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GEOMETRIC ERGODICITY AND EXISTENCE OF HIGHER-ORDER MOMENTS FOR DTARCH(p, q) PROCESS†

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ABSTRACT

We consider a double threshold AR-ARCH type process and give sufficient conditions under which the higher-order moments exist. Geometric ergodicity and strict stationarity are also studied.

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Keywords. Double threshold AR-ARCH model, irreducible, geometric ergodicity, stationarity, moment.

1. INTRODUCTION

Threshold autoregressive models of Tong (1978) and Tong and Lim (1980) are quite popular in the nonlinear time series literature. ARCH (autoregressive conditional heteroscedasticity) process was proposed by Engle (1982) to explain time series with conditional heteroscedastic variances. Tong (1990) suggested a threshold model with an ARCH error (SETAR-ARCH) which combines the advantage of the threshold AR model and ARCH model. Double threshold AR-ARCH model is a natural extension of SETAR-ARCH model. Family of ARCH model has proven useful in financial applications and have great attention in economics and statistical literature (see, *e.g.*, Engle and Bollerslev, 1986; Bollerslev *et al.*, 1992; Schwert, 1989; Nelson, 1991; Guégan and Diebolt, 1994; Li and Li, 1996; Wong and Li, 1997). Stationarity, geometric ergodicity (or strongly mixing/absolute regularity) and existence of higher-order moments are of importance in the statistical inference of model. There are many papers to discuss estimations of coefficients and threshold for threshold AR-ARCH models and

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distribution theory for the estimators. To study the limiting distribution of the estimator, it is assumed, in general, the absolute regularity and higher order moment conditions (see Chan, 1993; Masry and Tjøstheim, 1995; Li and Li, 1996; Lu, 1996; An *et al.*, 1997; Hansen, 1999 and 2000; Ling, 1999; Hwang and Woo, 2001; Lee, 2002).

In this paper, we consider the double threshold autoregressive model with ARCH errors (DTARCH(p,q)) $\{y_t\}$ given by

$$y_t = \sum_{i=1}^{r}\Big(\theta_{i0} + \sum_{j=1}^{p}\theta_{ij}y_{t-j}\Big)I_{it} + \varepsilon_t, \tag{1.1}$$

$$\varepsilon_t = \sqrt{h_t}e_t, \tag{1.2}$$

$$h_t = \sum_{k=1}^{s}\Big(\alpha_{k0} + \sum_{j=1}^{q}\alpha_{kj}\varepsilon_{t-j}^2\Big)J_{kt}, \tag{1.3}$$

where $\{e_t\}$ is a sequence of independent and identically distributed (*iid*) random variables with mean 0 and variance σ^2, $I_{it} = I(a_{i-1} \le y_{t-b} < a_i)$ and $J_{kt} = I(b_{k-1} \le y_{t-d} < b_k)$, $-\infty = a_0 < a_1 < \cdots < a_r = \infty$, $-\infty = b_0 < b_1 < \cdots < b_s = \infty$, $b, d \in \{1, 2, \ldots, p+q\}$, $\alpha_{k0} > 0$, $\alpha_{kj} \ge 0$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, q$, $k = 1, 2, \ldots, s$, and $I(A)$ denotes the indicator function of a set A.

Goal of this paper is to find some region of coefficients on which y_t is strictly stationary and geometrically ergodic. Existence of higher-order moments is also considered.

In Section 2, we consider the geometric ergodicity and 4^{th} moment problem. Strict stationarity and higher-order moment condition are studied in Section 3.

2. Geometric Ergodicity and Moments

In this section, we define a Markov chain Y_t by

$$Y_t = (y_t, y_{t-1}, \ldots, y_{t-p-q+1}). \tag{2.1}$$

y_t given by (1.1)–(1.3) can be rewritten as

$$y_t = g_1(y_{t-1}, \ldots, y_{t-p}) + g_2(y_{t-1}, \ldots, y_{t-p-q})e_t. \tag{2.2}$$

