown to include most of currently used

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rotation designs as special cases and to be an effective design to remove biases arising from rotation design and to reflect two types of correlations into variance.

Recently the appropriateness of model based inference has been widely accepted in various fields. For example Rubin (1987) argued that any sensible analysis must be based on an assumed model for the nonresponse. This is somewhat related to our topic because nonresponses or missing values occur inevitably in a rotation sampling. See also Ghosh and Meeden (1986), Battese, Harter and Fuller (1988), Prasad and Rao (1990) for the model-based approach in small area sampling.

The methods that have been proposed use either a variance components approach or an empirical Bayes (EB) approach. Their methods commonly use certain mixed linear models for prediction or estimation purpose and are composed of two steps. First, assuming the variance components are known, certain best estimators or EB estimators are obtained for the unknown parameters of interest. Then the unknown variance components are estimated by Henderson's method of fitting of constants or the restricted maximum likelihood method. This approach is usually quite satisfactory for point estimation. However, due to the lack of closed–form expressions for the mean squared errors of the estimators, it is difficult to estimate the standard errors associated with the estimators. In this work we propose a model based approach for two-way balanced rotation design.

2. A BAYESIAN MODEL FOR THE 2-WAY ROTATION DESIGN

A rotation system $r_1 - r_2 - r_1 - \cdots - r_2 - r_1$ represents the rotation scheme that a sample unit is in the sample for the first r_1 months, not in the sample for the following r_2 months, and returns to the sample for the following r_1 month. This procedure is repeated until it is interviewed for the mr_1 times and then the sample unit is retired from the sample. This rotation system is symbolized as $r_1^m - r_2^{m-1}$.

In the two-way balanced rotation design adapting $r_1^m - r_2^{m-1}$ rotation system, all sample units are grouped into mr_1 rotation groups. These mr_1 rotation groups and all interview times from the first time to the $(mr_1)^{th}$ time are represented in every monthly sample. Typical examples with such properties are U.S. Current Population Survey, Canadian Labor Force Survey and Australian Labor Force Survey. Because some of the $r_1^m - r_2^{m-1}$ designs lack such two-way balancing (*e.g.* Rao and Graham, 1964; Cantwell, 1990), we refer to Park, Kim and Choi (2001) for the necessary and sufficient condition of the 2-way balancing. From now on, we only concentrate on the two-way balanced design and thus, all observation are obtained through a two-way balanced rotation sampling.

We assume that k sample units are taken from each rotation group during n months. Thus, total Gnk observations are available to us where $G = mr_1$ is the number of group. Let $\mathbf{x}_g = (x_{g11}, \dots, x_{g1k}, x_{g21}, \dots, x_{gnk})'$. A typical element x_{gtj} of $\mathbf{x}_g$ represents the j^{th} sample unit from the g^{th} rotation group at time t. Note that two important biases may arise in rotation sampling; they are the interview time bias and the rotation group bias. Bailer (1975) discussed the interview time bias arising from different interview times in the same survey month. Cantwell and Caldwell (1998) investigated the rotation group bias arising from different rotation groups. From these two studies, the two biases are important factors in rotation sampling design and lead us to consider the following model.

$$E(x_{gtj}|\boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{\alpha}) = \mu_t + \gamma_{gt} + \alpha_{lt} \tag{2.1}$$

where μ_t, γ_{gt} and α_{lt} are the monthly level at month t, the g^{th} rotation group bias and the l^{th} interview time bias, respectively. Here, we assumed x_{gtj} is the measurement from the sample unit interviewed for the l^{th} time at month t.

Park, Kim and Choi (2001) derived two matrices to identify interview time of a particular rotation group surveyed at month t and to determine whether or not two sample units are same when they are interviewed at two different months. The first one is $G \times G$ matrix, L_1^t defined as $L_1^t = L_1^{t-1} L_1$ where $L_1^0 = I$ and for $i, j = 1, 2, \dots, g$, $(L_1)_{ij} = 1$ if $j = (\text{mod}_m\{m - m^* + (i/r_1)\}r_1 + 1)$ for $i = r_1, 2r_1, \dots, mr_1$ or if $j = i + 1$ for $i \neq r_1, 2r_1, \dots, mr_1$, and 0 otherwise. Here, m^* is the integer satisfying $\text{mod}_m\{m^*(l+1) - l\} = 0$ for $1 \le m^* \le m$ where $l = r_2/r_1$. The second one is another $G \times G$ matrix, L_2^t where its $(i,j)^{th}$ element is 1 if $t_i = t_j - t$ for $j \ge i$ and is 0 otherwise. Here, $t_i = (i-1) + \sum_{k=1}^{m-1} r_2 I_{[i > kr_1]}$ where $I_{[\cdot]}$ is the usual indicator function. $(L_1^{t_1})_{ij} = 1$ implies that the rotation group interviewed for the i^{th} time at month t is again interviewed for the j^{th} time at month $t + t_1$. $(L_2^{t_1})_{ij} = 1$ implies that two rotation groups contain same sample units when they are interviewed for the i^{th} and j^{th} times at respective months t and $t + t_1$.

We assume without loss of generality that the G^{th} rotation group at the initial month $t = 1$ is interviewed for the G^{th} time. We express this interview time with $G \times 1$ elementary vector $\mathbf{u}_g$ where the g^{th} element is 1 and the remaining elements are 0. Then, by L_1, $(\mathbf{u}_g' L_1^{t-1})'$ indicates the interview time of the g^{th} rotation group at month t in which the g^{th} rotation group is interviewed for the g^{th} time at the initial month. Thus, the interview time bias of x_{tgj} is $\mathbf{u}_g' L_1^{t-1} \boldsymbol{\alpha}_t$ where

$\boldsymbol{\alpha}_t = (\alpha_{1t}, \alpha_{2t}, \ldots, \alpha_{Gt})'$ and hence

$$E(\mathbf{x}_{gt}|\boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{\alpha}) = \mu_t \mathbf{1}_k + \gamma_{gt}\mathbf{1}_k + \mathbf{u}_g' L_1^{t-1} \boldsymbol{\alpha}_t \mathbf{1}_k \quad \text{for} \quad t = 1, \ldots, n \tag{2.2}$$

where $\mathbf{x}_{gt} = (x_{gt1}, \ldots, x_{gtk})'$.

Define $\mathbf{x} = (\mathbf{x}_1', \ldots, \mathbf{x}_G')'$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)'$, $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1', \ldots, \boldsymbol{\alpha}_n')'$ and $\boldsymbol{\gamma}_g = (\gamma_{g1}, \ldots, \gamma_{gn})'$ for $g = 1, 2, \ldots, G$. Then, from (2.2), we have

$$E(\mathbf{x}_g|\boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{\alpha}) = (\boldsymbol{\mu} + \boldsymbol{\gamma}_g + \mathbf{M}_g\boldsymbol{\alpha}) \otimes \mathbf{1}_k = (\boldsymbol{\mu} + \boldsymbol{\gamma}_g) \otimes \mathbf{1}_k + \mathcal{M}_g\boldsymbol{\alpha} \tag{2.3}$$

where $\mathbf{1}_k$ is the $k \times 1$ unit vector, $\otimes$ is the Kronecker product, $\mathcal{M}_g = \mathbf{M}_g \otimes \mathbf{1}_k$, and

$$\mathbf{M}_g = \begin{pmatrix} \mathbf{u}_g' L_1^0 & \mathbf{0}' & \cdots & \mathbf{0}' \\ \mathbf{0}' & \mathbf{u}_g' L_1^1 & \cdots & \mathbf{0}' \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}' & \mathbf{0}' & \cdots & \mathbf{u}_g' L_1^{n-1} \end{pmatrix}.$$

