Bull. Korean Math. Soc. **40** (2003), No. 2, pp. 341–349

A WEAK LAW FOR WEIGHTED SUMS OF ARRAY OF ROW NA RANDOM VARIABLES

Jong Il Baek, Han-Ying Liang and Jeong Yeol Choi

Abstract. Let $\{X_{nk} \mid 1 \le k \le n, n \ge 1\}$ be an array of random variables and $\{a_n | n \ge 1\}$ and $\{b_n | n \ge 1\}$ be a sequence of constants with $a_n > 0,\ b_n > 0, n \ge 1$. In this paper, for array of row negatively associated(NA) random variables, we establish a general weak law of large numbers ($WLLN$) of the form $\left(\sum_{k=1}^{n} a_k X_{nk} - \nu_{nk}\right)/b_n$ converges in probability to zero, as $n \to \infty$, where $\{\nu_{nk} | 1 \le k \le n, n \ge 1\}$ is a suitable array of constants.

1. Introduction

Alam and Lal Saxena ([4]) and Joag-Dev and Proschan ([9]) introduced the notion of negatively associated (NA) random variables. Concepts of NA random variables are of considerable uses in multivariate statistical analysis and system reliability. Many authors ([12], [13]) have studied the limit properties for them. We start this section with definition as follows.

Definition ([9]). Random variables $X_1, \cdots, X_n$ are said to be negatively associated (NA) if for any two disjoint nonempty subsets A_1 and A_2 of $\{1, \cdots, n\}$ and f_1 and f_2 are any two coordinatewise nondecreasing functions,

$$Cov(f_1(X_i,\ i \in A_1),\ f_2(X_j,\ j \in A_2)) \le 0,$$

whenever the covariance is finite. If for every $n \ge 2,\ X_1, \cdots, X_n$ are NA, then the sequence $\{X_i | i \in N\}$ is said to be NA.

Received February 26, 2002.

2000 Mathematics Subject Classification: Primary 60F05; Secondary 62E10, 45E10.

Key words and phrases: negatively associated random variables, weak law of large numbers, weighted sum.

This paper was supported by grant NO.R01-2000-000-00010 from the Basic Research Program of KOSEF and Wonkwang University Research Grant in 2003.

Let $\{X_{nk}|1 \le k \le n, n \ge 1\}$ be an array of row NA random variables and let $\{a_n|n \ge 1\}$ and $\{b_n|n \ge 1\}$ be a sequence of constants with $a_n > 0,\ 0 < b_n \to \infty, n \ge 1$. Then we establish a general weak law of large numbers $(WLLN)$ of the form

$$(1.1)\quad (\sum_{k=1}^{n} a_k X_{nk} - \nu_{nk})/b_n \text{ converges in probability to zero as } n \to \infty,$$

where $\{\nu_{nk}|1 \le k \le n, n \ge 1\}$ is a suitable array of constants.

The $WLLN$s of the form (1.1) for array of random variables have been established by Gut ([7]), Hong and Oh ([8]), Kowalski and Rychlik ([11]), and Sung ([15]). Our purpose establish a general weak law of large numbers $(WLLN)$ for weighted sums of array of row NA random variables which satisfy $P(|X_{nk}| > x) = O(1)P(|X| > x), \forall x \ge 0$. In section 2, we study some preliminary results and in section 3, we derive the main results for weighted sums of array of row NA random variables satisfying $P(|X_{nk}| > x) = O(1)P(|X| > x), \forall x \ge 0$.

2. Preliminaries

LEMMA 2.1. *Let $\{X_{nk}|1 \le k \le n, n \ge 1\}$ be an array of random variables which satisfy $P(|X_{nk}| > x) = O(1)P(|X| > x), \forall x \ge 0$. Let $\{a_n|n \ge 1\}$ and $\{b_n|n \ge 1\}$ be a sequence of constants with $a_n > 0,\ 0 < b_n \to \infty, n \ge 1$. Put*

$$c_n = b_n/a_n$$

and define $X'_{nk} = -c_n I(X_{nk} < -c_n) + X_{nk} I(|X_{nk}| \le c_n) + c_n I(X_{nk} > c_n)$. If

$$(2.1)\qquad nP\{|X| > c_n\} = o(1)$$

then the $WLLN$

$$(2.2)\qquad \frac{\sum_{k=1}^{n} a_k (X_{nk} - X'_{nk})}{b_n} \to 0 \text{ in probability as } n \to \infty.$$

