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On the Teaching Linear Algebra at the University Level: The Role of Visualization in the Teaching Vector Spaces

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In linear algebra course, the theory of vector space is usually presented in a very formal setting, which causes severe difficulties to many students. In this study, the effect of teaching the theory of vector space in linear algebra from the geometrical point of view on students' learning was investigated. It was found that the teaching of the theory of vector space in linear algebra from the geometrical point of view increases the meaningful learning since it increases the visualization.

Key words: vector spaces, visualization.

INTRODUCTION

Linear algebra, which concerns with abstract system called vector spaces and origins from the solutions of systems of linear equations, is a branch of modern algebra. Harel (1989) described that linear algebra deals with abstract structures representing various concepts and systems with wide variety of different properties. Linear algebra is an indispensable course for university students because of its widespread applicability in natural and social sciences. Since the topics in linear algebra are important it was even suggested that high school students study selected aspects of it as well (Begle 1970).

Harel (1989) explained that topics in linear algebra, as well as the subject itself, are ubiquitous in nature, it has became a required course in most curricula. In this topic, Knopp's conviction (Knopp 1974) was that the ideas of linear algebra are valuable for many students who will never progress to higher level mathematics courses or who will never find a need for the formal style of communication usually associated with courses in abstract mathematics. Also Antigue (1999) pointed out that mathematical work in linear algebra mobilizes several system of representations, including graphics, pictures, symbolic writing, natural language, and others.

Linear algebra is usually perceived as a difficult subject by university students. The main reason of this is probably linear algebra is a curious blend of notions at once more abstract and more concrete than those one finds in calculus. Staib (1969) stated that the applications of linear algebra to 'real problems' are not so immediate as in the case of calculus and thus students are faced in this course. In a survey, Robert and Robinet (1994) showed that the main criticisms made by the students toward linear algebra concern the use of formalism, the overwhelming amount of new definitions and the lack of connection with what they already know in mathematics (Dorier 1994). A number of creative and radical studies about improving the teaching of linear algebra have been performed in recent years. The researches contain goals of instruction, materials and methods of instruction, technology of instruction, levels of abstraction and concretion, student diversity and connection with others courses etc. The visualization studies of abstract linear algebra are the most urgent thing in the research agenda with respect to instruction methods (Dorier 1994; Harel 1987; 1989; Long 1983; Presmeg 1986 & Wang 1989). It is possible to present with algebraically and geometrical models of abstract concepts in linear algebra. Harel (1989) suggested a progressive increase in abstractions: from geometry to $\boldsymbol{R}^2$, $\boldsymbol{R}^3$ and then $\boldsymbol{R}^n$ and finally formal vector space (Harel 1989). In his teaching experiment, the first phase is the fundamental as it is the basis for visual representation. He pointed out that a instruction program included abstract process could be taught to students in three phases. The phases are visualizing concept and process, representation and the establishment of the dimension of $\boldsymbol{R}^n$ and abstract vector spaces.

An important component of forming concrete or at least semi-concrete of our mental representation of a concept is an external or physical reference. Bernardz & Kieran (1996) firstly considered concepts such as geometric structures and mathematical-physical models for meaningful teaching algebra. Since in the abstract system of linear algebra concrete embodiments are complex physical systems, which are not in the scope of undergraduate curricula, it is suitable usage of semi-concrete structure pointed out as geometric system in teaching of the structure. Even if there are not in all topic, most subject in linear algebra can be explained geometrically as well as algebraically. That is; geometric responses of algebraic concepts in linear algebra can be found. According to

Vinner (1983), the concept images of a person consist of the sets of the concept's properties that the person has mind and set of all pictures that have ever been associated with the concept in the person's mind. Wang (1989) suggested emphasis be placed on intuitive understanding and geometric visualization and interpretation of the key structures of linear algebra.

Visualization can be alternative method and powerful resource for students doing mathematics, a resource that can upon the way to different ways of thinking about mathematics than the linguistic and logico-propositional thinking of traditional and the symbol manipulation of traditional algebra. In order to handle concepts one needs a concept image, not only a concept definition. By point of view, linear algebra, in additition to practicability in application area, is an important course for students' cognitive development both geometric and algebraic.

Vector space is a subject-forming basis of linear algebra. Harel (1987) pointed out that linear systems, linear transformation, matrix arithmetic and vector spaces form the core content of linear algebra. In Euclidean geometry, vector space can be thought as generalizing of cartesian coordinate system with algebraic abstraction. Thompson and Yaqub (1970) pointed out the concept of vector space an abstraction certain simple geometrical ideas encountered in analytic geometry. Also, Mirsky explained the point of contact between linear algebra and geometry such as space, plane and coordinate system are numerous (Mirsky 1963).

Subjects of vector spaces describing on a field can be match subjects of geometry. In spite of this; in linear algebra textbooks and instruction notes of teaching staffs have been treated vector space subject with algebraic expression. In this study, the effect of teaching the theory of vector space in linear algebra from the geometrical point of view on students' learning was investigated. The teaching method can be considered as useful method for linear algebra course, which can perform both algebraically and geometrically.