There are two types of correlations in rotation sampling design. They are first-order and second-order correlations. The first order correlation arises between successive uses of the same sample unit and the second-order correlation arises between two sample units from the same rotation group (Kumar and Lee, 1983; Lee, 1990; Park, Kim and Choi, 2001). To reflect these two correlations and the expectation of (2.3) into a Bayesian perspective, we might setup a normal EB model as follows:

(I) Given $\boldsymbol{\mu}, \boldsymbol{\gamma}_g$ and $\boldsymbol{\alpha}$, the conditional distributions of $\mathbf{x}_g$, $g = 1, \ldots, G$, are independent multivariate normal with mean $E(\mathbf{x}_g|\boldsymbol{\mu}, \boldsymbol{\gamma}_g, \boldsymbol{\alpha}) = (\boldsymbol{\mu} + \boldsymbol{\gamma}_g) \otimes \mathbf{1}_k + \mathcal{M}_g\boldsymbol{\alpha}$ and variance-covariance matrix $\sigma^2\mathbf{I}_{nk}$.

(II) $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\gamma}_g$'s are independent with $\boldsymbol{\mu} \sim N(a\mathbf{1}_n, \sigma_\mu^2\mathbf{V}_\mu)$, $\boldsymbol{\alpha} \sim N(\mathbf{0}, \sigma_\alpha^2\mathbf{V}_\alpha)$ and $\boldsymbol{\gamma}_g \sim N(\mathbf{0}, \sigma_\gamma^2\mathbf{V}_\gamma)$.

If auxiliary variables are available for $\boldsymbol{\mu}$, we might choose a prior distribution whose mean is $\mathbf{A}\boldsymbol{\beta}$ where $\mathbf{A}$ is the matrix of auxiliary variables. However, it is reasonable, we believe, to assume that is a stationary time series for our purpose. For stationarity, both $\mathbf{V}_\mu$ and $\mathbf{V}_\gamma$ are $n \times n$ correlation matrices only depending on time lag, and $\mathbf{V}_\alpha$ is a $nG \times nG$ correlation matrix with the $(i,j)^{th}$ $G \times G$ block matrix $\rho_{\alpha,|i-j|}L_2^{j-i}$. The second-order correlation is reflected by correlation matrix $\mathbf{V}_\gamma$ while the first-order correlation is reflected by $\mathbf{V}_\alpha$. Because the first-order correlation occurs only between measurements from the same sample unit and

$L_2^{t_1-t_2}$ identifies which two measurements at month t_1 and t_2 are from the same sample unit only by respective interview times, $\mathrm{Cov}(\boldsymbol{\alpha}_{t_1}, \boldsymbol{\alpha}_{t_2}) = \sigma_\alpha^2 \rho_{\alpha,|t_1-t_2|} L_2^{t_2-t_1}$ to have the prior distribution of $\boldsymbol{\alpha}$ given above.

By stationarity of $\boldsymbol{\mu}$, $\boldsymbol{\gamma}_g$, and $\boldsymbol{\alpha}$, they follow some stationary processes such as ARMA(p,q). However, we may assume that they follow AR(1) processes because there is no theoretical difference between AR(1) and ARMA(p,q) processes except computational complexity and empirical studies (Lee, 1990; Yansaneh and Fuller, 1992) show that the first and second-order correlations have the typical correlation structure of AR(1) process. Thus, the $(i,j)^{th}$ elements of $\mathbf{V}_\mu$ and $\mathbf{V}_\gamma$ are $\rho_\mu^{|i-j|}$ and $\rho_\gamma^{|i-j|}$, respectively and $\rho_{\alpha,|i-j|} = \rho_\alpha^{|i-j|}$ for $\mathbf{V}_\alpha$.

Let $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1', \ldots, \boldsymbol{\gamma}_G')'$. Then, from (I) and (II), the joint distribution of $\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ is given by

$$
\begin{aligned}
f(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{\alpha}) \propto \exp\Bigg\{ &- \frac{1}{2\sigma_\mu^2}(\boldsymbol{\mu} - a\mathbf{1}_n)'\mathbf{V}_\mu^{-1}(\boldsymbol{\mu} - a\mathbf{1}_n) - \frac{1}{2\sigma_\gamma^2}\sum_{g=1}^{G} \boldsymbol{\gamma}_g'\mathbf{V}_\gamma^{-1}\boldsymbol{\gamma}_g \\
&- \frac{1}{2\sigma^2}\sum_{g=1}^{G} \left(\mathbf{x}_g - (\boldsymbol{\mu}+\boldsymbol{\gamma}_g+\mathcal{M}_g\boldsymbol{\alpha})\otimes\mathbf{1}_k\right)' \left(\mathbf{x}_g - (\boldsymbol{\mu}+\boldsymbol{\gamma}_g+\mathcal{M}_g\boldsymbol{\alpha})\otimes\mathbf{1}_k\right) \\
&- \frac{1}{2\sigma_\alpha^2}\sum_{g=1}^{G} \boldsymbol{\alpha}'\mathcal{M}_g'(\mathcal{M}_g\mathbf{V}_\alpha\mathcal{M}_g')^{-1}\mathcal{M}_g\boldsymbol{\alpha}\Bigg\}
\end{aligned}
$$

and it can be shown that the joint marginal distribution of $\mathbf{x}$ and $\boldsymbol{\mu}$

$$
\begin{aligned}
f(\mathbf{x}, \boldsymbol{\mu}) \propto \exp\Bigg\{ &- \frac{1}{2\sigma^2}\sum_{i=1}^{g} (\mathbf{x}_g - \boldsymbol{\mu}\otimes\mathbf{1}_k)'\,\mathcal{B}_g^{-1}\,(\mathbf{x}_g - \boldsymbol{\mu}\otimes\mathbf{1}_k) \\
&- \frac{1}{2\sigma_\mu^2}(\boldsymbol{\mu} - a\mathbf{1}_n)'\mathbf{V}_\mu^{-1}(\boldsymbol{\mu} - a\mathbf{1}_n)\Bigg\}
\end{aligned}
\tag{2.4}
$$

where $\lambda_\alpha = \sigma_\alpha^2/\sigma^2$, $\lambda_\gamma = \sigma_\gamma^2/\sigma^2$ and

$$
\mathcal{B}_g = \mathbf{I}_{nk} + \lambda_\alpha\mathcal{M}_g\mathbf{V}_\alpha\mathcal{M}_g' + \lambda_\gamma\mathbf{V}_\gamma\otimes\mathbf{J}_k.
$$

From (2.4) we can obtain following result.