Here

$$g_1(y_{t-1}, \ldots, y_{t-p}) = \sum_{i=1}^{r}\Big(\theta_{i0} + \sum_{j=1}^{p}\theta_{ij}y_{t-j}\Big)I_{it}, \tag{2.3}$$

$$g_2^2(y_{t-1},\dots,y_{t-p-q}) = \sum_{k=1}^{s}\Big(\alpha_{k0} + \sum_{j=1}^{q}\alpha_{kj}\varepsilon_{t-j}^2\Big)J_{kt}, \tag{2.4}$$

where

$$\varepsilon_{t-j} = y_{t-j} - \sum_{i=1}^{r}\Big(\theta_{i0} + \sum_{l=1}^{p}\theta_{il}y_{t-j-l}\Big)I_{i(t-j)}, \quad j = 1,\dots,q.$$

Throughout this paper, we assume that

$$\theta_j = \max_{1\le i\le r}|\theta_{ij}|, \quad 0 \le j \le p, \tag{2.5}$$

$$\alpha_l = \max_{1\le k\le s}\alpha_{kl}, \quad 0 \le l \le q. \tag{2.6}$$

LEMMA 2.1. *Let $v(z) = \sum_{i=1}^{n}\gamma_i z_i^m$ and $z = (z_1,\dots,z_n)$ for $n, m \in Z^+$. If $\sum_{i=1}^{n}\xi_i < 1$, we may choose $\gamma_i > 0$, $i = 1,\dots,n$ so that for some positive constant $r < 1$,*

$$\gamma_1\Big(\sum_{i=1}^{n}\xi_i z_i^m\Big) + \sum_{i=2}^{n}\gamma_i z_{i-1}^m \le rv(z). \tag{2.7}$$

PROOF. Let $\delta > 0$ be such that $\sum_{i=1}^{n}\xi_i + \delta = 1$. Choose $\gamma_1 > 0$ arbitrary and define

$$\gamma_{i+1} = \gamma_1\Big(1 - \xi_1 - \cdots - \xi_i - \frac{i\delta}{n}\Big), \quad i = 1,2,\dots,n-1. \tag{2.8}$$

Then following two inequalities hold:

$$\gamma_1\xi_i + \gamma_{i+1} \le \gamma_i\Big(1 - \frac{\delta}{n}\Big), \quad 1 \le i \le n-1, \tag{2.9}$$

$$\gamma_1\xi_n \le \gamma_n\Big(1 - \frac{\delta}{n}\Big). \tag{2.10}$$

Using (2.9) and (2.10), we have that

$$\sum_{i=1}^{n-1}(\gamma_1\xi_i + \gamma_{i+1})z_i^m + \gamma_1\xi_n z_n^m \le \Big(1 - \frac{\delta}{n}\Big)\sum_{i=1}^{n}\gamma_i z_i^m. \tag{2.11}$$

Thus (2.7) holds with $r = (1 - \delta/n)$. □

ASSUMPTION 1.1. The *iid* random variables $\{e_t\}$ have a probability density function that is positive over R^1.

ASSUMPTION 1.2. θ_j and α_l $(1 \le j \le p,\ 1 \le l \le q)$ hold the following:

(a) $(\sum_{j=1}^{p}\theta_j)^2 + (1+\sum_{j=1}^{p}\theta_j)^2(\sum_{l=1}^{q}\alpha_l) < 1$;
(b) $(1+3\sigma^2)(\sum_{j=1}^{p}\theta_j)^4 + (Ee_t^4 + 3\sigma^2)(1+\sum_{j=1}^{p}\theta_j)^4(\sum_{l=1}^{q}\alpha_l)^2 < 1$.

THEOREM 2.1. *Let Assumption 1.1 hold.*

(1) *If Assumption 1.2(a) holds, then y_t is geometrically ergodic and $E_\pi(y_t^2) < \infty$, where π is the stationary distribution of y_t.*

(2) *If $E(e_t) = E(e_t^3) = 0$, $E(e_t^4) < \infty$ and Assumption 1.2(b) holds, then y_t is geometrically ergodic and $E_\pi(y_t^4) < \infty$.*

PROOF. Since g_1 and g_2 are bounded on compacts, ϕ-irreducibility and aperiodicity of Y_t is derived from Assumption 1.1. Moreover every compact set is a small set (see, *e.g.*, Bhattacharya and Lee, 1995).

