FRENET EQUATIONS OF NULL CURVES

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ABSTRACT. The purpose of this paper is to study the geometry of null curves in a 6-dimensional semi-Riemannian manifold $M_q$ of index $q$, since the general $n$-dimensional cases are too complicated. We show that it is possible to construct three types of Frenet equations of null curves in $M_q$, supported by one example. We find each type of Frenet equations invariant under any causal change. And we discuss some properties of null curves in $M_q$.

1. INTRODUCTION

Theory of space curves of a Riemannian manifold is fully developed and its local and global geometry is well-known. But its counter part of the curve theory of a semi-Riemannian manifold is relatively new and in a developing stage. In case of semi-Riemannian manifolds, there are three categories of curves, namely, spacelike, timelike and null, depending on their causal character. We know from O'Neill [10], that the study of timelike curves has many similarities with the spacelike curves. However, since the induced metric of a null curve is degenerate, this case is much more complicated and also different from the non-degenerate case.

Duggal & Bejancu [4, Chapter 3] published their work on “general theory of null curves in Lorentz manifolds”. They constructed a Frenet frame and proved the fundamental existence and uniqueness theorem for this class of null curves. Their study was restricted to Lorentz manifold, since for the general semi-Riemannian manifolds of index greater than one, they have shown (by an example) that their Frenet frame is not invariant with respect to causal change of any of its generating vector fields.

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The objective of this paper is to study on null curves in a 6-dimensional semi-Riemannian manifold $M_q$ of index $q$, since the general $n$-dimensional cases are too complicated. And we guess their Frenet equations from the 6-dimensional cases. We show that it is possible to construct three types of Frenet frames suitable for $M_q$ and cite one example, each invariant under any causal change. This is then followed by constructing general Frenet equations (called compound Frenet equations) which include all the possible forms of the three types. And we study some properties of null curves in $M_q$.

2. Transversal vector bundles

Let $M_q$ be a real 6-dimensional semi-Riemannian manifold of constant index $q$ ($1 \leq q \leq 3$) and $C$ be a smooth null curve in $M_q$ locally given by

$$x^i = x^i(t), \quad t \in \mathbb{I} \subset \mathbb{R}, \quad i \in \{0, 1, \ldots, 5\}$$

for a coordinate neighborhood $\mathcal{U}$ on $C$. Since $C$ is a null curve, the tangent vector field $\frac{d}{dt} = \lambda$ on $\mathcal{U}$ satisfies $g(\lambda, \lambda) = 0$. Denote by $TC$ the tangent bundle of $C$ and $TC^\perp$ the $TC$-perp. Clearly, $TC^\perp$ is a vector bundle over $C$ of rank 5 and $TC$ is a vector subbundle of $TC^\perp$ of rank 1 (cf. Duggal & Bejancu [4] and O'Neill [10]). This implies that $TC^\perp$ is not complementary to $TC$ in $TM_q|C$. Thus we must find complementary vector bundle to $TC$ in $TM_q$ which will play the role of the normal bundle $TC^\perp$ consistent with the classical non-degenerate theory. A few researchers have done research on this matter dealing with only specified problems (cf. Bonnor [2], Duggal [3], Graves [6], Ikawa [8]). Duggal & Bejancu [4] developed a general mathematical theory to deal with the null case, which we briefly as follows:

Suppose $S(TC^\perp)$ denotes the complementary vector subbundle to $TC$ in $TC^\perp$, i.e., we have

$$TC^\perp = TC \perp S(TC^\perp),$$

where $\perp$ means the orthogonal direct sum. It follows that $S(TC^\perp)$ is a non-degenerate 4-dimensional vector subbundle of $TM_q$. We call $S(TC^\perp)$ a screen vector bundle of $C$, which being non-degenerate, we have

$$TM_q|C = S(TC^\perp) \perp S(TC^\perp)^\perp, \quad (1)$$

where $S(TC^\perp)^\perp$ is a 2-dimensional complementary orthogonal vector subbundle to $S(TC^\perp)$ in $TM_q|C$. 
Throughout this paper we denote by $F(C)$ the algebra of smooth functions on $C$ and by $\Gamma(E)$ the $F(C)$ module of smooth sections of a vector bundle $E$ over $C$. We use the same notation for any other vector bundle.

**Theorem 2.1** (Duggal & Bejancu [4]). Let $C$ be a null curve on a semi-Riemannian manifold $M_q$ and $S(TC^\perp)$ a screen vector bundle of $C$. Then there exists a unique vector bundle $\text{nt}(C)$ over $C$ of rank 1, such that on each coordinate neighborhood $U \subset C$ there is a unique section $N \in \Gamma(\text{nt}(C)|_U)$ satisfying

$$g(\lambda, N) = 1, \quad g(N, N) = g(N, X) = 0$$

for every $X \in \Gamma(S(TC^\perp)|_U)$.

We call $\text{nt}(C)$ the null transversal bundle of $C$ with respect to $S(TC^\perp)$. Next consider the vector bundle

$$\text{tr}(C) = \text{nt}(C)\perp S(TC^\perp),$$

which according to (1) and (2) is complementary but not orthogonal to $TC$ in $TM_q|_C$. More precisely, we have

$$TM_q|_C = TC \oplus \text{tr}(C) = (TC \oplus \text{nt}(C)) \perp S(TC^\perp).$$

We call $\text{tr}(C)$ the transversal vector bundle of $C$ with respect to $S(TC^\perp)$. The vector field $N$ in Theorem 2.1 is called the null transversal vector field of $C$ with respect to $\lambda$. As $\{\lambda, N\}$ is a null basis of $\Gamma((TC \oplus \text{nt}(C))|_U)$ satisfying (2), we obtain

**Proposition 2.1** (Duggal & Bejancu [4]). Let $C$ be a null curve on a semi-Riemannian manifold $M_q$. Then any screen vector bundle of $C$ is semi-Riemannian of index $q - 1$.

### 3. Frenet Equations of Type 1

Let $C$ be a null curve on $M_3$ and $N$ be the null transversal vector field of $C$. Denote $\nabla$ the Levi-Civita connection on $M_3$. In this section we study a class of null curves $C$ whose Frenet frame is made up of two null vector fields $\lambda$ and $N$, two timelike and two spacelike vector fields. We denote the Frenet equations of this particular class of $C$ by Type 1.

From $g(\lambda, \lambda) = 0$ and $g(\lambda, N) = 1$ we have

$$g(\nabla_\lambda \lambda, \lambda) = 0 \quad \text{and} \quad g(\nabla_\lambda \lambda, N) = -g(\lambda, \nabla_\lambda N) = h,$$
where \( h \) is a smooth function on \( \mathcal{U} \). These relations and the equation (3) imply that

\[
\nabla_{\lambda} \lambda = h \lambda + S_1,
\]

where \( S_1 \in S(TC^\perp) \). Thus \( S_1 \) is everywhere perpendicular to both \( \lambda \) and \( N \). Since \( S(TC^\perp) \) is a semi-Riemannian vector bundle of rank 4 and index 2, in general there are three cases (timelike, spacelike and null) by the causality of the vector field \( S_1 \).

In this section we assume that \( S_1 \) is non-null. Based on this restriction, we define the first curvature function \( \kappa_1 \) by \( \kappa_1 = \sigma_1 \varepsilon_1 \), where

\[
\sigma_1 = \|S_1\|
\]

and \( \varepsilon_1 = 1 \) or \(-1\) according as \( S_1 \) is spacelike or timelike, i.e., \( \varepsilon_1 \) is the signature of \( S_1 \). Now we set \( U_1 = S_1/\sigma_1 \) so that \( U_1 \) is a unit vector field along \( C \) which is everywhere perpendicular to \( \lambda \) and \( N \). Thus using above, we have

\[
\nabla_{\lambda} \lambda = h \lambda + \kappa_1 \varepsilon_1 U_1.
\]

Now, from \( g(\nabla_{\lambda} N, \lambda) = -h \), \( g(\nabla_{\lambda} N, N) = 0 \) and \( g(\nabla_{\lambda} N, U_1) = \kappa_2 \), where \( \kappa_2 \) denotes the second curvature function. We have

\[
\nabla_{\lambda} N = -h N + \kappa_2 \varepsilon_1 U_1 + S_2,
\]

where \( S_2 \) is a vector field on \( S(TC^\perp) \). Thus \( S_2 \) is perpendicular to \( \lambda, N \) and \( U_1 \). We assume that \( S_2 \) is also non-null. Define the third curvature function \( \kappa_3 \) by \( \kappa_3 = \sigma_2 \varepsilon_2 \), where

\[
\sigma_2 = \|T_2\|
\]

and \( \varepsilon_2 \) is the signature of \( S_2 \). Set \( U_2 = S_2/\sigma_2 \) so that \( U_2 \) is also a unit vector field along \( C \) and is everywhere parallel to \( S_2 \). Thus we have

\[
\nabla_{\lambda} N = -h N + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2.
\]

Repeating above process we have an orthonormal basis \( \{U_1, U_2, U_3, U_4\} \) of \( S(TC^\perp) \) which is made up of two timelikes and two spacelikes. Setting

\[
W_i = \varepsilon_i U_i, \quad i \in \{1, 2, 3, 4\},
\]
we obtain the following equations

\[
\begin{align*}
\nabla_\lambda \lambda &= h \lambda + \kappa_1 W_1, \\
\nabla_\lambda N &= -h N + \kappa_2 W_1 + \kappa_3 W_2, \\
\varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3, \\
\varepsilon_2 \nabla_\lambda W_2 &= -\kappa_3 \lambda - \kappa_4 W_1 + \kappa_6 W_3 + \kappa_7 W_4, \\
\varepsilon_3 \nabla_\lambda W_3 &= -\kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4, \\
\varepsilon_4 \nabla_\lambda W_4 &= -\kappa_7 W_2 - \kappa_8 W_3,
\end{align*}
\]  

(4)

where \( h \) and \( \{\kappa_1, \kappa_2, \ldots, \kappa_8\} \) are smooth functions on \( \mathcal{U} \), \( \{W_1, W_2, W_3, W_4\} \) is a certain orthonormal basis of \( \Gamma(S(TC^\perp)|\mathcal{U}) \) and \( \varepsilon_i = g(W_i, W_i) \) is the signature of each \( W_i \), such that \( \varepsilon_i = +1 \) or \(-1\). We call

\[ F_1 = \{\lambda, N, W_1, \ldots, W_4\} \]  

(5)

a Frenet frame of Type 1 on \( \mathcal{M}_3 \) along \( C \) with respect to a given screen vector bundle \( S(TC^\perp) \) and the equations (4) are called its Frenet equations of Type 1. Finally, the functions \( \{\kappa_1, \kappa_2, \ldots, \kappa_8\} \) are called curvature functions of \( C \) with respect to the frame \( F_1 \).