LEMMA 2.1. *Under the model (I) and (II) the conditional distribution of* $\boldsymbol{\mu}$ *given* $\mathbf{x}$ *and the marginal distribution of* $\mathbf{x}$ *are multivariate normals with mean and variance-covariance,* $(\mathbf{E}(a\mathbf{V}_\mu^{-1}\mathbf{1}_n + \lambda_\mu\sum_g \mathbf{B}_g\bar{\mathbf{x}}_g), \sigma_\mu^2\mathbf{E})$ *and* $(a\mathbf{1}_{Gnk}, \sigma^2\mathcal{Q})$, *re-*

spectively, where $\lambda_\mu = \sigma_\mu^2/\sigma^2, \bar{\mathbf{x}}_g = (\bar{x}_{g1}, \ldots, \bar{x}_{gn})'$ *with* $\bar{x}_{gt} = k^{-1}\sum_{j=1}^k x_{gtj}$,

$$\mathbf{E} = \left(\mathbf{V}_\mu^{-1} + \lambda_\mu \sum_{g=1}^G \mathbf{B}_g\right)^{-1}, \quad \mathbf{B}_g = \frac{1}{k}\mathbf{I}_n + \lambda_\alpha \mathbf{M}_g \mathbf{V}_\alpha \mathbf{M}_g' + \lambda_\gamma \mathbf{V}_\gamma$$

and

$$\mathcal{Q} = \mathrm{diag}(\mathcal{B}_1, \ldots, \mathcal{B}_G) + \lambda_\mu \mathbf{J}_g \otimes (\mathbf{V}_\mu \otimes \mathbf{J}_k).$$

Let $\alpha_g(t)$ be the label of subunit in group g at time t. Then x_{gtk} and $x_{gt'k'}$ are the samples from a same subunit if and only if $\alpha_g(t) = \alpha_g(t')$. With this notation we can summarize following covariance structure of x_{gtj}'s from Lemma 2.1:

$$\mathrm{Cov}(x_{gtj}, \bar{x}_{g't'j'}) = \begin{cases} \sigma_\mu^2 \rho_\mu^{|t-t'|}, & \text{if } g \neq g', \\ \sigma_\gamma^2 \rho_\gamma^{|t-t'|} + \sigma_\mu^2 \rho_\mu^{|t-t'|}, & \text{if } g = g' \text{ but } \alpha_g(t) \neq \alpha_g(t'), \\ \sigma_\alpha^2 \rho_\alpha^{|t-t'|} + \sigma_\gamma^2 \rho_\gamma^{|t-t'|} + \sigma_\mu^2 \rho_\mu^{|t-t'|}, & \text{if } g = g', \alpha_g(t) = \alpha_g(t') \text{ but } k \neq k', \\ \sigma^2 + \sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\mu^2, & \text{if } g = g', t = t' \text{ and } k = k'. \end{cases} \tag{2.5}$$

In a frequentist point of view each group is independent and because of the independence we have that the sampled group or population give us no information about the unsampled group or population. In order to relate the unsampled group to the sampled group we need a prior which makes the groups dependent. This is somewhat related to the superpopulation approach. The model (I) and (II) introduce μ as a hyperparameter.

ρ_γ is the second-order correlation occurring from the same rotation group, while ρ_α account for the correlation due to the successive use of same subunit. Because samples from the same subunit always belong to the same rotation group, the correlation within subunit is the compound effects of rotation group and subunit. The first-order correlation, which is originally defined to these compound effects, can be explained by ρ_α in our model.

3. ESTIMATION

In a rotation sampling it might be a main focus to estimate the monthly levels or monthly changes. If we assume the squared error loss, it is well known that the conditional mean is the Bayes estimator. Thus the Bayes estimator of monthly levels $\boldsymbol{\mu}$ is $\mathbf{E}(a\mathbf{V}_\mu^{-1}\mathbf{1}_n + \lambda_\mu \sum_{g=1}^{G} \mathbf{B}_g^{-1}\bar{\mathbf{x}}_g)$. Also the Bayes estimator of a monthly change can be obtained easily from the Bayes estimator of $\boldsymbol{\mu}$. However, many unknown parameters are involved in the Bayes estimator of $\boldsymbol{\mu}$. As we mentioned earlier this leads us to rely on the EB method. The unknown variance components are estimated by the analysis of variance method or the method of moment. First we will have a consistent estimator of σ^2 by the following theorem.

THEOREM 3.1. *Under the model (I) and (II)* $\sum_g \sum_t \sum_j (x_{gtj} - \bar{x}_{gt})^2 / \sigma^2$ *follows a* χ^2*-distribution with* $Gn(k-1)$ *degrees of freedom.*

The proof of Theorem 3.1 is given in Appendix. By the theorem we see that

$$\text{MSE} = \frac{1}{Gn(k-1)} \sum_g \sum_t \sum_j (x_{gtk} - \bar{x}_{gt})^2$$

is a consistent estimator of σ^2.

Next the estimation problems of the other variance components and correlations will be considered. Since $\bar{\mathbf{x}}_g$'s are sufficient for the problems, we start with the marginal distribution of $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_1', \bar{\mathbf{x}}_2', \ldots, \bar{\mathbf{x}}_G')'$. Because Lemma 3.1 is an obvious extension of Lemma 2.1, the proof will be omitted.

LEMMA 3.1. *Under the model (I) and (II) the marginal distribution of* $\bar{\mathbf{x}}$ *is a multivariate normal with mean* $a\mathbf{1}_{Gn}$ *and variance-covariance matrix* $\sigma^2\mathbf{Q}$ *where* $\mathbf{Q} = \mathbf{D} + \lambda_\mu \mathbf{J}_G \otimes \mathbf{V}_u$ *and* $\mathbf{D} = \text{diag}(\mathbf{B}_1, \ldots, \mathbf{B}_G)$.

THEOREM 3.2. *Let* $\mathbf{R}$ *is a quadratic form of* $\bar{x}_{gt}$*'s, then* $\mathbf{R}$ *is independent of* MSE.

To obtain estimator of variance components, various methods can be applied. For example, Datta and Ghosh (1991) used Henderson's fitting constant method for their mixed model. Although Henderson's fitting constant method gives unbiased estimators for mixed models, it is required to calculate generalized inverses of very large matrices. The complication of model (I) and (II) might cause difficulties in calculating not only reductions in sums of squares but also coefficients of the variance components and correlations. If we note that the suggested Bayes model is not a mixed model, we have a plenty of choice for employing analysis of variance type estimation. Because we should estimates various correlations as well as variance components, we believe that a method of estimating variance components based on symmetric sums of products of observations, rather than sums of squares, which was suggested by Koch (1967, 1968) is appropriate. Koch's method uses the fact that expected values of products of observations are linear functions of the variance components and correlations. Even though there are infinitely many quadratic forms that can be used in the manner of Koch's method and it gives no criteria for selecting the quadratic forms to be used, this procedure is widely used in the context of variance component estimation.