We consider that geometrical structures supported algebraically on the teaching of vector spaces increase in meaningful teaching of the students and develop vector space concept and concept's properties in the students' understanding. Since it is provided our hypothesis which the students comprehend well geometrically vector space concept in linear algebra, it is required to teach precondition learning such as point setting in $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$, vector construction, vector addition and scalar multiplication. It should be consider three components because students clarify relations between the theory of vector spaces and geometry in teaching process. These are:

i) Usage of vector geometry as an effective method
ii) Usage of general structures in two subjects and relations between these subjects.
iii) Usage of geometric descriptions for clarify vectoral structures.

METHOD

The students used in this study were 60 sophomores enrolled in linear algebra course designed for the professional teaching of mathematics program. All students have had the same formal education in mathematics. They took calculus and set theory course in the first year. Those courses did not include linear algebra content. The students were divided randomly into two groups consist of 30 students. All students were given basis knowledge deal with vector concept required for linear algebra. They were taught vector space concept by one instructor for three one-hour lectures per week. In the process, vector space concept to Group A was presented in the two hours geometrically and one hour algebraically. Group B was also presented in the two hours algebraically and one hour geometrically. At the end of four weeks, the two groups were given same test. The question in the test was chosen to be simple problems on the vector space concept, which can be solved directly by applying the vector space definition.

Problem 1: Which of the following sets is a subspace of vector space V? Give your answer with explanation.

a) $W = \{(x, y) \in \mathbf{R}^2 : y = x+1\}$
b) $W = \{(x, y) \in \mathbf{R}^2 : y = x^2\}$

Problem 2: W_1 and W_2 are non-trivial subspaces of a vector space V. Is $W_1 \cup W_2$ a subspace of vector space V?

Problem 3: Let V be a vector space and ω a fixed vector in V. If W is a set of all scalar multiplications of ω, is W a subspace?

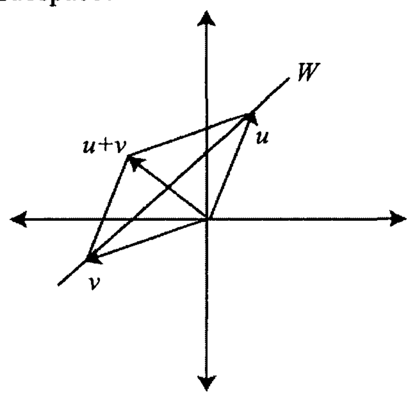

Figure 1. Addition of two vectors

The students were asked to answers these question both algebraically and geometrically. It has been explained V is equal to R^2 or R^3 for geometric descriptions. Geometric description (GD) is the students' answers involved only written geometric descriptions, such as: "The set of W in problem 1-a is not a subspace of R^2" has been geometrically shown in Figure 1.

Algebraic description (AD) is the students' answers involved algebraic descriptions without any geometric interpration. For example, any element of W set in problem 1-a is an ordered pair of the form $(x,x+1)$. The set of W is not a subspace of R^2 since it cannot obtain an ordered pair of the form (0,0) for no value of x variable.

Students' correct responses were categorized in two groups. These are:

a) Correct answer with explanation.
b) Correct answer without explanation

Correct answer with explanation: In problem 1-a and 1-b, students were required to explain their answers. Proofs given problem 2 and 3 were also considered as explanations.

Correct answer without explanation: The variable refers to the correct answer that students provided to problems 1-a, 1-b and to the solutions without proof in problems 2 and 3.

The responses, which the students handed to the questions, have been submitted at following tables. On the end of four weekly course processes, two groups were asked to answer three questions given above. The results analyzed by SPSS packet program. The results are presented by percentages, frequencies and t-test is carried out. Significance level was taken as $p = 0.005$.

Findings

It is allowed to their correct and incorrect answers without interesting in algebraic and geometric descriptions of the students at Table-1. Furthermore, it is clarified that the students could not reply to the question on others column.

Table-1: The students' general responses

Problem	Group	Correct Answer	Incorrect Answer	No response	p-value
1-a	A	%86.6	%6.7	%6.7	0.497
	B	%80.0	%13.3	%6.7	
1-b	A	%80.0	%13.3	%6.7	0.549
	B	%73.3	%20.0	%6.7	
2	A	%63.3	%30.0	%6.7	0.004
	B	%26.7	%60.0	%13.3	
3	A B	%86.6	%3.4	%10.0	0.004

The students' responses are given in Table-2 separating from algebraic and geometric description. In the tables "incorrect response" category includes "no responses" as well.