Remark 1. Since the screen bundle is semi-Riemannian of index 2, this implies that two of \( W_i \)'s are timelikes and another two are spacelikes. We know that the choice of different timelikes \( W_i \) generates different Frenet equations of the same type.

4. FRENNET EQUATIONS OF TYPE 2

In this section we study a class of null curves \( C \) whose Frenet frame is generated by a pseudo-orthonormal basis consisting of the two null vector fields \( \lambda \) and \( N \), additional two null vector fields \( L_i \) and \( L_{i+1} \) such that \( g(L_i, L_{i+1}) = 1 \), one timelike vector field \( U_j \) and one spacelike vector field \( U_k \), \( \{j, k\} \neq \{i, i + 1\} \). If we set

\[
U_i = \frac{1}{\sqrt{2}}(L_i - L_{i+1}), \quad \text{and} \quad U_{i+1} = \frac{1}{\sqrt{2}}(L_i + L_{i+1})
\]  

(6)

then \( U_i \) and \( U_{i+1} \) are timelike and spacelike vector fields, respectively, and \( F = \{\lambda, N, U_1, \ldots, U_4\} \) is a Frenet frame of \( C \), but have Frenet equations of another type. We denote the Frenet equations of this particular class of \( C \) by Type 2. There are three choices for \( \{L_i, L_{i+1}\} \): \( \{L_1, L_2\} \), \( \{L_2, L_3\} \) and \( \{L_3, L_4\} \).
To choose \( \{L_1, L_2\} \), we let the vector field \( \nabla_{\lambda} \lambda - h \lambda \) be null and define the curvature function \( K_1 \) by
\[
\nabla_{\lambda} \lambda = h \lambda + K_1 L_1,
\]
where \( L_1 \in \Gamma(S(TC^\perp)) \). Thus \( L_1 \) is a null vector field along \( C \) which is everywhere perpendicular to \( \lambda \) and \( N \). Since \( S(TC^\perp) \) is semi-Riemannian vector bundle of rank 4, we can take a vector field \( V \) along \( C \) such that \( g(L_1, V) \neq 0 \), otherwise \( S(TC^\perp) \) is degenerate. Set
\[
L_2 = \frac{1}{g(L_1, V)} \left\{ V - \frac{g(V, V)}{g(L_1, V)} L_1 \right\},
\]
then \( g(L_1, L_2) = 1 \) along \( C \). Set this case so that the equation (6) holds for \( i = 1 \). Therefore \( U_1 \) and \( U_2 \) are perpendicular to \( \lambda \) and \( N \) and we have
\[
\nabla_{\lambda} \lambda = h \lambda + \kappa_1 \varepsilon_1 U_1 + \tau_1 \varepsilon_2 U_2
\]
where \( \kappa_1 = -\tau_1 = \frac{K_1}{\sqrt{2}} \) and \( \varepsilon_i \) is the signature of each \( U_i \). Also,
\[
g(\nabla_{\lambda} N, \lambda) = -h, \quad g(\nabla_{\lambda} N, N) = 0, \quad g(\nabla_{\lambda} N, U_1) = \kappa_2, \quad g(\nabla_{\lambda} N, U_2) = \kappa_3
\]
implies that
\[
\nabla_{\lambda} N = -h N + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2 + S_3,
\]
where \( S_3 \) is a vector field perpendicular to \( \lambda, N, U_1 \) and \( U_2 \). Since \( S(TC^\perp) \) is a semi-Riemannian vector bundle of index 2, there are three cases by the causality of the vector field \( S_3 \). In this section we assume that \( S_3 \) is non-null. Now define a torsion function \( \tau_3 \) by \( \tau_3 = \sigma_3 \varepsilon_3 \) where \( \sigma_3 = \|S_3\| \) and \( \varepsilon_3 \) is the signature of \( S_3 \). Set \( U_3 = S_3 / \sigma_3 \), then \( U_3 \) is a unit non-null vector field along \( C \) which is also perpendicular to \( \lambda, N, U_1 \) and \( U_2 \). Thus we obtain
\[
\nabla_{\lambda} N = -h N + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2 + \tau_3 \varepsilon_3 U_3.
\]
(7)

Also from the following results
\[
g(\nabla_{\lambda} U_1, \lambda) = -\kappa_1,
g(\nabla_{\lambda} U_1, N) = -\kappa_2,
g(\nabla_{\lambda} U_1, U_1) = 0,
g(\nabla_{\lambda} U_1, U_2) = \kappa_4,
g(\nabla_{\lambda} U_1, U_3) = \kappa_5,
\]
we obtain
\[
\nabla_{\lambda} U_1 = -\kappa_2 \lambda - \kappa_1 N + \kappa_4 \varepsilon_2 U_2 + \kappa_5 \varepsilon_3 U_3 + S_4
\]
where $S_4$ is a non-null vector field perpendicular to $\lambda, N, U_1, U_2$ and $U_3$. Now we define another torsion function $\tau_5$ by $\tau_5 = \sigma_4 \varepsilon_4$ where $\sigma_4 = \|S_4\|$. Set $U_4 = S_4/\sigma_4$, then $U_4$ is also a unit non-null vector field along $C$ which is everywhere perpendicular to $\lambda, N, U_1, U_2$ and $U_3$. Thus we obtain
\[
\nabla_\lambda U_1 = -\kappa_2 \lambda - \kappa_1 N + \kappa_4 \varepsilon_2 U_2 + \kappa_5 \varepsilon_3 U_3 + \tau_5 \varepsilon_4 U_4.
\]
(8)

In a similar way we get
\[
\begin{align*}
\nabla_\lambda U_2 &= -\kappa_3 \lambda - \tau_1 N - \kappa_4 \varepsilon_1 U_1 + \kappa_6 \varepsilon_3 U_3 + \kappa_7 \varepsilon_4 U_4, \\
\nabla_\lambda U_3 &= -\tau_3 \lambda - \kappa_5 \varepsilon_1 U_1 - \kappa_6 \varepsilon_2 U_2 + \kappa_8 \varepsilon_4 U_4, \\
\nabla_\lambda U_4 &= -\tau_5 \varepsilon_1 U_1 - \kappa_7 \varepsilon_2 U_2 - \kappa_8 \varepsilon_3 U_3,
\end{align*}
\]
(9)

where
\[\kappa_6 = g(\nabla_\lambda U_2, U_3), \quad \kappa_7 = g(\nabla_\lambda U_2, U_4), \quad \kappa_8 = g(\nabla_\lambda U_3, U_4).\]

Setting
\[W_i = \varepsilon_i U_i, \quad i \in \{1, 2, 3, 4\}\]
we have the following equations
\[
\begin{align*}
\nabla_\lambda \lambda &= h \lambda + \kappa_1 W_1 + \tau_1 W_2, \\
\nabla_\lambda N &= -h N + \kappa_2 W_1 + \kappa_3 W_2 + \tau_3 W_3, \\
\varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3 + \tau_6 W_4, \\
\varepsilon_2 \nabla_\lambda W_2 &= -\kappa_3 \lambda - \tau_1 N - \kappa_4 W_1 + \kappa_6 W_3 + \kappa_7 W_4, \\
\varepsilon_3 \nabla_\lambda W_3 &= -\tau_3 \lambda - \kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4, \\
\varepsilon_4 \nabla_\lambda W_4 &= -\tau_5 W_1 - \kappa_7 W_2 - \kappa_8 W_3.
\end{align*}
\]
(10)

In the above case, we call
\[F_{2}^{(1)} = \{\lambda, N, W_1, W_2, W_3, W_4\}\]
(11)
a Frenet frame of Type 2 on $M_3$ along $C$ with respect to a given screen vector bundle $S(TC^\perp)$ and the equations (10) its Frenet equations of Type 2. The functions \{\kappa_1, \kappa_2, \ldots, \kappa_8\} and \{\tau_1, \tau_3, \tau_5\} are called the curvature functions and the torsion functions of $C$ with respect to the frame $F_{2}^{(1)}$.

On the other hand, using \{L_1, L_2\} such that $g(L_1, L_2) = 1$ and
\[g(\nabla_\lambda N, \lambda) = -h, \quad g(\nabla_\lambda N, L_1) = K_3, \quad g(\nabla_\lambda N, L_2) = K_2,\]
we can write
\[
\nabla_\lambda N = -h N + K_2 L_1 + K_3 L_2 + Q_3
\]
where $Q_3$ is perpendicular to $\lambda, N, L_1$ and $L_2$. Also

$$\nabla_{\lambda}N + hN - K_2L_1 - K_3L_2$$

$$= \nabla_{\lambda}N + hN - \frac{K_2}{\sqrt{2}}(U_2 + U_1) - \frac{K_3}{\sqrt{2}}(U_2 - U_1)$$

$$= \nabla_{\lambda}N + hN + \left(\frac{K_3 - K_2}{\sqrt{2}}\right)U_1 - \left(\frac{K_3 + K_2}{\sqrt{2}}\right)U_2$$

and

$$\kappa_2 = g(\nabla_{\lambda}N, U_1) = \frac{1}{\sqrt{2}}\{g(\nabla_{\lambda}N, L_1 - L_2)\} = \frac{K_3 - K_2}{\sqrt{2}},$$

$$\kappa_3 = g(\nabla_{\lambda}N, U_2) = \frac{1}{\sqrt{2}}\{g(\nabla_{\lambda}N, L_1 + L_2)\} = \frac{K_2 + K_3}{\sqrt{2}}.$$

Using above results and the equation (7) we conclude that $Q_3 = \tau_3\varepsilon_3U_3$. Therefore

$$\nabla_{\lambda}N = -hN + K_2L_1 + K_3L_2 + \tau_3\varepsilon_3U_3. \quad (12)$$

In a similar way we obtain

$$\nabla_{\lambda}L_1 = -K_3\lambda + K_4L_1 + K_5\varepsilon_3U_3 + \overline{K}_6\varepsilon_4\overline{U}_4, \quad (13)$$

$$\nabla_{\lambda}L_2 = -K_2\lambda - K_1N - K_4L_2 + K_7\varepsilon_3U_3 + \overline{K}_8\varepsilon_4\overline{U}_4, \quad (14)$$

where $\overline{U}_4$ is a unit non-null vector field perpendicular to $\lambda, N, L_1, L_2$ and $U_3$, and the smooth functions $K_i$ $(i = 4, 5, 7)$ and $\overline{K}_j$ $(j = 6, 8)$ are defined by

$$K_4 = g(\nabla_{\lambda}L_1, L_2) = -g(L_1, \nabla_{\lambda}L_2),$$

$$K_5 = g(\nabla_{\lambda}L_1, U_3) = -g(L_1, \nabla_{\lambda}U_3),$$

$$\overline{K}_6 = g(\nabla_{\lambda}L_1, \overline{U}_4) = -g(L_1, \nabla_{\lambda}\overline{U}_4),$$

$$K_7 = g(\nabla_{\lambda}L_2, U_3) = -g(L_2, \nabla_{\lambda}U_3),$$

$$\overline{K}_8 = g(\nabla_{\lambda}L_2, \overline{U}_4) = -g(L_2, \nabla_{\lambda}\overline{U}_4).$$