Proof. For arbitrary $\epsilon > 0$,

$$\begin{aligned} & P\left\{\frac{|\sum_{k=1}^{n} a_k(X_{nk} - X'_{nk})|}{b_n} > \epsilon\right\} \\ \leq\ & P\left\{\bigcup_{k=1}^{n}(X_{nk} \neq X'_{nk})\right\} \\ \leq\ & \sum_{k=1}^{n} P\{|X_{nk}| > c_n\} \\ \leq\ & O(1)nP\{|X| > c_n\} = o(1) \text{ by (2.1).} \end{aligned}$$

□

LEMMA 2.2. *Let $\{X_{nk}|1 \leq k \leq n, n \geq 1\}$ be an array of random variables which satisfy $P(|X_{nk}| > x) = O(1)P(|X| > x), \forall x \geq 0$, in a real number $p = 1, 2$. Let $\{a_n|n \geq 1\}$ and $\{b_n|n \geq 1\}$ be a sequence of constants with $a_n > 0,\ 0 < b_n \to \infty, n \geq 1$, and suppose that either*

$$\frac{b_n}{a_n}\uparrow,\ \frac{b_n}{na_n}\downarrow,\ \sum_{k=1}^{n} a_k^p = o(b_n^p),\ \text{and} \sum_{k=1}^{n}\frac{b_k^p}{k^2a_k^p} = O\left(\frac{b_n^p}{\sum_{k=1}^{n} a_k^p}\right) \tag{2.3}$$
$$\text{or } \frac{b_n}{a_n}\uparrow,\ \frac{b_n}{na_n} \to \infty,$$

$$\sum_{k=1}^{n} a_k^p = O(na_n^p),\ \text{and} \sum_{k=1}^{n}\frac{b_k^p}{k^2{a_k}^p} = O\left(\frac{b_n^p}{\sum_{k=1}^{n} a_k^p}\right) \tag{2.4}$$

or

$$\frac{b_n}{na_n}\uparrow,\ \text{and} \sum_{k=1}^{n} a_k = O(\frac{na_n}{\log n}) \tag{2.5}$$

hold. If (2.1) *holds, then*

$$\sum_{k=1}^{n} a_k^p P\{|X_{nk}| > c_n\} = o(a_n^p) \tag{2.6}$$

and

$$\sum_{k=1}^{n} a_k^p E|X_{nk}|^p I(|X_{nk}| \leq c_n) = o(b_n^p) \tag{2.7}$$

obtain, where $c_n = b_n/a_n$.

Proof. It is omitted because the proof is similar to the proof of [3]. □

REMARK 1. Note that assumption of array of row NA random variables is not required in Lemmas 2.1 and 2.2.

3. Main results

Applying Lemma 2.1 and Lemma 2.2, we establish some limit theorems as follows.

THEOREM 3.1. *Let $\{X_{nk}|1 \leq k \leq n, n \geq 1\}$ be an array of row NA random variables satisfying $P(|X_{nk}| > x) = O(1)P(|X| > x), x \geq 0$. Let $\{a_n|n \geq 1\}$ and $\{b_n|n \geq 1\}$ be a sequence of constants with $a_n > 0$, $0 < b_n \to \infty, n \geq 1$, and suppose that either (2.3) or (2.4) or (2.5) hold. If (2.1) holds, then the $WLLN$*

$$\frac{\sum_{k=1}^{n} a_k(X'_{nk} - EX'_{nk})}{b_n} \longrightarrow 0 \text{ in probability as } n \to \infty,$$

where $X'_{nk} = -c_n I(X_{nk} < -c_n) + X_{nk} I(|X_{nk}| \leq c_n) + c_n I(X_{nk} > c_n)$.

Proof. Let $X'_{nk} = -c_n I(X_{nk} < -c_n) + X_{nk} I(|X_{nk}| \leq c_n) + c_n I(X_{nk} > c_n)$.