Noting (2.5), we might equate observed values of following symmetric sums of product to their expected values and solve the resulting equations to get estimators of the variance components and correlations:

$$
\begin{aligned}
\mathrm{T_S} &= \frac{1}{nG}\sum_{g=1}^{G}\sum_{t=1}^{n}\bar{x}_{gt}^{2}, \\
\mathrm{T_M} &= \frac{1}{G(G-1)}\sum_{g\neq g'}^{G}\bar{x}_{g}\bar{x}_{g'}, \\
\mathrm{T_{BG}} &= \frac{1}{nG(G-1)}\sum_{g\neq g'}^{G}\sum_{t=1}^{n}\bar{x}_{gt}\bar{x}_{g't}, \\
\mathrm{T_{BGL}} &= \frac{1}{(n-1)G(G-1)}\sum_{g\neq g'}^{G}\sum_{t=1}^{n-1}\bar{x}_{gt}\bar{x}_{g't+1}, \qquad (3.1)\\
\mathrm{T_{BSL1}} &= \frac{1}{L_1}\sum_{g=1}^{G}\sum_{t=1}^{n-1}\bar{x}_{gt}\bar{x}_{gt+1}I_g(t,t+1), \\
\mathrm{T_{BSL2}} &= \frac{1}{L_2}\sum_{g=1}^{G}\sum_{t=1}^{n-2}\bar{x}_{gt}\bar{x}_{gt+2}I_i(t,t+2), \\
\mathrm{T_{WSL}} &= \frac{1}{M}\sum_{g=1}^{G}\sum_{t=1}^{n-1}\bar{x}_{gt}\bar{x}_{gt+1}(1-I_i(t,t+1)),
\end{aligned}
$$

where $L_p = \sum_{g=1}^{G}\sum_{t=1}^{n-p} I_g(t,t+p)$ for $p=1,2$, $M = \sum_{g=1}^{G}\sum_{t=1}^{n-1}(1-I_g(t,t+p)) = G(n-1) - L_1$ and $I_i(t,s)$ is an indicator function defined by $I_g(t,s) = 1$, if $\alpha_g(t) \neq \alpha_g(s)$, and $I_g(t,s) = 0$, otherwise. It is, however, possible to use other symmetric sums of products.

Next we shall see how T's in (3.1) can be used in obtaining consistent estimators of various variance components and correlations. The following lemma gives the convergence result of those symmetric sums of products given in (3.1).

LEMMA 3.2. *Consider the model given in (I) and (II). Then each symmetric* *sums of products given in (3.1) converge in probability to their exact or asymptotic means,* a^2, $\sigma_\mu^2 + a^2$, $\sigma_\mu^2\rho_\mu + a^2$, $\sigma_\gamma^2\rho_\gamma + \sigma_\mu^2\rho_\mu + a^2$, $\sigma_\gamma^2\rho_\gamma^2 + \sigma_\mu^2\rho_\mu^2 + a^2$, $\sigma_\alpha^2\rho_\alpha + \sigma_\gamma^2\rho_\gamma + \sigma_\mu^2\rho_\mu + a^2$ *and* $\sigma^2/k + \sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\mu^2 + a^2$, *respectively.*

Now we equate T's in (3.1) to their expectations. Let

$$\mathrm{MS_{BG}} = \mathrm{T_{BG}} - \mathrm{T_M} = \frac{1}{nG(G-1)} \sum_{g \neq g'}^{G} \sum_{t=1}^{n} (\bar{x}_{gt} - \bar{x}_g)\left(\bar{x}_{g't} - \bar{x}_{g'}\right)$$

$\mathrm{MS_{BG}}$ is a cross covariance analogous to (6.5.2) in Fuller (1976) and by Lemma 3.2, $\mathrm{MS_{BG}}$ converges in probability to σ_μ^2. Hence, consistent estimators of σ_μ^2 and λ_μ are given by

$$\widehat{\sigma}_\mu^2 = \max(0, \mathrm{MS_{BG}}) \quad \text{and} \quad \widehat{\lambda}_\mu = \frac{\widehat{\sigma}_\mu^2}{\mathrm{MSE}}.$$

Also $(\mathrm{T_{BGL}} - \mathrm{T_M})/\mathrm{MS_{BG}}$ converges in probability to ρ_μ. However, $\mathrm{T_{BGL}} - \mathrm{T_M} = \mathrm{MS_{BGL}} + o_p(n^{-1})$ where

$$\mathrm{MS_{BGL}} = \frac{1}{nG(G-1)} \sum_{g \neq g'}^{G} \sum_{t=1}^{n-1} (\bar{x}_{gt} - \bar{x}_g)(\bar{x}_{g't+1} - \bar{x}_{g'}).$$

$\mathrm{MS_{BGL}}/\mathrm{MS_{BG}}$ and $(\mathrm{T_{BGL}} - \mathrm{T_M})/\mathrm{MS_{BG}}$ differ only a negligible term and have a common limit. It is preferable to adopt $\mathrm{MS_{BGL}}$ instead of $\mathrm{T_{BGL}} - \mathrm{T_M}$ because $\mathrm{MS_{BGL}}$ is a cross covariance analogous. (See also p.236 of Fuller, 1976). A consistent estimator of ρ_μ is given by

$$\widehat{\rho}_\mu = \begin{cases} \dfrac{\mathrm{MS_{BGL}}}{\mathrm{MS_{BG}}}, & \text{if } \mathrm{MS_{BG}} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The other estimations are rather complicate. For example, the estimators of σ_γ^2 and σ_α^2, suggested by Koch, are not linear functions of T's, and they are not unbiased. However, Koch's method still gives consistent estimators. It also noted that Koch's method suggests a little bit strange quantity $\mathrm{T_S} - \mathrm{T_M}$. Although it is a consistent estimator of $\sigma^2/k + \sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\mu^2$ and is necessary to estimate σ_α^2, $\mathrm{T_M}$ in $\mathrm{T_S} - \mathrm{T_M}$ can be replaced safely by $G^{-1}\sum_{g=1}^{G} \bar{x}_g^2$ to retain consistency. It also guarantees the positive estimation of $\sigma^2/k + \sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\mu^2$ and reduce variance slightly. Hence it is desirable to estimate $\sigma^2/k + \sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\mu^2$ by

$$\mathrm{MS_{WG}} = \frac{1}{nG} \sum_{g=1}^{G} \sum_{t=1}^{n} (\bar{x}_{gt} - \bar{x}_g)^2.$$

Accounting for negative estimation of a variance, consistent estimators of $\sigma_\gamma^2, \rho_\gamma, \sigma_\alpha^2$ and ρ_α can be obtained sequentially as follow:

$$\widehat{\sigma}_\gamma^2 = \max\left(0,\ \frac{(\mathrm{T}_{\mathrm{BSL1}} - \mathrm{T}_{\mathrm{BGL}})^2}{\mathrm{T}_{\mathrm{BSL2}} - \mathrm{T}_{\mathrm{BGL}} + \widehat{\sigma}_\mu^2 \widehat{\rho}_\mu (1 - \widehat{\rho}_\mu)}\right),$$

$$\widehat{\rho}_\gamma = \begin{cases} \dfrac{\mathrm{T}_{\mathrm{BSL1}} - \mathrm{T}_{\mathrm{BGL}}}{\widehat{\sigma}_\gamma^2}, & \text{if } \widehat{\sigma}_\gamma^2 \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\widehat{\sigma}_\alpha^2 = \max\left(0,\ \mathrm{MS}_{\mathrm{WG}} - \frac{1}{k}\mathrm{MSE} - \widehat{\sigma}_\gamma^2 - \widehat{\sigma}_\mu^2\right),$$

$$\widehat{\rho}_\alpha = \begin{cases} \dfrac{\mathrm{T}_{\mathrm{WSL}} - \mathrm{T}_{\mathrm{BSL1}}}{\widehat{\sigma}_\alpha^2}, & \text{if } \widehat{\sigma}_\alpha^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence consistent estimators of λ_γ and λ_α are given by

$$\widehat{\lambda}_\gamma = \frac{\widehat{\sigma}_\gamma^2}{\mathrm{MSE}},$$
$$\widehat{\lambda}_\alpha = \frac{\widehat{\sigma}_\alpha^2}{\mathrm{MSE}} = \frac{\mathrm{MS}_{\mathrm{WG}}}{\mathrm{MSE}} - \widehat{\lambda}_\gamma - \widehat{\lambda}_\mu - \frac{1}{k}.$$