Next, by the transformations (6) for $i = 1$ we have

$$\nabla_{\lambda}U_1 = \frac{1}{\sqrt{2}}(\nabla_{\lambda}L_1 - \nabla_{\lambda}L_2), \quad \nabla_{\lambda}U_2 = \frac{1}{\sqrt{2}}(\nabla_{\lambda}L_1 + \nabla_{\lambda}L_2).$$

Using (13) and (14) in above equations and the following results

$$\kappa_4 = g(\nabla_{\lambda}U_1, U_2) = g(\nabla_{\lambda}L_1, L_2) = K_4,$$

$$\kappa_5 = g(\nabla_{\lambda}U_1, U_3) = \frac{1}{\sqrt{2}}g(\nabla_{\lambda}L_1 - \nabla_{\lambda}L_2, U_3) = \frac{1}{\sqrt{2}}(K_5 - K_7),$$

$$\kappa_6 = g(\nabla_{\lambda}U_2, U_3) = \frac{1}{\sqrt{2}}g(\nabla_{\lambda}L_1 + \nabla_{\lambda}L_2, U_3) = \frac{1}{\sqrt{2}}(K_5 + K_7),$$
we obtain
\[ K_6 \varepsilon^4 U_4 = K_6 \varepsilon_4 U_4, \quad K_8 \varepsilon^4 U_4 = K_8 \varepsilon_4 U_4, \]
where \[ K_6 = \frac{\kappa_7 + \tau_5}{\sqrt{2}}, \quad K_8 = \frac{\kappa_7 - \tau_5}{\sqrt{2}}. \]

Thus (13) and (14) become
\[ \nabla_{\lambda} L_1 = -K_3 \lambda + K_4 L_1 + K_5 \varepsilon_3 U_3 + K_6 \varepsilon_4 U_4, \]
\[ \nabla_{\lambda} L_2 = -K_2 \lambda - K_1 N - K_4 L_2 + K_7 \varepsilon_3 U_3 + K_8 \varepsilon_4 U_4. \]

In a similar way we get
\[ \nabla_{\lambda} U_3 = -\tau_3 \lambda - K_7 L_1 - K_5 L_2 + \kappa_8 \varepsilon_4 U_4, \]
\[ \nabla_{\lambda} U_4 = -K_8 L_1 - K_6 L_2 - \kappa_8 \varepsilon_3 U_3. \]

Setting \[ W_i = \varepsilon_i U_i, \quad i \in \{3, 4\}, \]
we get the following equations
\begin{align*}
\nabla_{\lambda} \lambda & = h \lambda + K_1 L_1, \\
\nabla_{\lambda} N & = -h N + K_2 L_1 + K_3 L_2 + \tau_3 W_3, \\
\nabla_{\lambda} L_1 & = -K_3 \lambda + \kappa_4 L_1 + K_5 W_3 + K_6 W_4, \\
\nabla_{\lambda} L_2 & = -K_2 \lambda - K_1 N - \kappa_4 L_2 + K_7 W_3 + K_8 W_4, \\
\varepsilon_3 \nabla_{\lambda} W_3 & = -\tau_3 \lambda - K_7 L_1 - K_5 L_2 + \kappa_8 W_4, \\
\varepsilon_4 \nabla_{\lambda} W_4 & = -K_8 L_1 - K_6 L_2 - \kappa_8 W_3,
\end{align*}
\[ (15) \]

where
\[ K_1 = -\sqrt{2} \kappa_1 = \sqrt{2} \tau_1, \quad K_2 = \frac{\kappa_3 - \kappa_2}{\sqrt{2}}, \quad K_3 = \frac{\kappa_3 + \kappa_2}{\sqrt{2}}, \]
\[ K_5 = \frac{\kappa_6 + \kappa_5}{\sqrt{2}}, \quad K_6 = \frac{\kappa_7 + \tau_5}{\sqrt{2}}, \quad K_7 = \frac{\kappa_6 - \kappa_5}{\sqrt{2}}, \quad K_8 = \frac{\kappa_7 - \tau_5}{\sqrt{2}}. \]

In this case, we also call
\[ F_2^{(1)} = \{ \lambda, N, L_1, L_2, W_3, W_4 \} \]
\[ (16) \]
a Frenet frame of Type 2 on \( M_3 \) along \( C \) with respect to a given screen vector bundle \( S(TC^{\perp}) \) and the equations (15) its Frenet equations of Type 2.

In the next cases, to choose \( \{ L_2, L_3 \} \) and \( \{ L_3, L_4 \} \), we let the vector fields
\[ \nabla_{\lambda} N + h N - k_2 U_1 \quad \text{and} \quad \nabla_{\lambda} U_1 + k_2 \lambda + k_1 N - k_4 U_2 \]
be null in turn, then using a procedure same as above for each such cases, we obtain the equations of the form (10) with the torsion functions \( \{ \tau_1 = 0, \tau_3 = -\kappa_3, \tau_5 \} \), or equivalently,

\[
\begin{align*}
\nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1, \\
\nabla_\lambda N &= -hN + \kappa_2 W_1 + K_3 L_2, \\
\varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + K_4 L_2 + K_5 L_3 + \tau_5 W_4, \\
\nabla_\lambda L_2 &= -K_5 W_1 + \kappa_6 L_2 + K_8 W_4, \\
\nabla_\lambda L_3 &= -K_3 \lambda - K_4 W_1 - \kappa_6 L_3 + K_9 W_4, \\
\varepsilon_4 \nabla_\lambda W_4 &= -\tau_5 W_1 - K_9 L_2 - K_8 L_3,
\end{align*}
\]

where

\[
\begin{align*}
\frac{K_3}{\sqrt{2}} &= -\kappa_3, & \frac{1}{\sqrt{2}}(K_4 - K_5) &= -\kappa_4, & \frac{1}{\sqrt{2}}(K_4 + K_5) &= \kappa_5, \\
\frac{K_3}{\sqrt{2}} &= \tau_3, & \frac{1}{\sqrt{2}}(K_8 - K_9) &= \kappa_7, & \frac{1}{\sqrt{2}}(K_8 + K_9) &= \kappa_8,
\end{align*}
\]

and \( \{ \tau_1 = 0, \tau_3 = 0, \tau_5 = -\kappa_5 \} \), or equivalently,

\[
\begin{align*}
\nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1, \\
\nabla_\lambda N &= -hN + \kappa_2 W_1 + \kappa_3 W_2, \\
\varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + \kappa_4 W_2 + K_5 L_3, \\
\varepsilon_2 \nabla_\lambda W_2 &= -\kappa_3 \lambda - \kappa_4 W_1 + K_6 L_3 + K_7 L_4, \\
\nabla_\lambda L_3 &= -K_7 W_2 + \kappa_8 L_3, \\
\nabla_\lambda L_4 &= -K_5 W_1 - K_6 W_2 - \kappa_8 L_4,
\end{align*}
\]

where

\[
\begin{align*}
\frac{K_5}{\sqrt{2}} &= -\kappa_5 = \tau_5, & \frac{1}{\sqrt{2}}(K_6 - K_7) &= \kappa_6, & \frac{1}{\sqrt{2}}(K_6 + K_7) &= \kappa_7,
\end{align*}
\]

respectively. In the cases \( \{ L_2, L_3 \} \) and \( \{ L_3, L_4 \} \), we also call

\[
F_2^{(2)} = \{ \lambda, N, W_1, L_2, L_3, W_4 \} \quad \text{and} \quad F_2^{(3)} = \{ \lambda, N, W_1, W_2, L_3, L_4 \},
\]

respectively, Frenet frames of Type 2 on \( M_3 \) along \( C \) with respect to a given screen vector bundle \( S(TC^\perp) \), and the equations (17) and (18) their Frenet equations of Type 2, respectively.

**Remark 2.** We know that the Frenet equations (10) include all three different Frenet equations of Type 2. Hence we call the equations (10) the general Frenet equations.
of Type 2 of the null curve \( C \) and \( F_2 = \{\lambda, N, W_1, \ldots, W_4\} \) the general Frenet frame of Type 2 on \( M_3 \) along the null curve \( C \).

5. Frenet equations of Type 3

In this section we study a class of null curves \( C \) whose Frenet frame is generated by a pseudo-orthonormal basis consisting of the two null vector fields \( \lambda \) and \( N \) and additional four null vector fields \( L_1, L_2, L_3 \) and \( L_4 \) such that \( g(L_i, L_{i+1}) = 1, \ i = 1, 3 \). If we set also

\[
U_i = \frac{L_i - L_{i+1}}{\sqrt{2}}, \quad U_{i+1} = \frac{L_i + L_{i+1}}{\sqrt{2}}, \quad i = 1, 3.
\]

then \( \{U_1, U_3\} \) and \( \{U_2, U_4\} \) are timelike and spacelike vector fields respectively, and \( F = \{\lambda, N, U_1, \ldots, U_4\} \) is also a Frenet frame of \( C \), but have Frenet equations of the other type. We denote the Frenet equations of this particular class of \( C \) by Type 3. There is only one choice for \( \{L_1, L_2, L_3, L_4\} \).