In fact, from the definition of NA random variables, we know that $\{a_k X'_{nk}|1 \leq k \leq n, n \geq 1\}$ is still an array of row NA random variables. It follows from Chebyshev's inequality that for arbitrary $\epsilon > 0$,

$$\begin{aligned}
& P\left\{\frac{|\sum_{k=1}^{n} a_k(X'_{nk} - EX'_{nk})|}{b_n} > \epsilon\right\} \\
\leq\ & \frac{1}{\epsilon^2 b_n^2} E\left(\sum_{k=1}^{n} a_k(X'_{nk} - EX'_{nk})\right)^2 \\
\leq\ & C\frac{1}{b_n^2}\sum_{k=1}^{n} a_k^2 E(X'_{nk} - EX'_{nk})^2 \leq C\frac{1}{b_n^2}\sum_{k=1}^{n} a_k^2 EX'^2_{nk} \\
\leq\ & C\frac{1}{b_n^2}\sum_{k=1}^{n} a_k^2 EX^2_{nk} I(|X_{nk}| \leq c_n) + C\frac{1}{b_n^2}\sum_{k=1}^{n} a_k^2 c_n^2 P(|X_{nk}| \geq c_n) \\
\longrightarrow\ & 0 \text{ as } n \to \infty, \text{ by (2.6) and (2.7)},
\end{aligned}$$

where C is positive constant which may be different in various places. □

THEOREM 3.2. *Let $\{X_{nk}|1 \leq k \leq n, n \geq 1\}$ be an array of row NA random variables which satisfy $P(|X_{nk}| > x) = O(1)P(|X| > x), \forall x \geq 0$. Let $\{a_n|n \geq 1\}$ and $\{b_n|n \geq 1\}$ be a sequence of constants with*

$a_n > 0,\ 0 < b_n \to \infty, n \geq 1$, *and suppose that either* (2.3) *or* (2.4) *or* (2.5) *hold. If* (2.1) *holds, then*

$$\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_i X_{ni}|/b_n \to 0 \text{ in probability as } n \to \infty.$$

Proof. Let $X'_{nk} = -c_n I(X_{nk} < -c_n) + X_{nk} I(|X_{nk}| \leq c_n) + c_n I(X_{nk} > c_n)$ and $X''_{ni} = X_{ni} - X'_{ni}$. So,

$$\tag{2.8} \begin{aligned} &\max_{1\leq k\leq n} \frac{|\sum_{i=1}^{k} a_i X_{ni}|}{b_n} \\ &\leq \frac{\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_i X'_{ni}|}{b_n} + \frac{\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_i X''_{ni}|}{b_n} \\ &= I_1 + I_2. \end{aligned}$$

For any $\epsilon > 0$,

$$\tag{2.9} \begin{aligned} P(I_2 \geq \epsilon) &= P\left(\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_i X''_{ni}| \geq \epsilon b_n\right) \\ &\leq \sum_{i=1}^{n} P(|X_{ni}| > c_n) \\ &= O(1) n P(|X| > c_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Note that

$$\begin{aligned} &\frac{\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_i E X'_{ni}|}{b_n} \\ \leq\ & \frac{1}{b_n} \sum_{i=1}^{n} a_i [(c_n P|X_{ni}| > c_n) + E|X_{ni}| I(|X_{ni}| \leq c_n)] \\ \leq\ & \frac{1}{a_n} (\sum_{i=1}^{n} a_i) P(|X| > c_n) + \frac{1}{b_n} \sum_{i=1}^{n} a_i E|X_{ni}| I(|X_{ni}| \leq c_n) \\ \longrightarrow\ & 0 \text{ as } n \to \infty, \text{ by (2.6) and (2.7).} \end{aligned}$$

Thus, to prove $I_1 \longrightarrow 0$ in probability, it suffices to show that for arbitrary $\epsilon > 0$,

$$P\left(\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_i (X'_{ni} - EX'_{ni})| > b_n \epsilon\right) \longrightarrow 0 \text{ as } \to \infty.$$

In fact, from the definition of NA variables, we know that $\{a_i X'_{ni} | 1 \leq i \leq k, n \geq 1\}$ is still an array of row NA random variables. Thus, using lemma 4 of Matula [14], we get

$$
\begin{aligned}
& P\left(\max_{1\le k\le n} |\sum_{i=1}^{k} a_i(X'_{ni} - EX'_{ni})| > \epsilon b_n\right) \\
\le\ & C\frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2 E(X'_{ni} - EX'_{ni})^2 \\
\le\ & C\frac{1}{a_n^2}\sum_{i=1}^{n} a_i^2 P(|X_{ni}| > c_n) + C\frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2 EX_{ni}^2 I(|X_{ni}| \le c_n) \\
\longrightarrow\ & 0 \text{ as } n \to \infty, \text{ by using (2.6) and (2.7).}
\end{aligned}
$$