Throughout Section 3, we have obtained consistent estimators of various variance components and correlations. To estimate a based on the marginal distribution of the $\bar{x}_{gt}$, first $\mathbf{Q}$ would be estimated by substituting each λ's and ρ's in $\mathbf{Q}$ for their estimators, then applies the maximum likelihood method, which will give

$$\widehat{a} = \frac{\mathbf{1}'_{Gn}\widehat{\mathbf{Q}}^{-1}\bar{\mathbf{x}}}{\mathbf{1}'_{Gn}\widehat{\mathbf{Q}}^{-1}\mathbf{1}_{Gn}}.$$

Substituting the estimators $\widehat{a}, \widehat{\mathbf{E}}, \widehat{\mathbf{V}}_\mu$ and $\widehat{\mathbf{B}}_g$'s respectively for $a, \mathbf{E}, \mathbf{V}_\mu$ and $\mathbf{B}_g$'s, it follows that an EB estimator of monthly level μ is given by

$$\widehat{\mu} = \widehat{\mathbf{E}}\Big(\widehat{a}\widehat{\mathbf{V}}_\mu^{-1}\mathbf{1}_n + \lambda_\mu \sum_g^G \widehat{\mathbf{B}}_g^{-1}\bar{\mathbf{x}}_g\Big) \tag{3.2}$$

4. Conclusion

The procedure for (3.2) is widely accepted. See, for example, Ghosh and Meeden (1997). However, it is conceivable to think of procedures alternative to the proposed method of EB estimation. A natural candidate is an EB estimator based on MLE's of a, λ's and ρ's. It should be noted that the maximum likelihood equations do not admit any closed-form solutions. Also the proposed EB estimator is based on the consistent estimators, the large sample property would not differ much from that of the estimator based on MLE. Because the rotation sampling is used usually for long-terms, we believe, the proposed EB estimator would be competitive to the MLE based estimator.

One might criticize the assumption that the sample size k is the same for all subunits may not be realistic in practice. Note, however, the intra-subunit variance σ^2 may differ from each subunit, and it is natural to adjust the sample size so that $\bar{x}_{gt}$ have the same intra-subunit variance. In fact, the GCE or the MVLUE(Minimum Variance Linear Unbiased Estimator) is obtained under assumptions that the intra-subunit variance $\bar{x}_{gt}$ is the same, and the value is known, or can be estimated by other device. With these assumptions, $\bar{\mathbf{x}}$ is still sufficient statistic. A slight modification of formulas can enable the results to be usable.

Appendix

Proof of Theorem 3.1. Let $\mathbf{D}_{Gn}(\mathbf{J}_k) = \mathbf{I}_{Gn} \otimes \mathbf{J}_k$. Then it is easy to check that

$$\left(\mathbf{I}_{Gnk} - \frac{1}{k}\mathbf{D}_{Gn}(\mathbf{J}_k)\right) \mathcal{Q} = \mathbf{I}_{Gnk} - \frac{1}{k}\mathbf{D}_{Gn}(\mathbf{J}_k) \tag{5.1}$$

is an idempotent matrix. Also we have $\mathbf{1}'_{Gnk}(\mathbf{I}_{Gnk} - k^{-1}\mathbf{D}_{Gn}(\mathbf{J}_k))\mathbf{1}_{Gnk} = \mathbf{0}$. Thus $\sum_{g,t,j} (x_{gtj} - \bar{x}_{gt})^2 / \sigma^2 = \mathbf{x}'(\mathbf{I} - k^{-1}\mathbf{D}_{ng}(\mathbf{J}_k))\mathbf{x}/\sigma^2$ is a χ^2 random variable with $\text{rank}(\mathbf{I}_{Gnk} - k^{-1}\mathbf{D}_{Gn}(\mathbf{J}_k)) = Gn(k-1)$ degrees of freedom. □

Proof of Theorem 3.2. By (5.1), a quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ is independent of MSE if and only if

$$\left(\mathbf{I}_{Gnk} - \frac{1}{k}\mathbf{D}_{Gn}(\mathbf{J}_k)\right) \mathcal{Q}\mathbf{A} = \left(\mathbf{I}_{Gnk} - \frac{1}{k}\mathbf{D}_{Gn}(\mathbf{J}_k)\right) \mathbf{A} = \mathbf{0}.$$

Note that $\mathbf{I}_{Gnk} - k^{-1}\mathbf{D}_{Gn}(\mathbf{J}_k)$ is the orthogonal projection matrix of the column space of $\mathbf{D}_{Gn}(\mathbf{1}_k) = \mathbf{I}_{Gn} \otimes \mathbf{1}_k$. Thus $\mathbf{x}'\mathbf{A}\mathbf{x}$ is independent of MSE if and only if

$$\text{col}(\mathbf{A}) \subset \text{col}\left(\mathbf{D}_{Gn}(\mathbf{1}_k)\right)$$

where col($\mathbf{A}$) denotes the column space of $\mathbf{A}$. Since $\bar{\mathbf{x}} = k^{-1}\mathbf{D}_{Gn}(\mathbf{1}_k)'\mathbf{x}$, a quadratic form $\bar{\mathbf{x}}'\mathbf{B}\bar{\mathbf{x}}$ of $\bar{\mathbf{x}}$ can be written as $\bar{\mathbf{x}}'\mathbf{B}\bar{\mathbf{x}} = k^{-2}\mathbf{x}'\mathbf{D}_{Gn}(\mathbf{1}_k)\mathbf{A}\mathbf{D}_{Gn}(\mathbf{1}_k)'\mathbf{x}$

and

$$\operatorname{col}\left(\mathbf{D}_{Gn}(\mathbf{1}_k)\mathbf{B}\mathbf{D}_{Gn}(\mathbf{1}_k)'\right) \subset \operatorname{col}\left(\mathbf{D}_{Gn}(\mathbf{1}_k)\right).$$

We see that a quadratic form of $\bar{x}_{gt}$'s is independent of MSE. □

The following results are useful to prove subsequent theorems and lemmas.

LEMMA 5.1. *Under the model (I) and (II),*

1. $\mathbf{1}_n'\mathbf{V}_\gamma\mathbf{1}_n = O(n), \quad \operatorname{tr}(\mathbf{V}_\gamma^2) = O(n), \quad \operatorname{tr}(\mathbf{V}_\gamma\mathbf{V}_\mu) = O(n).$
2. $\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n = O(n), \quad \operatorname{tr}(\mathbf{V}_\mu^2) = O(n).$
3. $\mathbf{1}_n'\mathbf{B}_g\mathbf{1}_n = O(n), \quad \mathbf{1}_n'\mathbf{B}_g\mathbf{B}_{g'}\mathbf{1}_n = O(n).$
4. $\operatorname{tr}(\mathbf{B}_g\mathbf{B}_{g'}) = O(n)$, $\operatorname{tr}(\mathbf{B}_g\mathbf{V}_\mu) = O(n)$.