In the section 4 for Type 2, there exist two null vector fields \( \{L_1, L_2\} \), or equivalently, one timelike vector field and one spacelike vector field \( \{U_1, U_2\} \) satisfying

\[
\begin{align*}
\nabla_\lambda \lambda &= h\lambda + K_1 L_1 = h\lambda + \kappa_1 \varepsilon_1 U_1 + \tau_1 \varepsilon_2 U_2, \\
\nabla_\lambda N &= -hN + K_2 L_1 + K_3 L_2 + S_3 = -hN + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2 + S_3.
\end{align*}
\]

In this section we let the vector field \( S_3 \) be null and define the curvature function \( T_3 \) by

\[
\nabla_\lambda N = -hN + K_2 L_1 + K_3 L_2 + T_3 L_3 = -hN + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2 + T_3 L_3,
\]

where \( L_3 \) is a null vector field along \( C \) perpendicular to \( \lambda, N, L_1, L_2, U_1 \) and \( U_2 \). Then there exists another null vector field \( L_4 \) along \( C \) such that \( g(L_3, L_4) = 1 \) and is everywhere perpendicular to \( \lambda, N, L_1, L_2, U_1 \) and \( U_2 \). Set this case so that the equation (19) hold for \( i = 3 \). Therefore \( U_3 \) and \( U_4 \) are perpendicular to \( \lambda, N, L_1, L_2, U_1 \) and \( U_2 \) and we have

\[
\nabla_\lambda N = -hN + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2 + \tau_3 \varepsilon_3 U_3 + \tau_2 \varepsilon_4 U_4,
\]

where \( \tau_2 = -\tau_3 = \frac{T_3}{\sqrt{2}} \). Also from the following results

\[
\begin{align*}
g(\nabla_\lambda U_1, \lambda) &= -\kappa_1, & g(\nabla_\lambda U_1, N) &= -\kappa_2, & g(\nabla_\lambda U_1, U_1) &= 0, \\
g(\nabla_\lambda U_1, U_2) &= \kappa_4, & g(\nabla_\lambda U_1, U_3) &= \kappa_5, & g(\nabla_\lambda U_1, U_4) &= \tau_5,
\end{align*}
\]

\[
\begin{align*}
g(\nabla_\lambda U_2, \lambda) &= -\kappa_4, & g(\nabla_\lambda U_2, N) &= -\kappa_5, & g(\nabla_\lambda U_2, U_1) &= 0, \\
g(\nabla_\lambda U_2, U_2) &= -\kappa_1, & g(\nabla_\lambda U_2, U_3) &= -\kappa_2, & g(\nabla_\lambda U_2, U_4) &= \tau_3.
\end{align*}
\]
we obtain

\[ \nabla_\lambda U_1 = -\kappa_2 \lambda - \kappa_1 N + \kappa_4 \varepsilon_2 U_2 + \kappa_6 \varepsilon_3 U_3 + \tau_5 \varepsilon_4 U_4. \]  

(22)

In a similar way we get

\[
\begin{align*}
\nabla_\lambda U_2 &= -\kappa_3 \lambda - \tau_1 N - \kappa_4 \varepsilon_1 U_1 + \kappa_6 \varepsilon_3 U_3 + \kappa_7 \varepsilon_4 U_4, \\
\nabla_\lambda U_3 &= -\tau_3 \lambda - \kappa_5 \varepsilon_1 U_1 - \kappa_6 \varepsilon_2 U_2 + \kappa_8 \varepsilon_4 U_4, \\
\nabla_\lambda U_4 &= -\tau_2 \lambda - \tau_5 \varepsilon_1 U_1 - \kappa_7 \varepsilon_2 U_2 - \kappa_8 \varepsilon_3 U_3.
\end{align*}
\]  

(23)

Setting

\[ W_i = \varepsilon_i U_i, \quad i \in \{1, 2, 3, 4\} \]

we have the following equations

\[
\begin{align*}
\nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1 + \tau_1 W_2, \\
\nabla_\lambda N &= -hN + \kappa_2 W_1 + \kappa_3 W_2 + \tau_3 W_3 + \tau_2 W_4, \\
\varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3 + \tau_5 W_4, \\
\varepsilon_2 \nabla_\lambda W_2 &= -\kappa_3 \lambda - \tau_1 N - \kappa_4 W_1 + \kappa_6 W_3 + \kappa_7 W_4, \\
\varepsilon_3 \nabla_\lambda W_3 &= -\tau_3 \lambda - \kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4, \\
\varepsilon_4 \nabla_\lambda W_4 &= -\tau_2 \lambda - \tau_5 W_1 - \kappa_7 W_2 - \kappa_8 W_3.
\end{align*}
\]  

(24)

In this case, we call

\[ F_3 = \{\lambda, N, W_1, W_2, W_3, W_4\} \]  

(25)

a Frenet frame of Type 3 on \( M_3 \) along \( C \) with respect to a give screen vector bundle \( S(\pi \mathcal{C}^\perp) \) and the equations (24) its Frenet equations of Type 3. The functions \( \{\kappa_1, \kappa_2, \ldots, \kappa_8\} \) and \( \{\tau_1, \tau_2, \tau_3, \tau_5\} \) are called the curvature functions and the torsion functions of \( C \) with respect to the frame \( F_3 \).

On the other hand, using the set of null vector fields \( \{L_1, L_2, L_3, L_4\} \) such that \( g(L_1, L_2) = 1, g(L_3, L_4) = 1 \) and all other \( g(L_i, L_j) = 0, i \leq j \), we have

\[
\begin{align*}
\nabla_\lambda \lambda &= h\lambda + K_1 L_1, \\
\nabla_\lambda N &= -hN + K_2 L_1 + K_3 L_2 + T_3 L_3, \\
\nabla_\lambda L_1 &= -K_3 \lambda + K_4 L_1 + T_4 L_3 + T_5 L_4, \\
\nabla_\lambda L_2 &= -K_2 \lambda - K_1 N - K_4 L_2 + T_6 L_3 + T_7 L_4, \\
\nabla_\lambda L_3 &= -T_7 L_1 - T_5 L_2 + T_8 L_3, \\
\nabla_\lambda L_4 &= -T_3 \lambda - T_6 L_1 - T_8 L_4.
\end{align*}
\]  

(26)

In this case, we also call

\[ F_3 = \{\lambda, N, L_1, L_2, L_3, L_4\} \]  

(27)
a Frenet frame of Type 3 on $\mathbf{M}$ along $C$ with respect to a given screen vector bundle $S(TC^\perp)$ and the equations (26) its Frenet equations of Type 3. The functions $\{K_1, \ldots, K_4\}$ and $\{T_5, \ldots, T_8\}$ are called the curvature functions and the torsion functions of $C$ with respect to the frame $F_3$.

Next, by the transformations (19) for $i = 3$, we have

$$\nabla_\lambda U_3 = \frac{1}{\sqrt{2}}(\nabla_\lambda L_3 - \nabla_\lambda L_4), \quad \nabla_\lambda U_4 = \frac{1}{\sqrt{2}}(\nabla_\lambda L_3 + \nabla_\lambda L_4).$$

Using the following relations

$$\frac{T_4 - T_5}{\sqrt{2}} = \frac{1}{\sqrt{2}}g(\nabla_\lambda L_1, L_4 - L_3) = -g(\nabla_\lambda L_1, U_3) = -K_5,$$

$$\frac{T_4 + T_5}{\sqrt{2}} = \frac{1}{\sqrt{2}}g(\nabla_\lambda L_1, L_4 + L_3) = g(\nabla_\lambda L_1, U_4) = K_6,$$

$$\frac{T_6 - T_7}{\sqrt{2}} = \frac{1}{\sqrt{2}}g(\nabla_\lambda L_2, L_4 - L_3) = -g(\nabla_\lambda L_1, U_3) = -K_7,$$

$$\frac{T_6 + T_7}{\sqrt{2}} = \frac{1}{\sqrt{2}}g(\nabla_\lambda L_2, L_4 + L_3) = g(\nabla_\lambda L_1, U_4) = K_8,$$

$$T_8 = g(\nabla_\lambda L_3, L_4) = g(\nabla_\lambda U_3, U_4) = K_8 = \kappa_8,$$

$$T_4 L_3 + T_5 L_4 = -K_5 U_3 + K_6 U_4,$$

$$T_6 L_3 + T_7 L_4 = -K_7 U_3 + K_8 U_4,$$

we have

$$\begin{align*}
\nabla_\lambda \lambda &= h\lambda + K_1 L_1, \\
\nabla_\lambda N &= -hN + K_2 L_1 + K_3 L_2 + T_3 W_3 + T_2 W_4, \\
\nabla_\lambda L_1 &= -K_3 \lambda + \kappa_4 L_1 + K_5 W_3 + K_6 W_4, \\
\nabla_\lambda L_2 &= -K_2 \lambda - K_1 N - \kappa_4 L_2 + K_7 W_3 + K_8 W_4, \\
\varepsilon_3 \nabla_\lambda W_3 &= -T_3 \lambda - K_7 L_1 - K_5 L_2 + \kappa_8 W_4, \\
\varepsilon_4 \nabla_\lambda W_4 &= -T_2 - K_8 L_1 - K_6 L_2 - \kappa_8 W_3,
\end{align*}$$

where

$$K_1 = -\sqrt{2}\kappa_1, \quad K_2 = \frac{\kappa_3 - \kappa_2}{\sqrt{2}}, \quad K_3 = \frac{\kappa_3 + \kappa_2}{\sqrt{2}}, \quad K_4 = \kappa_4,$$

$$K_5 = \frac{\kappa_6 + \kappa_5}{\sqrt{2}}, \quad K_6 = \frac{\kappa_7 + \tau_3}{\sqrt{2}}, \quad K_7 = \frac{\kappa_6 - \kappa_5}{\sqrt{2}}, \quad K_8 = \frac{\kappa_7 - \tau_3}{\sqrt{2}}.$$

In this case, we also call

$$F_3 = \{\lambda, N, L_1, L_2, W_3, W_4\}$$
a Frenet frame of Type 3 on $M_3$ along $C$ with respect to a given screen vector bundle $S(TC^{-1})$ and the equations (28) its Frenet equations of Type 3.

Remark 3. We know that the Frenet equations (24) include all of six different Frenet equations of Type 1 (in the case $\tau_1 = \tau_2 = \tau_3 = \tau_5 = 0$), three different Frenet equations of Type 2 (in the case $\tau_2 = 0$) and one Frenet equations of Type 3. By the same calculation, we find that $M_2$ have Frenet equations of two types, named by Type 1 and Type 2, up to the signatures of $W_i$'s and $M_1$ have Frenet equations of only one type, named by Type 1, up to the signatures of $W_i$'s. Hence we call the equations (24) the compound Frenet equations of the null curve $C$ and $F = \{\lambda, N, W_1, \ldots, W_4\}$ the compound Frenet frame on $M_q$ ($1 \leq q \leq 3$) along $C$.

Remark 4. In general, let $(M, g)$ be a real $(m + 2)$-dimensional semi-Riemannian manifold of index $q$ and $C$ be a smooth null curve in $M$. We know that $C$ has $k$-type Frenet equations, namely Type 1, Type 2, \ldots, Type $k$, where $k = \min\{q, m + 2 - q\}$.