To prove part (1), we define a test function $V : R^{p+q} \to R$ by

$$V(z_1, \ldots, z_{p+q}) = \sum_{i=1}^{p+q} \gamma_i z_i^2 + 1, \tag{2.12}$$

where γ_i, $1 \le i \le p+q$, are to be defined later. We first assume that $p = q = 2$. By (2.3)–(2.6) and simple calculation, we obtain that

$$\begin{aligned} |g_1(y_{t-1}, y_{t-2})| &\le \theta_0 + \theta_1|y_{t-1}| + \theta_2|y_{t-2}|, \\ g_2^2(y_{t-1}, \ldots, y_{t-4}) &\le \sum_{i=1}^{4} d_i y_{t-i}^2 + h_1(y_{t-1}, \ldots, y_{t-4}), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \alpha_1(1+\theta_1+\theta_2), \\ d_2 &= \alpha_1\theta_1(1+\theta_1+\theta_2) + \alpha_2(1+\theta_1+\theta_2), \\ d_3 &= \alpha_1\theta_2(1+\theta_1+\theta_2) + \alpha_2\theta_1(1+\theta_1+\theta_2), \\ d_4 &= \alpha_2\theta_2(1+\theta_1+\theta_2) \end{aligned}$$

and

$$\begin{aligned} h_1(y_{t-1}, \ldots, y_{t-4}) = \theta_0^2(\alpha_1+\alpha_2) &+ 2\alpha_1\theta_0\big(|y_{t-1}| + \theta_1|y_{t-2}| + \theta_2|y_{t-3}|\big) \\ &+ 2\alpha_2\theta_0\big(|y_{t-2}| + \theta_1|y_{t-3}| + \theta_2|y_{t-4}|\big). \end{aligned}$$

For $z = (z_1, \ldots, z_4)$, we have that

$$
\begin{aligned}
&E\big(V(Y_t)|Y_{t-1}=z\big)\\
&= E\Big[\gamma_1\big(g_1(z_1,z_2)+g_2(z_1,\ldots,z_4)e_t\big)^2+\sum_{i=2}^{4}\gamma_i z_{i-1}^2+1\,\Big|\,Y_{t-1}=z\Big]\\
&\le \gamma_1\Big(\sum_{i=1}^{2}\beta_i z_i^2+\sigma^2\sum_{i=1}^{4}d_i z_i^2\Big)+\sum_{i=2}^{4}\gamma_i z_{i-1}^2+1+h_1(z). \qquad (2.13)
\end{aligned}
$$

Here $\sum_{i=1}^{2}\beta_i=(\sum_{i=1}^{2}\theta_i)^2$, $\sum_{i=1}^{4}d_i=(1+\sum_{i=1}^{2}\theta_i)^2(\sum_{i=1}^{2}\alpha_i)$. Use mathematical induction to obtain that, for arbitrary $p,q\in Z^+$, and $z=(z_1,\ldots,z_{p+q})$,

$$g_1^2(z_1,\ldots,z_p)\le\sum_{i=1}^{p}\beta_i z_i^2+o(\|z\|^2), \qquad (2.14)$$

$$g_2^2(z_1,\ldots,z_{p+q})\le\sum_{i=1}^{p+q}d_i z_i^2+o(\|z\|^2) \qquad (2.15)$$

with

$$\sum_{i=1}^{p}\beta_i=\Big(\sum_{i=1}^{p}\theta_i\Big)^2 \quad\text{and}\quad \sum_{i=1}^{p+q}d_i=\Big(1+\sum_{i=1}^{p}\theta_i\Big)^2\Big(\sum_{i=1}^{q}\alpha_i\Big). \qquad (2.16)$$

Therefore

$$
\begin{aligned}
&E\big[V(Y_t)|Y_{t-1}=(z_1,\ldots,z_{p+q})\big]\\
&\le\gamma_1\Big(\sum_{i=1}^{p}\beta_i z_i^2+\sigma^2\sum_{i=1}^{p+q}d_i z_i^2\Big)+\sum_{i=2}^{p+q}\gamma_i z_{i-1}^2+o(\|z\|^2). \qquad (2.17)
\end{aligned}
$$

Define γ_i, $i=1,2,\ldots,p+q$, by the same manner as given in (2.8) with $n=p+q$, $\sum\xi_i=\sum\beta_i+\sigma^2\sum d_i$. Then it follows from (2.17), Assumption 1.2(a) and Lemma 2.1 that there exists some constant $r<1$ such that

$$E\big[V(Y_t)\,\big|\,Y_{t-1}=z\big]\le rV(z)+o(\|z\|^2),$$

from which inequalities (3.7) and (3.8) in Tjøstheim (1990) hold with some $\epsilon>0, M<\infty$ and compact set $K=\{\|z\|\le k\}$ for sufficiently large $k<\infty$. Hence the geometric ergodicity and existence of the second moment of y_t follow (see, *e.g.*, Meyn and Tweedie, 1993; An *et al.*, 1997).