PROOF. Since $\mathbf{1}_n'\mathbf{V}_\gamma\mathbf{1}_n = n(1+\rho_\gamma)/(1-\rho_\gamma) - 2\rho_\gamma(1-\rho_\gamma^n)/(1-\rho_\gamma)$, $\operatorname{tr}(\mathbf{V}_\gamma^2) = n(1+\rho_\gamma^2)/(1-\rho_\gamma^2) - 2\rho_\gamma^2(1-\rho_\gamma^{2n})/(1-\rho_\gamma^2)$, $\operatorname{tr}(\mathbf{V}_\gamma\mathbf{V}_\mu) = n(1+\rho_\gamma\rho_\mu)/(1-\rho_\gamma\rho_\mu) - 2\rho_\gamma\rho_\mu(1-\rho_\gamma^n\rho_\mu^n)/(1-\rho_\gamma\rho_\mu)$, and $\mathbf{V}_\gamma$ and $\mathbf{V}_\mu$ have the same structure, the first and the second assertions are obvious.

Next consider $\mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g'$ in $\mathbf{B}_g$. It should be noted that in a 2-way balanced $r_1^m - r_2^{m-1}$ design each column and each row of $\mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g'$ has at most mr_1 nonzero elements and the absolute value of each nonzero element is less than or equal to 1. Thus each elements of $\mathbf{B}_g\mathbf{1}_n$ as well as $\mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g'\mathbf{1}_n$ is $O(1)$ and we have

$$\mathbf{1}_n'\mathbf{B}_g\mathbf{1}_n = O(n) \quad \text{and} \quad \mathbf{1}_n'\mathbf{B}_g\mathbf{B}_{g'}\mathbf{1}_n = O(n).$$

It remains to show that $\operatorname{tr}(\mathbf{B}_{g'}\mathbf{B}_g) = O(n)$ and $\operatorname{tr}(\mathbf{B}_g\mathbf{V}_\mu) = O(n)$. Note that $\operatorname{tr}(\mathbf{M}_{g'}\mathbf{V}_\alpha\mathbf{M}_{g'}'\mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g') = O(n)$, $\operatorname{tr}(\mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g'\mathbf{V}_\gamma) = O(n)$ and $\operatorname{tr}(\mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g'\mathbf{V}_\mu) = O(n)$. These also follow from the fact that each columns and rows of $\mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g'$ has at most r_1m nonzero elements. Hence

$$\begin{aligned}
\operatorname{tr}\left(\mathbf{B}_{g'}\mathbf{B}_g\right) &= \operatorname{tr}\left[\frac{1}{k^2}\mathbf{I}_n + \frac{\lambda_\alpha}{k}\left(\mathbf{M}_{g'}\mathbf{V}_\alpha\mathbf{M}_{g'}' + \mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g'\right) + 2\frac{\lambda_\gamma}{k}\mathbf{V}_\gamma\right] \\
&\quad + \operatorname{tr}\left[\lambda_\alpha^2\mathbf{M}_{g'}\mathbf{V}_\alpha\mathbf{M}_{g'}'\mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g'\right. \\
&\quad \left. + \lambda_\alpha\lambda_\gamma\left(\mathbf{M}_{g'}\mathbf{V}_\alpha\mathbf{M}_{g'}' + \mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g'\right)\mathbf{V}_\gamma + \lambda_\gamma^2\mathbf{V}_\gamma^2\right] \\
&= n\Big(\frac{1}{k^2} + 2\frac{\lambda_\alpha}{k} + 2\frac{\lambda_\gamma}{k}\Big) + O(n) \\
&= O(n),
\end{aligned}$$

$$\begin{aligned}\operatorname{tr}(\mathbf{B}_g\mathbf{V}_\mu) &= \operatorname{tr}\left[\frac{1}{k}\mathbf{V}_\mu + \lambda_\mu\mathbf{M}_g\mathbf{V}_\alpha\mathbf{M}_g'\mathbf{V}_\mu + \lambda_\alpha\lambda_\mu\mathbf{V}_\gamma\mathbf{V}_\mu\right] \\ &= O(n).\end{aligned}$$

□

PROOF OF LEMMA 3.2. Consider the convergence of $\mathrm{T_M}$. Using Corollary 1.1, Corollary 1.2 of Searl (1971) and previous results, it follows that

$$\begin{aligned}&E(\bar{x}_g\bar{x}_{g'}) \\ &= \frac{\sigma^2}{n^2}\operatorname{tr}\left[\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}\mathbf{1}_n'(\mathbf{B}_g+\lambda_\mu\mathbf{V}_\mu)\mathbf{1}_n & \lambda_\mu\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n\\ \lambda_\mu\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n & \mathbf{1}_n'(\mathbf{B}_{g'}+\lambda_\mu\mathbf{V}_\mu)\mathbf{1}_n\end{pmatrix}\right] + a^2\mathbf{1}_2'\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\mathbf{1}_2 \\ &= \frac{\sigma_\mu^2}{n^2}\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n + a^2 \\ &= a^2 + O(n^{-1}),\end{aligned}$$

$$\begin{aligned}&\operatorname{Var}(\bar{x}_g'\bar{x}_{g'}) \\ &= 2\frac{\sigma^4}{n^4}\operatorname{tr}\left[\left\{\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}\mathbf{1}_n'(\mathbf{B}_g+\lambda_\mu\mathbf{V}_\mu)\mathbf{1}_n & \lambda_\mu\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n\\ \lambda_\mu\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n & \mathbf{1}_n'(\mathbf{B}_{g'}+\lambda_\mu\mathbf{V}_\mu)\mathbf{1}_n\end{pmatrix}\right\}^2\right] \\ &\quad + 4\frac{a^2\sigma^2}{n^2}\mathbf{1}_2'\begin{pmatrix}\mathbf{0} & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}\mathbf{1}_n'(\mathbf{B}_g+\lambda_\mu\mathbf{V}_\mu)\mathbf{1}_n & \lambda_\mu\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n\\ \lambda_\mu\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n & \mathbf{1}_n'(\mathbf{B}_{g'}+\lambda_\mu\mathbf{V}_\mu)\mathbf{1}_n\end{pmatrix}\begin{pmatrix}\mathbf{0} & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\mathbf{1}_2 \\ &= \frac{\sigma^4}{n^4}\left\{\mathbf{1}_n'\mathbf{B}_g\mathbf{1}_n\mathbf{1}_n'\mathbf{B}_{g'}\mathbf{1}_n + \lambda_\mu\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n\left(\mathbf{1}_n'\mathbf{B}_{g'}\mathbf{1}_n + \mathbf{1}_n'\mathbf{B}_g\mathbf{1}_n\right) + 2\lambda^2(\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n)^2\right\} \\ &\quad + \frac{a^2\sigma^2}{n^2}\left(\mathbf{1}_n'\mathbf{B}_g\mathbf{1}_n + \mathbf{1}_n'\mathbf{B}_{g'}\mathbf{1}_n + 4\lambda_\mu\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n\right) \\ &= \frac{O(n^2)}{n^4} + \frac{O(n)}{n^2} \\ &= O(n^{-1}).\end{aligned}$$

These show that $\bar{x}_g\bar{x}_{g'} \to a^2$ in probability for every $g \neq g'$, and hence $\mathrm{T_M} \to a^2$ as $n \to \infty$.