**Example 1.** Let $R^8_3$ be a 6-dimensional semi-Riemannian space of index 3 with the semi-Riemannian metric

$$g(x, y) = -x^0 y^0 - x^1 y^1 - x^2 y^2 + x^3 y^3 + x^4 y^4 + x^5 y^5.$$ 

Suppose $C$ is a null curve in $R^8_3$ given by the equations

$$C : (A \cos t, A \sin t, B \sinh t, B \cosh t, At, Bt)$$

where $A, B, t \in R$ such that $(A, B) \neq (0, 0)$. Then,

$$\lambda = (-A \sin t, A \cos t, B \cosh t, B \sinh t, A, B),$$

$$\nabla_\lambda \lambda = (-A \cos t, -A \sin t, B \sinh t, B \cosh t, 0, 0).$$

If we take a spacelike vector field $V$ along $C$ such that

$$V = \begin{cases} 
(0, 0, 0, 0, 1, 0), & \text{if } A \neq 0, \\
(0, 0, 0, 0, 0, 1), & \text{if } B \neq 0, 
\end{cases}$$

then $g(\lambda, V) = A$ or $B$ and $g(V, V) = 1$. By the relation

$$N = \frac{1}{g(\lambda, V)} \left\{ V - \frac{g(V, V)}{2g(\lambda, V)} \lambda \right\},$$

we obtain the following null transversal vector field

$$N = \begin{cases} 
\frac{1}{2A^2}(A \sin t, -A \cos t, -B \cosh t, -B \sinh t, A, -B), & \text{if } A \neq 0, \\
\frac{1}{2B^2}(A \sin t, -A \cos t, -B \cosh t, -B \sinh t, -A, B), & \text{if } B \neq 0. 
\end{cases}$$
We need to know the causal character of the vector field $H(t) = \nabla_\lambda \lambda - h \lambda$ along $C$. By direct calculations we obtain

$$g(H(t), H(t)) = B^2 - A^2.$$ 

Hence $H(t)$ is spacelike, timelike or lightlike according as $B^2 > A^2$, $B^2 < A^2$ or $B^2 = A^2$ respectively.

Choose $A = 1$ and $B = \sqrt{2}$, then $H(t)$ is spacelike and the curve

$$C : (\cos t, \sin t, \sqrt{2} \sinh t, \sqrt{2} \cosh t, t, \sqrt{2}t)$$

falls in the Type 1 with the Frenet frame $F = \{\lambda, N, W_1, W_2, W_3, W_4\}$ as follows

$$\lambda = (-\sin t, \cos t, \sqrt{2} \cosh t, \sqrt{2} \sinh t, 1, \sqrt{2}),$$

$$N = \frac{1}{2}(\sin t, -\cos t, -\sqrt{2} \cosh t, -\sqrt{2} \sinh t, 1, -\sqrt{2}),$$

$$W_1 = (-\cos t, -\sin t, \sqrt{2} \sinh t, \sqrt{2} \cosh t, 0, 0),$$

$$W_2 = (-\sqrt{2} \cos t, -\sqrt{2} \sin t, \sinh t, \cosh t, 0, 0),$$

$$W_3 = (-\sqrt{2} \sin t, \sqrt{2} \cos t, 0, 0, 1),$$

$$W_4 = (\sqrt{2} \sin t, -\sqrt{2} \cos t, -\cosh t, -\sinh t, 0, -2).$$

The Frenet equations (4) and the curvature functions are given by

$$\nabla_\lambda \lambda = W_1, \quad \nabla_\lambda N = -\frac{1}{2} W_1, \quad \nabla_\lambda W_1 = \frac{1}{2} \lambda - N - \sqrt{2} W_3,$$

$$\nabla_\lambda W_2 = -2 W_3 - W_4, \quad \nabla_\lambda W_3 = -\sqrt{2} W_1 + 2 W_2, \quad \nabla_\lambda W_4 = -W_2,$$

with

$$h = 0, \quad \kappa_1 = 1, \quad \kappa_2 = -\frac{1}{2}, \quad \kappa_3 = 0, \quad \kappa_4 = 0,$$

$$\kappa_5 = -\sqrt{2}, \quad \kappa_6 = 2, \quad \kappa_7 = 1, \quad \kappa_8 = 0.$$ 

Next we set $A = \sqrt{2}$ and $B = 1$. The curve

$$C : (\sqrt{2} \cos t, \sqrt{2} \sin t, \sinh t, \cosh t, \sqrt{2}t, t)$$

also falls in Type 1 with the Frenet frame $F = \{\lambda, N, W_1, W_2, W_3, W_4\}$ as follows

$$\lambda = (-\sqrt{2} \sin t, \sqrt{2} \cos t, \cosh t, \sinh t, \sqrt{2}, 1),$$

$$N = \frac{1}{2}(-\sqrt{2} \sin t, -\sqrt{2} \cos t, -\cosh t, -\sinh t, -\sqrt{2}, 1),$$
\[ W_1 = (-\sqrt{2} \cos t, -\sqrt{2} \sin t, \sinh t, \cosh t, 0, 0), \]
\[ W_2 = (-\cos t, -\sin t, \sqrt{2} \sinh t, \sqrt{2} \cosh t, 0, 0), \]
\[ W_3 = (0, 0, \sqrt{2} \cosh t, \sqrt{2} \sinh t, 1, 0), \]
\[ W_4 = (\sin t, -\cos t, -\sqrt{2} \cosh t, -\sqrt{2} \sinh t, -2, 0). \]

The Frenet equations (4) and the curvature functions are given by
\[ \nabla_\lambda \lambda = W_1, \quad \nabla_\lambda N = -\frac{1}{2} W_1, \quad \nabla_\lambda W_1 = -\frac{1}{2} \lambda + N + \sqrt{2} W_3 \]
\[ \nabla_\lambda W_2 = 2W_3 + W_4, \quad \nabla_\lambda W_3 = -\sqrt{2}W_1 + 2W_2, \quad \nabla_\lambda W_4 = -W_2, \]
with
\[ h = 0, \quad \kappa_1 = 1, \quad \kappa_2 = -\frac{1}{2}, \quad \kappa_3 = 0, \quad \kappa_4 = 0, \]
\[ \kappa_5 = -\sqrt{2}, \quad \kappa_6 = 2, \quad \kappa_7 = 1, \quad \kappa_8 = 0. \]

Finally we set \( A = B = 1 \), then \( H(t) \) is lightlike. Therefore, the curve
\[ C : (\cos t, \sin t, \sinh t, \cosh t, t, t) \]
falls in the Type 3 with the Frenet frame \( F = \{ \lambda, N, L_1, L_2, L_3, L_4 \} \) as follows
\[ \lambda = (-\sin t, \cos t, \cosh t, \sinh t, 1, 1), \]
\[ N = \frac{1}{2}(\sin t, -\cos t, -\cosh t, -\sinh t, 1, -1), \]
\[ L_1 = (-\cos t, -\sin t, \sinh t, \cosh t, 0, 0), \]
\[ L_2 = \frac{1}{2}(\cos t, \sin t, \sinh t, \cosh t, 0, 0), \]
\[ L_3 = (-\sin t, \cos t, 0, 0, 0, 1), \]
\[ L_4 = (0, 0, \cosh t, \sinh t, 0, 1). \]

The Frenet equations (26) and the curvature and torsion functions are given by
\[ \nabla_\lambda \lambda = L_1, \quad \nabla_\lambda N = -\frac{1}{2} L_1, \]
\[ \nabla_\lambda L_1 = -L_3 + L_4, \quad \nabla_\lambda L_2 = \frac{1}{2} \lambda - N - \frac{1}{2}(L_3 + L_4), \]
\[ \nabla_\lambda L_3 = \frac{1}{2} L_1 - L_2, \quad \nabla_\lambda L_4 = \frac{1}{2} L_1, \]
with
\[ h = 0, \quad K_1 = 1, \quad K_2 = -\frac{1}{2}, \quad K_3 = 0, \quad K_4 = 0, \]
\[ T_3 = 0, \quad T_4 = -1, \quad T_5 = 1, \quad T_6 = T_7 = -\frac{1}{2}, \quad T_8 = 0. \]
The Frenet equations (28) gives

\[ \nabla_\lambda \lambda = L_1, \quad \nabla_\lambda N = -\frac{1}{2} L_1, \quad \nabla_\lambda L_1 = \sqrt{2} W_3, \]

\[ \nabla_\lambda L_2 = \frac{1}{2} \lambda - N - \frac{1}{\sqrt{2}} W_4, \quad \nabla_\lambda W_3 = \frac{1}{\sqrt{2}} L_2, \quad \nabla_\lambda W_4 = \frac{1}{\sqrt{2}} (L_1 - L_2). \]

Also the Frenet equations (24) gives

\[ \nabla_\lambda \lambda = \frac{1}{\sqrt{2}} (W_2 - W_1), \]

\[ \nabla_\lambda N = \frac{1}{2 \sqrt{2}} (W_1 - W_2) \]

\[ \nabla_\lambda W_1 = \frac{1}{2 \sqrt{2}} \lambda - \frac{1}{\sqrt{2}} N - W_3 - \frac{1}{2} W_4, \]

\[ \nabla_\lambda W_2 = \frac{1}{2 \sqrt{2}} \lambda - \frac{1}{\sqrt{2}} N + W_3 - \frac{1}{2} W_4, \]

\[ \nabla_\lambda W_3 = \frac{1}{2} (W_1 + W_2), \]

\[ \nabla_\lambda W_4 = -W_1. \]

6. INVARiance OF Frenet FRAMES

In this section we show that each type of the Frenet frames always transform to the same type by the canonical parameter transformations of the coordinate neighborhood of C and the screen vector bundle. And we discuss some properties of null curves in \( M_4 \).

First, with respect to a given screen vector bundle \( S(TC^\perp) \), we consider two Frenet frames \( F \) and \( F^* \) along two neighborhoods \( U \) and \( U^* \) respectively with non-null intersection. Then we have

\[ \lambda^* = \frac{dt}{dt^*} \lambda, \quad N^* = \frac{dt}{dt} N, \quad (30) \]

\[ W^*_\alpha = \sum_{\beta=1}^{4} A^\beta_\alpha W_\beta, \quad 1 \leq \alpha \leq 4, \quad (31) \]

where \( A^\beta_\alpha \) are smooth functions on \( U \cap U^* \) and the matrix \((A^\beta_\alpha(x))\) is an element of the semi-orthogonal group \( O(q - 1, 4 - q + 1) \) for any \( x \in U \cap U^* \).
If we write the first and second equations of the compound Frenet equations (24) for both $F$ and $F^*$ and use (30) and (31), we obtain
\begin{equation}
\frac{d^2 t}{dt^2} + h \left( \frac{dt}{dt^*} \right)^2 = h^* \frac{dt}{dt^*},
\end{equation}
\begin{equation}
(\kappa_1^*, \tau_1^*, 0, 0) \left( A_0^\beta(x) \right) = (\kappa_1, \tau_1, 0, 0) \left( \frac{dt}{dt^*} \right)^2,
\end{equation}
\begin{equation}
(\kappa_2^*, \kappa_3^*, \tau_3^*, \tau_2^*) \left( A_\alpha^\beta(x) \right) = (\kappa_2, \kappa_3, \tau_3, \tau_2).
\end{equation}
From these relations we have

**Lemma 1.** Let $C$ be a null curve of a semi-Riemannian manifold $M_q$, and $F$ and $F^*$ be two Frenet frames of Type 1 on $U$ and $U^*$ with their respective curvature functions, induced by the same screen vector bundle $S(TC^\perp)$. Suppose $\kappa_1 \kappa_3 \neq 0$ on $U \cap U^* \neq \emptyset$. Then at any point of $U \cap U^*$ we have

\begin{align}
\kappa_1^* &= \kappa_1 A_1 \left( \frac{dt}{dt^*} \right)^2, \\
\kappa_2^* &= \kappa_2 A_1, \\
\kappa_3^* &= \kappa_3 A_2, \\
\kappa_\alpha^* &= \kappa_\alpha A_{\alpha-1} \frac{dt}{dt^*},
\end{align}

where $4 \leq \alpha \leq 8$ and $A_i = \pm 1$.