To prove part (2), define $V:R^{p+q}\to R$ by

$$V(z_1,\ldots,z_{p+q})=\sum_{i=1}^{p+q}\gamma_i z_i^4+1. \qquad (2.18)$$

Note that

$$E(g_1 + g_2 e_t)^4 \le (1+3\sigma^2)g_1^4 + (Ee_t^4 + 3\sigma^2)g_2^4, \tag{2.19}$$

$$g_1^4(z_1,\dots,z_p) = \Big(\sum_{i=1}^{p}\theta_i z_i\Big)^4 + h(z) \le \sum_{i=1}^{p}\eta_i z_i^4 + h(z), \tag{2.20}$$

$$g_2^4 \le \Big(\sum_{i=1}^{p+q} d_i z_i^2\Big)^2 + h'(z) \le \sum_{i=1}^{p+q}\delta_i z_i^4 + h'(z), \tag{2.21}$$

where $\sum \eta_i = (\sum\theta_i)^4$,$\sum\delta_i = (\sum d_i)^2 = (1+\sum\theta_i)^4(\sum\alpha_i)^2$, $h(z)$ and $h'(z)$ are functions satisfying $h(z)/V(z)$, $h'(z)/V(z) \to 0$ as $\|z\| \to \infty$.

It follows from (2.18)–(2.21) that

$$\begin{aligned} &E[V(Y_t)|Y_{t-1} = (z_1,\dots,z_{p+q})] \\ &\le \gamma_1\big[(1+3\sigma^2)g_1^4 + (Ee_t^4+3\sigma^2)g_2^4\big] + \sum_{i=2}^{p+q}\gamma_i z_{i-1}^4 + 1 \\ &\le \gamma_1\sum_{i=1}^{p+q}\Big[(1+3\sigma^2)\eta_i + (Ee_t^4+3\sigma^2)\delta_i\Big]z_i^4 + \sum_{i=2}^{p+q}\gamma_i z_{i-1}^4 + f(z), \end{aligned} \tag{2.22}$$

where $\eta_{p+1} = \cdots = \eta_{p+q} = 0$ and $f(z)/V(z) \to 0$ as $\|z\| \to \infty$. Since

$$\begin{aligned} &(1+3\sigma^2)\sum_{i=1}^{p+q}\eta_i + (Ee_t^4+3\sigma^2)\sum_{i=1}^{p+q}\delta_i \\ &= (1+3\sigma^2)\Big(\sum_{i=1}^{p+q}\theta_i\Big)^4 + (Ee_t^4+3\sigma^2)\Big(1+\sum_{i=1}^{p+q}\theta_i\Big)^4\Big(\sum_{i=1}^{p+q}\alpha_i\Big)^2 \\ &< 1, \end{aligned}$$

applying Lemma 2.1 yields the conclusion. □

EXAMPLE 1. Suppose $e_t \sim N(0,1)$. Then $E(e_t) = E(e_t^3) = 0, E(e_t^2) = 1$ and $E(e_t^4) = 3$. When $\sum\theta_i = 0.5$, then $\sum\alpha_i \le 0.149$ guarantees the geometric ergodicity and existence of 4^{th} moment of y_t. If $p = 0$, *i.e.*, y_t is a threshold ARCH model, then $\sum\alpha_i < 1$ ensures $E(\varepsilon_t^2) < \infty$ and $6(\sum\alpha_i)^2 < 1$, *i.e.*, $\sum\alpha_i \le 0.4$ implies that $E(\varepsilon_t^4) < \infty$. Lemma 3.3 in An *et al.* (1997) emerges as a special case of Theorem 2.1(1).

REMARK 1. Recall that the geometric ergodicity of a Markov chain implies the absolute regularity and strongly mixing of the process.