Next it is easy to check that

$$\begin{aligned}E(\mathrm{T_{BG}}) &= \sigma_\mu^2 + a^2, & E(\mathrm{T_{BGL}}) &= \sigma_\mu^2\rho_\mu + a^2, \\ E(\mathrm{T_{BSL1}}) &= \sigma_\gamma^2\rho_\gamma + \sigma_\mu^2\rho_\mu + a^2, & E(\mathrm{T_{BSL2}}) &= \sigma_\gamma^2\rho_\gamma^2 + \sigma_\mu^2\rho_\mu^2 + a^2, \\ E(\mathrm{T_{WSL}}) &= \sigma_\alpha^2\rho_\alpha + \sigma_\gamma^2\rho_\gamma + \sigma_\mu^2\rho_\mu + a^2, & E(\mathrm{T_S}) &= \sigma^2/k + \sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\mu^2 + a^2.\end{aligned}$$

Thus it remains to show that each T's converges to its expected value.

As before, to prove the convergence of $\mathrm{T_{BG}}$, it suffices to show that the variance of $n^{-1}\sum_{t=1}^{n}\bar{x}_{gt}\bar{x}_{g't}$ has order $O(n^{-1})$ for every $g \neq g'$. Since

$$\begin{aligned}
&\mathrm{Var}(\bar{\mathbf{x}}_g'\bar{\mathbf{x}}_{g'}) \\
&= 2\sigma^4 \mathrm{tr}\left[\left\{\begin{pmatrix}\mathbf{0} & \frac{1}{2}\mathbf{I}_n\\ \frac{1}{2}\mathbf{I}_n & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{B}_i+\lambda_\mu\mathbf{V}_\mu & \lambda_\mu\mathbf{V}_\mu\\ \lambda_\mu\mathbf{V}_\mu & \mathbf{B}_{g'}+\lambda_\mu\mathbf{V}_\mu\end{pmatrix}\right\}^2\right] \\
&\quad + 4\sigma^2a^2\mathbf{1}_{2n}'\begin{pmatrix}\mathbf{0} & \frac{1}{2}\mathbf{I}_n\\ \frac{1}{2}\mathbf{I}_n & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{B}_g+\lambda_\mu\mathbf{V}_\mu & \lambda_\mu\mathbf{V}_\mu\\ \lambda_\mu\mathbf{V}_\mu & \mathbf{B}_{g'}+\lambda_\mu\mathbf{V}_\mu\end{pmatrix}\begin{pmatrix}\mathbf{0} & \frac{1}{2}\mathbf{I}_n\\ \frac{1}{2}\mathbf{I}_n & \mathbf{0}\end{pmatrix}\mathbf{1}_{2n} \\
&= \sigma^4\mathrm{tr}\left(\mathbf{B}_g\mathbf{B}_{g'}+\lambda_\mu\mathbf{B}_g\mathbf{V}_\mu+\lambda_\mu\mathbf{B}_{g'}\mathbf{V}_\mu+2\mathbf{V}_\mu^2\right) \\
&\quad +\sigma^2a^2\left(\mathbf{1}_n'\mathbf{B}_g\mathbf{1}_n+\mathbf{1}_n'\mathbf{B}_{g'}\mathbf{1}_n+4\lambda_\mu\mathbf{1}_n'\mathbf{V}_\mu\mathbf{1}_n\right) \\
&= O(n),
\end{aligned}$$

we see that $\mathrm{Var}\left(n^{-1}\sum_{t=1}^{n}\bar{x}_{gt}\bar{x}_{g't}\right)=\mathrm{Var}\left(n^{-1}\bar{\mathbf{x}}_g'\bar{\mathbf{x}}_{g'}\right)=O(n^{-1})$. Thus $\mathrm{T_{BG}}$ as well as $n^{-1}\sum_{t=1}^{n}\bar{x}_{gt}\bar{x}_{g't}$ converge in probability to $\sigma_\mu^2\rho_\mu+a^2$. Let $\mathbf{L}$ be a $n\times n$ matrix such that

$$\mathbf{L}=\begin{pmatrix}\mathbf{0} & \mathbf{I}_{n-1}\\ 0 & \mathbf{0}'\end{pmatrix}.$$

Then

$$\begin{aligned}
&\mathrm{Var}\left(\frac{1}{n}\sum_{t=1}^{n-1}\bar{x}_{gt}\bar{x}_{g't+1}\right) \\
&= 2\frac{\sigma^4}{n^2}\mathrm{tr}\left[\left\{\begin{pmatrix}\mathbf{0} & \frac{1}{2}\mathbf{L}\\ \frac{1}{2}\mathbf{L}' & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{B}_g+\lambda_\mu\mathbf{V}_\mu & \lambda_\mu\mathbf{V}_\mu\\ \mathbf{V}_\mu & \mathbf{B}_{g'}+\lambda_\mu\mathbf{V}_\mu\end{pmatrix}\right\}^2\right] \\
&\quad + 4\frac{a^2\sigma^2}{n^2}\mathbf{1}_{2n}'\left[\begin{pmatrix}\mathbf{0} & \frac{1}{2}\mathbf{L}\\ \frac{1}{2}\mathbf{L}' & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{B}_g+\lambda_\mu\mathbf{V}_\mu & \lambda_\mu\mathbf{V}_\mu\\ \mathbf{V}_\mu & \mathbf{B}_{g'}+\lambda_\mu\mathbf{V}_\mu\end{pmatrix}\begin{pmatrix}\mathbf{0} & \frac{1}{2}\mathbf{L}\\ \frac{1}{2}\mathbf{L}' & \mathbf{0}\end{pmatrix}\right]\mathbf{1}_{2n} \\
&= \frac{\sigma^4}{n^2}\left[\mathrm{tr}(\mathbf{L}'\mathbf{B}_g\mathbf{L}\,\mathbf{B}_{g'})+\lambda_\mu\mathrm{tr}(\mathbf{L}'\mathbf{B}_g\mathbf{L}\mathbf{V}_\mu)+\lambda_\mu\mathrm{tr}(\mathbf{L}'\mathbf{V}_\mu\mathbf{L}\mathbf{B}_{g'})\right. \\
&\quad \left.+\lambda_\mu^2\mathrm{tr}(\mathbf{L}\mathbf{V}_\mu\mathbf{L}\mathbf{V}_\mu)+\lambda_\mu^2\mathrm{tr}(\mathbf{L}'\mathbf{V}_\mu\mathbf{L}\mathbf{V}_\mu)\right] \\
&\quad + \frac{a^2\sigma^2}{n^2}\left[\mathbf{1}'\mathbf{L}\,\mathbf{B}_{g'}\mathbf{L}'\mathbf{1}+\mathbf{1}'\mathbf{L}'\mathbf{B}_g\mathbf{L}\mathbf{1}\right. \\
&\quad \left.+\lambda_\mu\left\{\mathbf{1}'\mathbf{L}\mathbf{V}_\mu\mathbf{L}'\mathbf{1}+2\mathbf{1}'\mathbf{L}\mathbf{V}_\mu\mathbf{L}\mathbf{1}+\mathbf{1}'\mathbf{L}'\mathbf{V}_\mu\mathbf{L}\,\mathbf{1}\right\}\right]
\end{aligned}$$