**Proof.** From the relations (33) satisfying $\tau_1 = \tau_1^* = 0$, we have $k_1^* \neq 0$ on $U \cap U^*$ and $A_1^2 = A_1^3 = A_1^4 = 0$. Since $(A_0^\beta(x))$ is a semi-orthogonal matrix, we infer that $A_1^1 = A_1 = \pm 1$ and $A_3^1 = A_3^1 = A_4^1 = 0$. Then from the second equation of Type 1 with respect to $F$ and $F^*$, and taking into account that $\kappa_3 \neq 0$, we obtain $k_3^* \neq 0$ on $U \cap U^*$ which implies $A_3^2 = A_3^3 = A_3^4 = 0$ and $A_2^2 = A_2 = \pm 1$. Repeating this process for all other equations we obtain all the relations in (35), which completes the proof. \qed

**Proposition 6.1.** Let $C$ be a null curve of a semi-Riemannian manifold $M_q$, and $F$ and $F^*$ be two Frenet frames of Type 1 on $U$ and $U^*$ with their respective curvature functions, induced by the same screen vector bundle $S(TC^\perp)$. Suppose $\kappa_1 \kappa_3 \neq 0$ on $U \cap U^* \neq \emptyset$. Then the second and third curvatures $\kappa_2$ and $\kappa_3$ are invariant to the coordinate transformations.
Next, we let $F$ and $F^*$ be Frenet frames of Type 2 or 3 on $\mathcal{U}$ and $\mathcal{U}^*$ and assume that the orthonormal basis of a screen vector bundle $S(TC^\perp)$ of $C$ have the signature $(-, +, (\pm), +)$, where $(\pm)$ is $+$ or $-$, according to the index $q = 2$ or $q = 3$. From the equation (33) we have

$$A_1^4 - A_2^4 = A_2^2 - A_1^2, \quad A_1^3 = A_2^3, \quad A_1^4 = A_2^4,$$  

(36)

because $\tau_1^* = -\kappa_1^*$ and $\tau_1 = -\kappa_1$. Using (31), (36) and $L_1 = \frac{1}{\sqrt{2}}(W_2 - W_1)$ due to (6) and (19), we obtain

$$L_1^* = (A_1^4 - A_2^4)L_1$$  

(37)

where $A_1^4 - A_2^4 \neq 0$, otherwise the matrix $(A_\alpha(x))$ is singular.

Since $W_1^*$ is a timelike vector field and $W_2^*$ is a spacelike vector fields, the first row $(A_1^1, \ldots, A_4^1)$ of $(A_\alpha(x))$ is a timelike vector field and the second row $(A_1^2, \ldots, A_4^2)$ is a spacelike vector field of $\mathbf{R}^{4-1}$ and these vectors are perpendicular to each other. Thus, we have

$$(A_1^1)^2 - (A_2^1)^2 - 1 = (A_1^2)^2 - (A_2^2)^2 + 1 = A_1^2A_2^2 - A_1^2A_2^2.$$

From this relation we have the following two relations

$$A_1^2A_2^2 - A_1^1A_2^1 = 1, \quad (A_1^1 - A_2^1)(A_1^1 + A_2^1) = 1.$$  

(38)

Using the relations

$$L_1 = \frac{1}{\sqrt{2}}(W_2 - W_1), \quad L_2 = \frac{1}{\sqrt{2}}(W_2 + W_1),$$

$$W_1 = \frac{1}{\sqrt{2}}(L_2 - L_1), \quad W_2 = \frac{1}{\sqrt{2}}(L_2 + L_1),$$

we have

$$W_\alpha^* = \frac{1}{\sqrt{2}}(A_\alpha^2 - A_\alpha^1)L_1 + \frac{1}{\sqrt{2}}(A_\alpha^2 + A_\alpha^1)L_2 + A_\alpha^3W_3 + A_\alpha^4W_4, \quad \alpha \in \{1, 2\}$$  

(39)

$$L_2^* = \frac{1}{2}(A_2^2 + A_2^1 - A_1^1 - A_1^2)L_1 + (A_1^1 + A_2^1)L_2 + \sqrt{2}A_1^3W_3 + \sqrt{2}A_1^4W_4.$$  

(40)

The scalar product of $L_1^*$ and $W_\alpha^*$, $L_2^*$ and $L_2^*$, and $L_2^*$ and $W_\alpha^*$ provide

$$A_3^1 = -A_3^1,$$

$$A_4^1 = -A_4^1,$$

$$(A_2^1 + A_2^2 - A_1^1 - A_1^2)(A_1^1 + A_2^1 + A_2^2 + A_2^3) = 4\{(A_1^1)^2 - (A_1^2)^2\},$$

$$(A_\alpha^1 - A_\alpha^2)(A_1^2 + A_1^3 + A_2^1 + A_1^4) = 4\{A_\alpha^3A_\alpha^2 - A_\alpha^1A_\alpha^4\}, \quad \alpha \in \{3, 4\},$$
respectively. Using (36) and (38), the last two equations reduce
\[
A_2^2 + A_1^2 - A_1^1 - A_2^1 = 2( (A_2^3)^2 - (A_1^3)^2 ) (A_1^1 - A_2^1),
\]
\[
A_\alpha^2 - A_\alpha^1 = 2( A_\alpha^3 A_\alpha^3 - A_\alpha^1 A_\alpha^1 ) (A_1^1 - A_2^1), \quad \alpha \in \{3, 4\},
\]
respectively. From these two equations, we have
\[
A_2^2 - A_1^1 = \{ (A_1^3)^2 - (A_2^3)^2 \} (A_1^1 - A_2^1),
\]
\[
A_1^2 - A_2^1 = 2\{ (A_1^3)^2 - (A_2^3)^2 \} (A_1^1 - A_2^1),
\]
\[
A_\alpha^2 = \{ A_\alpha^3 A_\alpha^3 - A_\alpha^1 A_\alpha^1 \} (A_1^1 - A_2^1), \quad \alpha \in \{3, 4\}.
\]
Thus if \( A_1^3 = A_1^4 = 0 \), then
\[
A_2^3 = A_2^4 = A_1^3 = A_1^4 = A_3^3 = A_3^4 = 0 \quad \text{and} \quad A_1^1 = A_2^2, \quad A_2^1 = A_2^2.
\]
Since the matrix \((A_\alpha^\beta(x))\) is an element of semi-orthogonal group \(O(q-1, 4-q+1)\) and the signature of the orthonormal basis \(\{W_1, W_2, W_3, W_4\}\) of the screen vector bundle \(S(\mathbb{C}^4)\) is \((-+, (+), +)\), the third and the fourth rows of \((A_\alpha^\beta(x))\) satisfy
\[
(\pm)(A_3^3)^2 + (A_4^3)^2 = (+)1, \quad (\pm)(A_3^4)^2 + (A_4^4)^2 = 1, \quad (\pm)A_3^3 A_4^3 + A_3^4 A_4^4 = 0,
\]
\[
(\pm)(A_3^3)^2 + (A_4^4)^2 = (+)1, \quad (\pm)(A_3^4)^2 + (A_4^3)^2 = 1, \quad (\pm)A_3^3 A_4^3 + A_3^4 A_4^4 = 0,
\]
This relations provide the following two relations:
\[
A_3^3 = A_4^4, \quad A_3^3 = A_4^4 \quad \text{or} \quad A_3^3 = -A_4^4, \quad A_3^3 = -A_4^4
\]
with the aid of the fact that \(A_3^3 A_4^4 - A_3^4 A_4^3 = (+)1\).

Since the matrix \((A_\alpha^\beta(x))\) is a semi-orthogonal, this matrix is orthogonally diagonalizable by an orthogonal diagonalization \(P\), i. e.,
\[
(^tPA(x)P) = \begin{pmatrix}
\cosh \theta_1 & \sinh \theta_1 & 0 & 0 \\
\sinh \theta_1 & \cosh \theta_1 & 0 & 0 \\
0 & 0 & \cosh \theta_2 & \sinh \theta_2 \\
0 & 0 & (\pm) \sinh \theta_2 & (\pm) \cosh \theta_2
\end{pmatrix}.
\]
Assume yhat, no loss generality, this matrix also denote
\[
(A_\alpha^\beta(x)) = \begin{pmatrix}
A_1^1 & A_1^2 & 0 & 0 \\
A_2^1 & A_2^2 & 0 & 0 \\
0 & 0 & A_3^3 & A_3^4 \\
0 & 0 & A_4^3 & A_4^4
\end{pmatrix}.
\]
The coordinate transformation (31) of this form is called canonical. For the canonical transformation, from the equations (39) and (40), we obtain

\[ L_2^* = (A_1^1 + A_2^1)L_2, \]

\[ W_\alpha^* = \sum_{\beta=3}^{4} A_\alpha^\beta W_\beta, \quad \alpha \in \{3, 4\}. \]

Exchange the equations (15) for \( i = 2, 3 \) and \( i = 3, 4 \) we obtain the following general relations

\[ L_i^* = (A_i^1 - A_{i+1}^1)L_i, \quad L_{i+1}^* = (A_i^1 + A_{i+1}^1)L_{i+1}, \]

\[ W_\alpha^* = \sum_{\beta=1}^{4} A_\alpha^\beta W_\beta, \quad \alpha \in \{1, 2\}. \]

Thus we have

**Proposition 6.2.** Let \( C \) be a null curve of a semi-Riemannian manifold \( M_q \) and \( F \) and \( F^* \) be two Frenet frames of Type 2 on \( U \cap U^* \), induced by the same screen vector bundle \( S(TC^\perp) \). Suppose \( \kappa_1 \neq 0 \) on \( U \cap U^* \). Then \( S(TC^\perp) \) is an orthogonal direct sum of two invariant subspaces \( \text{Span}\{L_i, L_{i+1}\} = \text{Span}\{W_i, W_{i+1}\} \) and \( \text{Span}\{W_1, W_i, W_{i+1}, W_4\} \) by the canonical transformation of coordinate neighborhoods of \( C \), where overhat (‘) denotes the deleted symbol for that term.