3. Stationarity and Moments

In this section we consider the process y_t given by (1.1)–(1.3) *via* the different Markov chain. Let

$$W_t = (y_t, y_{t-1}, \dots, y_{t-p+1}, \varepsilon_t^2, \dots, \varepsilon_{t-q+1}^2). \tag{3.1}$$

It is an awkward problem to prove the irreducibility of W_t, which is a crucial step to prove the geometric ergodicity of W_t.

Throughout this section we assume that W_t is a Feller chain, that is, for each bounded continuous function f on R^{p+q}, $E[f(W_t)|W_{t-1} = w]$ is continuous in w. Note that if piecewise continuous functions $\sum_{i=1}^{r}(\theta_{i0} + \sum_{j=1}^{p} \theta_{ij} y_{t-j}) I_{it}$ and $\sum_{k=1}^{s}(\alpha_{k0} + \sum_{j=1}^{q} \alpha_{kj}\varepsilon_{t-j}^2) J_{kt}$ are continuous, then W_t is a Feller chain.

The goal of this section is to find a region on which the process has a strictly stationary solution and has the finite m^{th} moment according to the stationary distribution. Many authors have studied the stationarity and moment conditions for nonlinear time series (*e.g.* Tweedie, 1988; Meyn and Tweedie, 1993; An *et al.*, 1997; Ling, 1999).

It is proved that if $\max_i \sum_{i=1}^{p} |\theta_{ij}| < 1$ and $\sum_{j=1}^{q} \max_i\{\alpha_{ij}\} < 1$, then the strictly stationary solution y_t of (1.1)–(1.3) exists (see Lee and Kim, 2002). It is shown that $E(e_t^{2m}) < \infty$, $\sum_{j=1}^{p} \max_i |\theta_{ij}| < 1$ and $\rho(\max_j E[\tilde{\alpha}_t{}^{(j)\otimes m}]) < 1$ ensure the existence of the m^{th} moment of y_t, where $\rho(A)$ designates the spectral radius of a matrix A and the definition of $\tilde{\alpha}_t{}^{(j)}$ is given in equation (4.6) in Ling (1999). But it is not simple to find a region of α_{ij} on which $\rho(\max_j E[\tilde{\alpha}_t{}^{(j)\otimes m}]) < 1$ is satisfied as well as the moment condition $E(e_t^{2m}) < \infty$ on e_t can be weakened to obtain the finite m^{th} moment of y_t.

Lemma 3.1. *If* $\sum_{i=1}^{p} \xi_i < 1$ *and* $\sum_{i=1}^{q} \phi_i < 1$, $n, m \in Z^+$, *then we can choose* γ_i, $1 \le i \le p+q$ *such that for some positive constant* $r < 1$,

$$\begin{aligned} &\gamma_1\Big(\sum_{i=1}^{p} \xi_i z_i^m\Big) + \sum_{i=2}^{p} \gamma_i z_{i-1}^m + \gamma_{p+1}\Big(\sum_{i=1}^{q} \phi_i u_i^n\Big) + \sum_{i=2}^{q} \gamma_{p+i} u_{i-1}^n \\ &\quad \le r\Big(\sum_{i=1}^{p} \gamma_i z_i^m + \sum_{i=1}^{q} \gamma_{p+i} u_i^n\Big). \end{aligned} \tag{3.2}$$

Proof. Since $\sum_{i=1}^{p} \xi_i < 1$ and $\sum_{i=1}^{q} \phi_i < 1$, choose $\delta_1 > 0$ and $\delta_2 > 0$ by $\sum \xi_i + \delta_1 = 1$ and $\sum \phi_i + \delta_2 = 1$. Now choose $\gamma_1 > 0$ and $\gamma_{p+1} > 0$ arbitrary and define $\gamma_i,\ 2 \le i \le p,\ p+2 \le i \le p+q$, by

$$\gamma_{i+1} = \gamma_1\Big(1 - \varsigma_1 - \cdots - \xi_i - \frac{i\delta}{p}\Big),\ 1 \le i \le p-1, \tag{3.3}$$

$$\gamma_{p+1+i} = \gamma_{p+1}\Big(1 - \phi_1 - \cdots - \phi_i - \frac{i\delta}{q}\Big),\ \ 1 \le i \le q-1. \tag{3.4}$$