$$= \frac{\sigma^4}{n^2}\left[\mathrm{tr}(\mathbf{B}_g^{11}\mathbf{B}_{g'}^{22}) + \lambda_\mu \mathrm{tr}(\mathbf{B}_g^{11}\mathbf{V}_\mu^{22}) + \lambda_\mu \mathrm{tr}(\mathbf{V}_\mu^{11}\mathbf{B}_{g'}^{22}) \right.$$
$$\left. +\lambda_\mu^2 \mathrm{tr}(\mathbf{V}_\mu^{21}\mathbf{V}_\mu^{21}) + \lambda_\mu^2 \mathrm{tr}(\mathbf{V}_\mu^{11}\mathbf{V}_\mu^{11})\right]$$
$$+ \frac{a^2\sigma^2}{n^2}\left[\mathbf{1}_{n-1}'\mathbf{B}_{g'}^{22}\mathbf{1}_{n-1} + \mathbf{1}_{n-1}'\mathbf{B}_g^{11}\mathbf{1}_{n-1}\right.$$
$$\left. +2\lambda_\mu \left\{\mathbf{1}_{n-1}'\mathbf{V}_\mu^{11}\mathbf{1}_{n-1} + \mathbf{1}_{n-1}'\mathbf{V}_\mu^{21}\mathbf{1}_{n-1}\right\}\right]$$

where $\mathbf{C}^{11}, \mathbf{C}^{22}$ and $\mathbf{C}^{21}$ represent the upper-left, the lower-right, and the lower-left $(n-1)\times(n-1)$ matrices of $\mathbf{C}$, respectively. It can be shown by a similar argument given in the proof of Lemma 5.1 that each term in the brackets of the above equation is $O(n)$. Thus we have

$$\mathrm{Var}\left(\frac{1}{n}\sum_{t=1}^{n-1}\bar{x}_{gt}\bar{x}_{g't+1}\right) = O(n^{-1}).$$

This shows that for every $g \neq g'$, $n^{-1}\sum_{t=1}^{n-1}\bar{x}_{gt}\bar{x}_{g't+1}$ converges in probability to $\sigma_\mu^2\rho_\mu + a^2$. Therefore $\mathrm{T_{BGL}}$ converges in probability to $\sigma_\mu^2\rho_\mu + a^2$.

Next, to prove the convergence of $\mathrm{T_{BSL1}}$, we first show that, for each $g = 1,2,\ldots,G$,

$$\frac{1}{L_g}\sum_{t=1}^{n-1}\bar{x}_{gt}\bar{x}_{gt+1}I_g(t,t+1) \tag{5.2}$$

where $L_g = \sum_{t=1}^{n-1} I_g(t,t+1)$, converges in probability to its mean, but it can be done by showing that, for every $i = 1,2,\ldots,g$,

$$\sum_{t=1}^{n-1}\sum_{t'>t}\mathrm{Cov}\left(\bar{x}_{gt}\bar{x}_{gt+1}, \bar{x}_{gt'}\bar{x}_{gt'+1}\right) I_g(t,t+1)I_g(t',t'+1) = O(L_g), \tag{5.3}$$

because (5.3) is the dominant part of $\mathrm{Var}(\sum_{t=1}^{n-1}\bar{x}_{gt}\bar{x}_{gt+1}I_g(t,t+1))$.

Let S_g be the set of all time t such that $\alpha_g(t) \neq \alpha_g(t+1)$. That is, S_g is the set of all t such that a subunit is replaced between time t and $t+1$ in group g. Suppose that t and t' are the elements of S_g. According to the nature of 2-way balanced $r_1^m - r_2^{m-1}$ design, we should consider five cases, (i) $\alpha_g(t) = \alpha_g(t')$, (ii) $\alpha_g(t) = \alpha_g(t'+1)$, (iii) $\alpha_g(t+1) = \alpha_g(t')$, (iv) $\alpha_g(t+1) = \alpha_g(t'+1)$ and (v) none of the above 4 cases, to obtain the covariance of $\bar{x}_{gt}\bar{x}_{gt+1}$ and $\bar{x}_{gt'}\bar{x}_{gt'+1}$. For example if (t, t') satisfies the condition (v), and $t' > t$, then

$$\begin{aligned}&\mathrm{Cov}(\bar{x}_{gt}\bar{x}_{gt+1}, \bar{x}_{gt'}\bar{x}_{gt'+1}) \\ &= \sigma^4\left[\lambda^2(t'-t) + \lambda(t'-t-1)\lambda(t'+1-t)\right] \\ &\quad + a^2\sigma^2\left[2\lambda(t'-t) + \lambda(t'-t-1) + \lambda(t'+1-t)\right]\end{aligned} \tag{5.4}$$

where $\lambda(t) = \lambda_\gamma \rho_\gamma^t + \lambda_\mu \rho_\mu^t$. The other conditions produce a slightly different covariance formula. However, if $t' - t$ is sufficiently large, the pair (t, t') satisfies the condition (v), *i.e.,* there are only finite number of pairs (t, t') which does not satisfy the condition (v). Thus for each fixed $t \in S_g$,

$$\begin{aligned}&\sum_{t'\in S_g, t'>t} \mathrm{Cov}(\bar{x}_{gt}\bar{x}_{gt+1}, \bar{x}_{gt'}\bar{x}_{gt'+1}) \\ &= \sum_{t'\in S_g, t'>t} \Big[\sigma^4\left\{\lambda^2(t'-t) + \lambda(t'-t-1)\lambda(t'+1-t)\right\} \\ &\quad + a^2\sigma^2\left\{2\lambda(t'-t) + \lambda(t'-t-1) + \lambda(t'+1-t)\right\}\Big] + O(1).\end{aligned}$$

Note that if $t, t' \in S_g$ and $t' > t$, then $t' = t + r_1 l$ for some $l = 1, 2, \ldots$ Thus writing $\sigma_{\max} = \max(\sigma_\gamma, \sigma_\mu)$ and $\rho_{\max} = \max(|\rho_\gamma|, |\rho_\mu|)$, we have

$$\begin{aligned}&\Big|\sum_{t'\in S_g, t'>t} \mathrm{Cov}(\bar{x}_{gt}\bar{x}_{gt+1}, \bar{x}_{gt'}\bar{x}_{gt'+1})\Big| \\ &\leq \sum_{t'\in S_g, t'>t} \left[4\sigma_{\max}^4 \rho_{\max}^{2(t'-t)} + 2a^2\sigma_{\max}^2\left\{2\rho_{\max}^{t'-t} + \rho_{\max}^{t'-t-1} + \rho_{\max}^{t'+1-t}\right\}\right] + O(1) \\ &\leq \sum_{l=1}^{\infty} \left[4\sigma_{\max}^4 \rho_{\max}^{2r_1 l} + 2a^2\sigma_{\max}^2\left\{2\rho_{\max}^{r_1 l} + \rho_{\max}^{r_1 l-1} + \rho_{\max}^{r_1 l+1}\right\}\right] + O(1) = O(1).\end{aligned} \tag{5.5}$$

Since the number of elements in S_g is L_g, (??) implies (5.3). We have seen that (5.2) converges as $n \to \infty$. However, $L_g \to \infty$ as $n \to \infty$, and $L_1 \approx L_2 \approx \cdots \approx L_g$ for n sufficiently large, it follows that $\mathrm{T_{BSL1}}$ converges in probability to its mean $\sigma_\gamma^2\rho_\gamma + \sigma_\mu^2\rho_\mu + a^2$. This completes the proof of the convergence of $\mathrm{T_{BSL1}}$. Similar argument can be applied to the cases of $\mathrm{T_{BSL2}}$ and $\mathrm{T_{WSL}}$. Thus it remains to show that $\mathrm{T_S}$ converges to $\sigma^2/k + \sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\mu^2 + a^2$. This also can be done by applying Corollary 1.1, Corollary 1.2 of Searl (1971). □

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