Similarly, from the Frenet equations (26), we obtain the following general relations

\[ L_i^* = (A_i^1 - A_{i+1}^1)L_i, \quad L_{i+1}^* = (A_i^1 + A_{i+1}^1)L_{i+1}, \quad i = 1, 3. \]

**Proposition 6.3.** Let \( C \) be a null curve of a semi-Riemannian manifold \( M_q \) and \( F \) and \( F^* \) be two Frenet frames of Type 3 on \( U \cap U^* \), induced by the same screen vector bundle \( S(TC^\perp) \). Suppose \( \kappa_1 \neq 0 \) on \( U \cap U^* \). Then \( S(TC^\perp) \) is a direct orthogonal direct sum of two invariant subspaces \( \text{Span}\{L_1, L_2\} = \text{Span}\{W_1, W_2\} \) and \( \text{Span}\{L_3, L_4\} = \text{Span}\{W_3, W_4\} \) by the canonical transformation of coordinate neighborhoods of \( C \).

**Proposition 6.4.** Let \( C \) be a null curve of a semi-Riemannian manifold \( M_q \), and \( F \) and \( F^* \) be two Frenet frames on \( U \) and \( U^* \) with curvatures \( \{\kappa_1, \kappa_2, \ldots, \kappa_8\} \) and \( \{\kappa_1^*, \kappa_2^*, \ldots, \kappa_8^*\} \) and torsion functions \( \{\tau_1, \tau_2, \tau_3, \tau_5\} \) and \( \{\tau_1^*, \tau_2^*, \tau_3^*, \tau_5^*\} \) induced by the same screen vector bundle \( S(TC^\perp) \) respectively. Suppose \( \kappa_1 \kappa_3 \neq 0 \) on \( U \cap U^* \neq \emptyset \). Then the type of Frenet equations is invariant to the coordinate transformations.
Proof. In the first case suppose \( F^* = F_2^* \) or \( F_3^* \) and \( F = F_1 \). Then we have \( \tau_1^* = -\kappa_1 \) and \( \tau_1 = 0 \). This means from equation (33) that \( A_1^2 = A_2^3 \). Since \( W_1^* \) and \( W_2^* \) are the timelike and spacelike vector fields respectively, the first row \( (A_1^1, A_1^2, 0, 0) \) of \( (A_G^2(x)) \) is timelike vector field and the second row \( (A_2^1, A_2^2, 0, 0) \) is a spacelike vector field of \( \mathbb{R}^4_{q-1} \) and these vectors are perpendicular to each other. Thus, we have

\[
(A_1^1)^2 - 1 = (A_2^1)^2 + 1 = A_1^1 A_2^1.
\]

From this relation we have the contradictory relation \( A_1^1 = A_2^1 \). Hence this case is not possible.

Conversely, if \( F^* = F_1^* \) and \( F = F_2 \) or \( F_3 \), then \( \tau_1^* = 0 \) and \( \tau_1 = -\kappa_1 \). From the equations (33) we have \( A_1^1 = -A_1^2 \). This means that the first row \( (A_1^1, A_1^2, 0, 0) \) of the matrix \( (A_G^2(x)) \) is an null vector, hence the vector field \( W_1^* \) is a null vector field, thus we conclude that this case is also not possible.

In the next cases suppose

\[
F^* = F_3^*, F = F_2 \text{ and } F^* = F_2^*, F = F_3
\]

respectively. From the equation (34), we have

\[
\begin{align*}
\kappa_2^* A_1^1 + \kappa_3^* A_2^1 &= \kappa_2, \\
\kappa_2^* A_1^2 + \kappa_3^* A_2^2 &= \kappa_3, \\
\tau_3^* A_3^3 + \tau_2^* A_3^4 &= \tau_3, \\
\tau_3^* A_3^4 + \tau_2^* A_3^1 &= \tau_2.
\end{align*}
\]

(41)

In case \( F^* = F_3^*, F = F_2 \), we have \( A_3^3 = A_3^4 \) and

\[
(A_3^3)^2 - 1 = (A_3^3)^2 + 1 = A_3^3 A_3^4.
\]

From this relation we have the contradictory relation \( A_3^3 = A_3^4 \). Hence this case is not possible. In another case \( F^* = F_2^*, F = F_3 \). From the equation (41) we conclude that this case is also not possible, which complete the proof. \( \square \)

Using the Frenet equations of Type 2 and Type 3 in (15) and (26) and the method of Lemma 1, we have the following lemmas.

**Lemma 2.** Let \( C \) be a null curve of a semi-Riemannian manifold \( M_q \), and \( F \) and \( F^* \) be two Frenet frames of Type 2 on \( U \) and \( U^* \) with respective curvature and torsion functions, induced by the same screen vector bundle \( S(TC^\perp) \). If \( \tau_3 \) is non-zero on
\( U \cap U^* \neq \emptyset \), then \( A_3^4 = A_4^3 = 0 \) and at any point of \( U \cap U^* \) we have

\[
\begin{aligned}
K_1^* &= K_1 D_1 \left( \frac{dt}{dt^*} \right)^2, & \tau_3^* &= \tau_3 A_3, \\
K_2^* &= K_2 D_1, & K_3^* &= K_3 C_1, \\
K_5^* &= K_5 C_2 \frac{dt}{dt^*}, & K_6^* &= K_6 C_3 \frac{dt}{dt^*}, \\
K_7^* &= K_7 D_2 \frac{dt}{dt^*}, & K_8^* &= K_8 D_3 \frac{dt}{dt^*}, \\
\kappa_4^* &= \kappa_4 \frac{dt}{dt^*} + D_1 \frac{dC_1}{dt^*}, & \kappa_5^* &= \kappa_5 A_4 \frac{dt}{dt^*},
\end{aligned}
\] (42)

where

\[
C_1 = A_1^1 - A_2^1, \quad C_2 = C_1 A_3^3, \quad C_3 = C_1 A_4^4, \\
D_1 = A_1^1 + A_2^1, \quad D_2 = E_1 A_3^3, \quad D_3 = E_1 A_4^4, \\
C_i D_i = 1, \quad i \in \{1, 2, 3\}, \\
A_3 = A_4 = \pm 1.
\]

**Lemma 3.** Let \( C \) be a null curve of a semi-Riemannian manifold \( M_q \), and \( F \) and \( F^* \) be two Frenet frames of Type 3 on \( U \) and \( U^* \) with respective curvature and torsion functions, induced by the same screen vector bundle \( S(T C^\perp) \). Then we have

\[
\begin{aligned}
K_1^* &= K_1 D_1 \left( \frac{dt}{dt^*} \right)^2, & K_2^* &= K_2 D_1, \\
K_3^* &= K_3 C_1, & K_4^* &= K_4 \frac{dt}{dt^*} + D_1 \frac{dC_1}{dt^*}, \\
T_3^* &= T_3 G_1, & T_4^* &= T_4 G_3 \frac{dt}{dt^*}, \\
T_5^* &= T_5 E_2 \frac{dt}{dt^*}, & T_6^* &= T_6 G_2 \frac{dt}{dt^*}, \\
T_7^* &= T_7 E_3 \frac{dt}{dt^*}, & T_8^* &= T_8 \frac{dt}{dt^*} + G_1 \frac{dE_1}{dt^*},
\end{aligned}
\] (43)

where

\[
E_1 = A_3^3 - A_4^3, \quad E_2 = C_1 E_1, \quad E_3 = D_1 E_1, \\
G_1 = A_3^3 + A_4^3, \quad G_2 = D_1 G_1, \quad G_3 = C_1 G_1, \\
E_i G_i = 1, \quad i \in \{1, 2, 3\}.
\]
Next, let \( F = \{ \lambda, N, W_1, \ldots, W_4 \} \) and \( \overline{F} = \{ \overline{\lambda}, \overline{N}, \overline{W}_1, \ldots, \overline{W}_4 \} \) be two Frenet frames with respect to \( (t, S(TC^1, U)) \) and \( (\overline{t}, \overline{S}(TC^1, \overline{U})) \), respectively. Then the general transformations that relate elements of \( F \) and \( \overline{F} \) on \( U \cap \overline{U} \), are

\[
\begin{align*}
\overline{\lambda} & = \frac{dt}{d\bar{t}} \lambda, \\
\overline{N} & = -\frac{1}{2} \frac{dt}{d\bar{t}} \sum_{\alpha=1}^{4} \varepsilon_{\alpha}(c_\alpha)^2 \lambda + \frac{d\bar{t}}{dt} N + \sum_{\alpha=1}^{4} c_\alpha W_\alpha, \\
\overline{W}_\alpha & = \sum_{\beta=1}^{4} B^\beta_\alpha \left( W_\beta - \varepsilon_\beta \frac{dt}{d\bar{t}} c_\beta \lambda \right), \quad 1 \leq \alpha \leq 4,
\end{align*}
\] (44)

where \( c_\alpha \) and \( B^\beta_\alpha \) are smooth functions on \( U \cap \overline{U} \) and the \( 4 \times 4 \) matrix \( (B^\beta_\alpha(x)) \) is an element of the semi-orthogonal group \( O(q-1, 4-q+1) \) for each \( x \in U \cap \overline{U} \). Then by using (44) and the first equation of the compound Frenet equation for both \( F \) and \( \overline{F} \) we obtain

\[
\begin{align*}
\hat{h} = \frac{d^2 t}{d\bar{t}^2} \frac{dt}{d\bar{t}} + \hat{h} \frac{dt}{d\bar{t}} + (c_2 \tau_1 - c_1 \kappa_1) \left( \frac{dt}{d\bar{t}} \right)^2, \\
(\hat{\kappa}_1, \overline{\tau}_1, 0, 0) (B^\beta_\alpha(x)) = (\kappa_1, \tau_1, 0, 0) \left( \frac{dt}{d\bar{t}} \right)^2.
\end{align*}
\] (45)

The screen vector bundle transformation (44) of the form

\[
(B^\beta_\alpha(x)) = \begin{pmatrix}
B^1_1 & B^1_2 & 0 & 0 \\
B^2_1 & B^2_2 & 0 & 0 \\
0 & 0 & B^3_3 & B^4_3 \\
0 & 0 & B^3_4 & B^4_4
\end{pmatrix}
\]

is called canonical. Using above relations and the method of Proposition 6.4, we obtain the following proposition.

**Proposition 6.5.** Let \( C \) be a null curve of a semi-Riemannian manifold \( M_q \) and \( F, \overline{F} \) be two Frenet frames with respect to \( (t, S(TC^1, U)) \) and \( (\overline{t}, \overline{S}(TC^1, \overline{U})) \) and their respective curvature functions. If \( \kappa_1 \neq 0 \) for all \( t \), then the type of Frenet equations is invariant of the canonical screen vector bundle transformations.