□

THEOREM 3.3. *Let $\{X_{nk}|1 \le k \le n, n \ge 1\}$ be an array of row NA random variables with $EX_{nk} = 0$ and $P(|X_{nk}| > x) = O(1)P(|X| > x), \forall x \ge 0$. Assume that $\{a_n|\ n \ge 1\}$ and $\{b_n|\ n \ge 1\}$ are sequence of constants satisfying $a_n \ne 0,\ b_n > 0$ and*

$$c_n = \frac{b_n}{|a_n|}, \frac{b_n}{n|a_n|} \to \infty, \sum_{i=1}^{n} a_i^2 = O(na_n^2).$$

If $E|X| < \infty$ and (2.1) holds, then

$$\max_{1\le k\le n} |\sum_{i=1}^{k} a_i X_{ni}|/b_n \to 0 \ \text{ in probability as } n \to \infty.$$

Proof. The definition of I_1 and I_2 is as that in Theorem 3.2. Similar to the arguments in Theorem 3.2 , we get $I_2 \longrightarrow 0$ in probability. Note that by $EX_{ni} = 0$,

$$
\begin{aligned}
& \frac{\max_{1\le k\le n} |\sum_{i=1}^{k} a_i EX'_{ni}|}{b_n} \\
\le\ & \frac{1}{b_n}\sum_{i=1}^{n} |a_i| [(c_n P|X_{ni}| > c_n) + E|X_{ni}|I(|X_{ni}| > c_n)] \\
\le\ & O(1)\left[\left(\frac{\sum_{i=1}^{n} |a_i|}{b_n}\frac{b_n}{|a_n|}P(|X| > c_n) + \frac{\sum_{i=1}^{n} |a_i|}{b_n}E|X|I(|X| > c_n)\right)\right] \\
=\ & O(1)\left[\frac{\sum_{i=1}^{n} |a_i|}{|a_n|}P(|X| > c_n) + \frac{\sum_{i=1}^{n} |a_i|}{b_n}c_n \int_1^{\infty} P(|X| > xc_n)dx\right]
\end{aligned}
$$

$$\begin{aligned}
&\leq O(1)\left[\frac{n^{1/2}(\sum_{i=1}^n a_i^2)^{1/2}}{|a_n|}P(|X|>c_n)\right.\\
&\left.+\frac{\sum_{i=1}^n|a_i|}{|a_n|}\sum_{k=1}^{\infty}\int_k^{k+1}P(|X|>xc_n)dx\right]\\
&\leq O(1)\left[n(P|X|>c_n)+\frac{\sum_{i=1}^n|a_i|}{|a_n|}\sum_{k=1}^{\infty}P(|X|>kc_n)\right]\\
&\leq O(1)\left[n(P|X|>c_n)+\frac{\sum_{i=1}^n|a_i|}{b_n}\right]\\
&\leq O(1)\left[nP(|X|>c_n)+\frac{n|a_n|}{b_n}\right]=o(1)\text{ as } n\longrightarrow\infty.
\end{aligned}$$

Thus, to prove $I_1 \longrightarrow 0$ in probability, we need only to prove that for arbitrary $\epsilon > 0$,

$$P\left(\max_{1\leq k\leq n}|\sum_{i=1}^k a_i(X'_{ni}-EX'_{ni}|\geq \epsilon b_n\right)\longrightarrow 0 \text{ as } n\to\infty.$$

In fact, without loss generality, we may assume $a_i > 0$ since $a_i = a_i^+ - a_i^-$.

Hence, by the definition of X'_{ni}, we know that $\{a_iX'_{ni}|1\leq i\leq n,\ n\geq 1\}$ is still an array of row NA random variables. Now, by using Lemma 4 of Matula ([14]), we have

$$\begin{aligned}
&P\left(\max_{1\leq k\leq n}|\sum_{i=1}^k a_i(X'_{ni}-EX'_{ni})|\geq \epsilon b_n\right)\\
&\leq \frac{C}{b_n^2}\sum_{i=1}^n a_i^2EX'^2_{ni}\\
&\leq \frac{1}{b_n^2}\sum_{i=1}^n a_i^2[c_n^2P(|X_{ni}|>c_n)+EX_{ni}^2(I|X_{ni}|\leq c_n)]\\
&\leq O(1)\left[\frac{\sum_{i=1}^n a_i^2}{b_n^2}\frac{b_n^2}{a_n^2}P(|X|>c_n)+\frac{1}{b_n^2}\sum_{i=1}^n a_i^2c_nE|X_{ni}|\right]\\
&\leq O(1)\left[\frac{\sum_{i=1}^n a_i^2}{a_n^2}P(|X|>c_n)+\frac{\sum_{i=1}^n a_i^2}{|a_n|b_n}E|X|\right]\\
&\leq O(1)\left[n(P|X|>c_n)+\frac{n|a_n|}{b_n}\right]=o(1)\text{ as } n\to\infty.
\end{aligned}$$