Then by the same reason adopted in the proof of Lemma 2.1, (3.2) holds with $r = \max\{1 - \delta_1/p,\ 1 - \delta_2/q\}$. □

THEOREM 3.1. *Let $m \in Z^+$ and $n = [m/2]+1$ where $[\cdot]$ denotes the Gaussian number. Suppose that $E(e_t^{2n}) < \infty$. If $\sum \theta_i < 1$ and $E(e_t^{2n})(\sum \alpha_i)^n < 1$, then $E_\pi(|y_t|^m) < \infty$.*

PROOF. To avoid the computational complexity, we assume that $p = q = 2$. The case $p > 2$ or $q > 2$ is entirely analogous, but involves messier notation. Define a test function

$$V(z_1, \ldots, z_p, u_1, \ldots, u_q) = \sum_{i=1}^{p} \gamma_i |z_i|^m + \sum_{i=1}^{q} \gamma_{p+i} u_i^n, \tag{3.5}$$

where γ_i is to be defined by the equations (3.3) and (3.4).

For $(z, u) = (z_1, z_2, u_1, u_2)$,

$$\begin{aligned}
&E\big[V(Y_t)|Y_{t-1} = (z,u)\big] \\
&= E\big[\gamma_1(\theta_0 + \theta_1|z_1| + \theta_2|z_2| + (\alpha_0 + \alpha_1 u_1 + \alpha_2 u_2)^{1/2} e_t)^m\big] + \gamma_2|z_1|^m \\
&\qquad + \gamma_3 E(e_t^{2n})(\alpha_0 + \alpha_1 u_1 + \alpha_2 u_2)^n + \gamma_4 u_1^n \\
&= \gamma_1\big(\theta_1|z_1| + \theta_2|z_2|\big)^m + \gamma_2|z_1|^m + \gamma_3 E(e_t^{2n})(\alpha_1 u_1 + \alpha_2 u_2)^n + \gamma_4 u_1^n + f(z,u) \\
&\le \gamma_1(\beta_1|z_1|^m + \beta_2|z_2|^m) + \gamma_2|z_1|^m + \gamma_3 E(e_t^{2n})(\eta_1 u_1^n + \eta_2 u_2^n) + \gamma_4 u_1^n + f(z,u).
\end{aligned}$$

Here

$$\frac{f(z,u)}{V(z,u)} \to 0 \ \text{ as } \ \mathrm{V(z,u)} \to \infty,$$

for $|z_i|^k u_j^{(m-k)/2} \le (|z_i|^m + u_j^n)^{k/m+(m-k)/(2n)}$ and $k/m + (m-k)/(2n) < 1$, $1 \le k \le m-1$.

Since $\beta_1 + \beta_2 = (\theta_1 + \theta_2)^m$ and $\eta_1 + \eta_2 = (\alpha_1 + \alpha_2)^n$, from assumptions $\sum \theta_i < 1$ and $E(e_t^{2n})(\sum \alpha_i)^n < 1$ and Lemma 3.1, we obtain the existence of $r < r' < 1$ such that

$$E[V(Y_t)|Y_{t-1} = (z,u)] \le rV(z,u) + f(z,u) \le r'V(z,u) - \epsilon, \ \ \|z\| > M \tag{3.6}$$

for some $\epsilon > 0$ and $M < \infty$. Conclusion can be deduced from Theorem 2 of Tweedie (1988). □

EXAMPLE 2. Suppose that $e_t \sim N(0,1)$. If $m = 4$, then $n = 3$ and $E(e_t^6) = 15$. Therefore $E|y_t|^4 < \infty$ provided that $\sum \theta_i < 1$ and $\sum \alpha_i \leq 0.4$.

REMARK 2. Let y_t be generated by (1.1)–(1.3) with $r = s = 2$, $I_{1t} = I(y_t - y_{t-b} \geq \lambda)$, $I_{2t} = 1 - I_{1t}$ and $J_{1t} = I(y_t - y_{t-d} \geq \lambda')$, $J_{2t} = 1 - J_{1t}$ for some constants $\lambda, \lambda' \in R^1$. Such a model is called a momentum threshold AR-ARCH model (MTARCH). We can easily show that Theorem 3.1 still holds for the MTARCH process $\{y_t\}$. The proof for MTARCH process follows essentially the same line of Theorem 3.1.

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