Table-2: The students' algebraic and geometric responses

Problem	Group	Correct		Incorrect		*p*-value	
		Algebraic	Geometric	Algebraic	Geometric	Algebraic	Geometric
1-a	A	%70.0	%80.0	%30.0	%20.0	1.000	0.053
	B	%70.0	%56.6	%30.0	%43.4		
1-b	A	%63.3	%80.0	%36.7	%20.0	0.791	0.0001
	B	%66.7	%36.7	%33.3	%63.3		
2	A	%43.3	%63.3	%56.7	%36.7	0.182	0.001
	B	%26.7	%23.4	%73.3	%76.6		
3	A	%50.0	%86.6	%50.0	%13.4	1.000	0.0001
	B	%50.0	%23.4	%50.0	%76.6		

Table-3 shows obtained information deal with algebraic and geometric descriptions of the students in detail. The correct descriptions of the students are specially given in the Table-3. Therefore the correct answers are evaluated on two dimensions as with justification and without justification. Since we are interested in correct answers in the Table-3, it has been regarded as equal to the descriptions of the students who gave on incorrect answers and could not reply.

Table-3: The students' responses with and without explanation

Problem	Group	Correct answer with explanation		Correct answer without explanation	
		Algebraic	Geometric	Algebraic	Geometric
1-a	A	%56.7	%70.0	%13.3	%10.0
	B	%46.7	%33.3	%23.3	%23.3
1-b	A	%60.0	%73.3	%3.3	%6.7
	B	%50.0	%20.0	%16.7	%16.7
2	A	%30.0	%43.3	%13.3	%20.0
	B	%6.7	%6.7	%20.0	%16.7
3	A	%26,7	%33.3	%23.3	%53.3
	B	%13.3	%3.3	%36.7	%20.0

(Continue)

Problem	Group	Incorrect answer and no response		p-value	
		Algebraic	Geometric	Algebraic	Geometric
1-a	A	%30.0	%20.0	0.447	0.004
	B	%30.0	%43.3		
1-b	A	%36.7	%20.0	0.445	0.0001
	B	%33.3	%63.3		
2	A	%56.7	%36.7	0.020	0.001
	B	%73.3	%76.6		
3	A	%50.0	%13.3	0.203	0.003
	B	%50.0	%76.6		

DISCUSSION

As seen from Table-1, it was found that the students in group A were more successful than the students in group B without regarding algebraic and geometric descriptions. Most of the students in both group answered problem 1 correctly by using only vector space description. This high percentage of correct answers may be due to solving similar exercises in the teaching process in the classroom. Although a different setting of problem 1 was discussed during the instruction, other problems were not discussed at any time during the instruction. Students in group A were found to be more successful than the students in group B in solving the problem 2 and 3 which required more conceptual understanding. When two groups were compared, students' responses in group A included less number of incorrect answers than group B students' responses. Although there is not a meaningful difference between algebraic descriptions of students from both groups, there is a meaningful difference between geometric descriptions except for problem 1-a (see Table-2). This difference between students' geometric description to problem 1-a and 1-b may be due to from difference between graphing of linear and quadratic equations in $\boldsymbol{R}^2$, because students are appeared to be able to draw more easily graphs of linear functions than graphs of quadratic functions. In addition, student's responses to problem 2 included less number of correct answers than the responses given to the other problems. The students' responses coded as correct answer were divided in two parts as answers with explanation and without explanation in order to analyses the responses in detail (see Table-3). The responses that showed sound understanding were coded as "correct answer with explanation". In this study, this group of responses were especially taken into consideration. Table-3 indicates that while there was no meaningful difference between both groups in responses coded as "correct answer with explanation" for algebraic description, there was a significant difference for geometric descriptions. This difference is especially apparent for problems 2 and 3. Although problems 2 and 3 required more conceptual understanding dealing with vector space concept, the success for these problems can be explained that students in group A than students in group B comprehend (understand) well institutionally vector space concept. The more geometrical description in the responses given by the students in group A and the more algebraic description in the responses given by the students in group B suggest that the content of the students responses are closely related with the style of instruction, as geometrical approach was used to group A and algebraic approach was used to group B. There is no difference in algebraic description between two groups although the student in Group A was taught supported geometrically. The distinction between concept definition and concept image supports our conclusion that differences between the achievements of two groups can be attributed to the different instructional approaches used. Although these concepts are given under equal circumstances to two groups, students' performances in solving these problems are found to be different. An explanation to this could be that though these two groups received the same concept definition of vector space, they were exposed to different experiences, which resulted in forming different concept image. Students from group A had accomplished these problems because they had well concept image deal with vector space.

CONCLUSION

The students' learning difficulties in acquiring the concept of vector spaces is more or less similar to the acquiring other concepts in linear algebra. One of the most important problems associated with the teaching the concept of vector space is risen from the students' understanding difficulties in establishing the relationship between their knowledge and intuition about concrete structures and abstract nature of axiomatic language of the vector space theory. There is a special importance of geometrical structures called as semi-concrete on being established of the connection. The results of this study indicate that students who took linear algebra course given from geometrical point of view can well establish this connection. In addition, the results suggest that students are more able to describe the concepts from geometrical point of view also understand the concepts from algebraic point of view. Moreover, teaching linear algebra from geometrical point of view provides students to look at the linear algebra course, which was seen as a cumulation of abstract structures and concepts from a different perspective. Including visualization into the teaching process increases the students' motivation towards the course. It could be suggested that the teaching method applied in this study could be extended to teaching the other concepts in mathematics

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