□

REMARK 2. We restrict $EX_{nk} = 0$ and $E|X| < \infty$ in Theorem 3.3, but the restriction for $\{a_n\}$ and $\{b_n\}$ is weakened compared with Theorem 3.2. Furthermore, if $\{X_{nk}|1 \leq k \leq n, n \geq 1\}$ is identically distributed in Theorem 3.3, then the condition of $E|X| < \infty$ can be cancelled, while the condition of $EX_{nk} = 0$ is mild. Hence, we can find that there is different advantage for Theorems 3.2 and 3.3.

By extending the index set for NA variables in Theorem 3.2 to the set Z of integers, the proof is similar.

THEOREM 3.4. *Let* $\{X_{nk}|k \in Z, n \geq 1\}$ *be an* NA *array of random variables satisfying* $P(|X_{nk}| > x) = O(1)P(|X| > x)$. *Let* $\{a_n|n \geq 1\}$ *and* $\{b_n|n \geq 1\}$ *be sequence of constants with* $a_n > 0$, $0 < b_n \to \infty, n \geq 1$ *and suppose that either* (2.3) *or* (2.4) *or* (2.5) *hold. If* (2.1) *holds, then* $|\sum_{k\in Z} a_k X_{nk}|/b_n \to 0$ *in probability as* $n \to \infty$.

Proof. It is omitted because the proof is similar to Theorem 3.2. □

References

[1] A. Adler and A. Rosalsky, *Some general strong laws for weighted sums of stochastically dominated random variables*, Stochastic Anal. Appl. **5** (1987), 1–16.

[2] ———, *On the weak law of large number for normed weighted sums of i.i.d. random variables*, Internat, J. Math. and Math. Sci. **14** (1991), 191–202.

[3] A. Adler and A. A. Rosalsky, and R. L. Taylor, *A weak law for normed weighted sums of random elements in Rademacher Type p Banach spaces*, J. Multivariate Anal. **37** (1991), 259–268.

[4] K. Alam and K. M. Lal Saxena, *Positive dependence in multivariate distributions*, Comm. Statist. **A 10** (1981), 1183–1196.

[5] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, martingales*, 2nd ed., Springer-Verlag, 1988.

[6] W. Feller, *A limit theorem for random variables with infinite moments*, Amer. J. Math. **68** (1946), 257–262.

[7] A. Gut, *The weak law of large numbers for arrays*, Statist. Probab. Lett. **14** (1992), 49–52.

[8] D. H. Hong and K. S. Oh, *On the weak law of large numbers for arrays*, Statist. Probab. Lett. **22** (1995), 55–57.

[9] K. Joag-Dev and F. Proschan, *Negative association of random variables with applications*, Ann. Statist. **11** (1983), 286–295.

[10] K. Knopp, *Theory and application of infinite series*, 2nd English ed., Blackie, London, 1951.

[11] P. Kowalski and Z. Rychlik, *On the weak law of large numbers for randomly indexed partial sums for arrays*, Ann. Univ. Mariae Curie-sklodowska Sect. A **51** (1997), 109–119.

[12] E. L. Lehmann, *Some concepts of dependence*, Ann. Math. Statist. **37** (1966), 1137–1153.

[13] M. Loeve, *Probability Theory I*, 4th ed. Springer-Verlag, New York, 1977.
[14] P. Matula, *A note on the almost sure convergence of sums of negatively dependent random variables*, Statist. Probab. Lett. **15** (1992), 209–213.
[15] S. H. Sung, *Weak law of large numbers for arrays*, Statist. Probab. Lett. **38** (1998), 101–105.

Jong Il Baek, School of Mathematics & Informational Statistics, and Institute of Basic Natural Science, Wonkwang University, IkSan 570-749, Korea
E-mail: jibaek@wonkwang.ac.kr

Han-Ying Liang, Department of Applied Mathematics, Tongji University, Shanhai 200092, P. R. China
E-mail: hyliang83@hotmail.com

Jeong Yeol Choi, School of Mathematics & Informational Statistics, and Institute of Basic Natural Science, Wonkwang University, IkSan 570-749, Korea
E-mail: jychoi@wonkwang.ac.kr