The following properties of the compound Frenet equations hold:

(a) The vanishing of the first curvature \( \kappa_1 \) on a neighborhood is independent of both the parameter transformations on \( C \) and the screen vector bundle transformations.
(b) It is possible to find a parameter on $C$ such that $h = 0$ in Frenet equations of all possible types, using the same screen bundle.

To prove (a) we let $\kappa_1 = 0$ (which implies that $\kappa_1 = 0$) on $U \cap \overline{U}$. Then, there exists a point $x \in U \cap \overline{U}$ such that, for Type 1,

$$B_1^1(x) = \cdots = B_1^4(x) = 0$$

and, for Type 2 and Type 3,

$$B_1^1(x) - B_2^1(x) = \cdots = B_1^4(x) - B_2^4(x) = 0.$$

This implies that the first and the second rows of the matrix $(\beta_0^\alpha(x))$ are linearly dependent, which is not possible since this matrix belongs to $O(q - 1, 4 - q + 1)$. Hence it follows from the relation of (45) that (a) holds.

To prove (b) we consider the following differential equation

$$\frac{d^2 t}{dt^*^2} - h^* \frac{dt}{dt^*} = 0$$

whose general solution comes from

$$t = a \int_{t_0}^{t^*} \exp \left( \int_{s_0}^{s} h^*(t^*) \, ds \right) \, ds + b, \quad a, b \in \mathbb{R}.$$

It follows from the relation (33) that any of these solutions, with $a \neq 0$, might be taken as a special parameter on $C$, such that $h = 0$. Denote one such solution by $p = \frac{t-b}{a}$, where $t$ is the general parameter as defined in above equation. We call $p$ a distinguished parameter of $C$, in terms for which $h = 0$. It is important to note that when $t$ is replaced by $p$ in the compound Frenet equations (24), the first two equations become

$$\begin{cases}
\nabla_{\frac{d}{dp}} \frac{d}{dp} = \kappa_1 W_1 + \kappa_2 W_2, \\
\nabla_{\frac{d}{dp}} N = \kappa_2 W_1 + \kappa_3 W_2 + \kappa_3 W_3 + \kappa_2 W_4,
\end{cases} \tag{46}$$

and the other equations remain unchanged.

In case $\kappa_1 = 0$, then, since $\kappa_1 = 0$ or $\kappa_1 = -\kappa_1$, the first equation of (46) takes the following familiar form

$$\frac{d^2 x^i}{dp^2} + \sum_{j,k=0}^{5} \Gamma_{jk}^i \frac{dx^j}{dp} \frac{dx^k}{dp} = 0, \quad i \in \{0, \ldots, 5\},$$

where $\Gamma_{jk}^i$ are the Christoffel symbols of the second type induced by $\nabla$. Hence $C$ is a null geodesic of $\mathbf{M}$. The converse follows easily. Thus we have the following theorem.
**Theorem 6.1.** Let \( C \) be a null curve of a semi-Riemannian manifold \( M_3 \). Then \( C \) is a null geodesic of \( M \) if and only if the first curvature \( \kappa_1 \) vanishes identically on \( C \).

Suppose \( F = \{ \lambda, N, W_1, \ldots, W_4 \} \) and \( \bar{F} = \{ \bar{\lambda}, \bar{N}, \bar{W}_1, \ldots, \bar{W}_4 \} \) are two Frenet frames of \( C \) with their respective screen spaces. Then, we know from Propositions 6.4 and 6.5 that they both belong to one of Type 1, Type 2 and Type 3.

**Lemma 4.** Let \( C \) be a null curve of \( M_3 \), with \( \kappa_1 \neq 0 \), and two Frenet frames \( F \) and \( \bar{F} \) of Type 1. Then, their curvature functions are related by

\[
\begin{align*}
\kappa_1 &= \kappa_1 B_1 \left( \frac{dt}{dt} \right)^2, \\
\kappa_2 &= B_1 \left\{ \kappa_2 + \kappa_1 c_1 + \frac{d\kappa_1}{dt} - \frac{1}{2} \kappa_1 \left( \frac{dt}{dt} \right)^2 \sum_{\alpha=1}^{4} (c_\alpha)^2 - (c_2 \kappa_4 + c_3 \kappa_5) \frac{dt}{dt} \right\}, \\
\kappa_3 &= \kappa_3 B_2^2 + \bar{\kappa} \sum_{\alpha=2}^{4} B_2^2 c_\alpha + \sum_{\alpha=2}^{4} B_2^2 \frac{dc_\alpha}{dt} + (c_1 \kappa_4 - c_3 \kappa_6 - c_4 \kappa_7) B_2^2 \frac{dt}{dt} + (c_1 \kappa_5 + c_2 \kappa_6 - c_4 \kappa_8) B_2^2 \frac{dt}{dt}, \\
\kappa_4 &= B_1 \left\{ B_2^2 \left( \kappa_4 + \kappa_1 \frac{dt}{dt} c_2 \right) + B_3^2 \left( \kappa_5 + \kappa_1 \frac{dt}{dt} c_3 \right) + B_4^2 \kappa_1 \frac{dt}{dt} c_4 \right\} \frac{dt}{dt}, \\
\kappa_5 &= B_1 \left\{ B_3^2 \left( \kappa_4 + \kappa_1 \frac{dt}{dt} c_2 \right) + B_3^2 \left( \kappa_5 + \kappa_1 \frac{dt}{dt} c_3 \right) + B_3^2 \kappa_1 \frac{dt}{dt} c_4 \right\} \frac{dt}{dt}, \\
\kappa_6 &= \left| \begin{array}{ccc} B_2^2 & B_3^2 & \kappa_8 \\
B_2^3 & B_3^3 & -\kappa_7 \\
B_2^4 & B_3^4 & \kappa_6 \end{array} \right| \frac{dt}{dt} + \sum_{\alpha=2}^{4} B_3^d \frac{dB_3^\alpha}{dt}, \\
\kappa_7 &= \left| \begin{array}{ccc} B_2^3 & -\kappa_8 & B_4^2 \\
B_2^4 & \kappa_7 & B_4^3 \\
B_2^4 & -\kappa_6 & B_4^4 \end{array} \right| \frac{dt}{dt} + \sum_{\alpha=2}^{4} B_4^d \frac{dB_4^\alpha}{dt}, \\
\kappa_8 &= \left| \begin{array}{ccc} \kappa_8 & B_3^2 & B_4^2 \\
-\kappa_7 & B_3^2 & B_3^4 \\
\kappa_6 & B_3^4 & B_4^4 \end{array} \right| \frac{dt}{dt} + \sum_{\alpha=2}^{4} B_4^d \frac{dB_4^\alpha}{dt}.
\end{align*}
\]

**Proof.** For the Type 1 it follows from (45) that

\[ B_1^1 = B_4^1 = 0 \quad (i \neq 1), \quad B_1^1 = B_4^1 = \pm 1. \]
Therefore the general transformations relating the elements of $F$ and $\overline{F}$ on $\mathcal{U} \cap \overline{\mathcal{U}}$ are given by

$$
\begin{align*}
\bar{\lambda} &= \frac{dt}{dt} \lambda, \\
\bar{N} &= -\frac{1}{2} \frac{dt}{dt} \sum_{i=1}^{4} \epsilon_i (c_i)^2 \lambda + \frac{dt}{dt} N + \sum_{i=1}^{4} c_i W_i, \\
\bar{W}_1 &= B_1 (W_1 + \frac{dt}{dt} c_1 \lambda), \\
\bar{W}_\alpha &= \sum_{\beta=2}^{4} B_\alpha^\beta (W_\beta - \frac{dt}{dt} c_\beta \lambda), \quad \alpha \in \{2, 3, 4\}.
\end{align*}
$$

(47)

The relations (*) follow by straightforward calculations from the Frenet equations of Type 1 and the use of (47).

\[\square\]

**Theorem 6.2.** Let $C$ be a null curve of $M_3$, with a Frenet frame $F$ of Type 1 and a screen vector bundle $S(TC)$ on $\mathcal{U} \subset C$ such that $\kappa_1 \neq 0$ on $\mathcal{U}$. Then there exists a screen vector bundle $\overline{S}(TC)$ which induces another Frenet frame $\overline{F}$ of Type 1 on $\mathcal{U}$ such that $\overline{\kappa}_4 = \overline{\kappa}_5 = 0$.

**Proof.** Define the following vector fields in terms of the elements of $F$ on $\mathcal{U}$:

$$
\begin{align*}
\bar{N} &= -\frac{1}{2} \left( \frac{\kappa_4^2 + \kappa_5^2}{\kappa_1^2} \right) \lambda + N - \frac{\kappa_4}{\kappa_1} W_2 - \frac{\kappa_5}{\kappa_1} W_3, \\
\bar{W}_2 &= W_2 + \frac{\kappa_4}{\kappa_1} \lambda, \\
\bar{W}_3 &= W_3 + \frac{\kappa_5}{\kappa_1} \lambda, \\
\bar{W}_i &= W_i, \quad i \in \{1, 4\}.
\end{align*}
$$

(48)

Let $\mathcal{U}^*$ be another coordinate neighborhood with parameter $t^*$ on $C$ such that $\mathcal{U} \cap \mathcal{U}^* \neq \emptyset$. By Lemma 1 we have the following on $\mathcal{U} \cap \mathcal{U}^*$

$$
\begin{align*}
\kappa_1^* &= \kappa_1 A_1 \left( \frac{dt}{dt^*} \right)^2, \\
\kappa_4^* &= \kappa_4 A_3 \frac{dt}{dt^*}, \\
\kappa_5^* &= \kappa_5 A_4 \frac{dt}{dt^*}, \\
W_i^* &= W_i, \quad i \in \{1, 2, 3, 4\}.
\end{align*}
$$

(49)
Define \( \{ \mathcal{N}^*, \mathcal{W}_1^*, \mathcal{W}_2^*, \mathcal{W}_3^*, \mathcal{W}_4^* \} \) by (48) but on \( \mathcal{U}^* \) with respect to \( F^* \), induced by the same \( S(\mathcal{T}C^\perp) \) on \( \mathcal{U}^* \). Then by using (30), (31), (48) and (49) we obtain
\[
\frac{d}{dt} \mathcal{N}^* = \mathcal{N}^*,
\mathcal{W}_i^* = \mathcal{A}_i \mathcal{W}_i, \quad i \in \{1, 2, 3, 4\}.
\]
Hence there exists a vector bundle \( \mathcal{S}(\mathcal{T}C^\perp) \) spanned on \( \mathcal{U} \) by \( \{ \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4 \} \) given by (48). Moreover, it is easy to check that this vector bundle is complementary to \( \mathcal{T}C \) in \( \mathcal{T}C^\perp \). The null transversal vector (constructed in Theorem 2.1), with respect to \( S(\mathcal{T}C^\perp) \), is locally represented by \( \mathcal{N} \) from (48). Finally taking into account that \( t = \tilde{t} \) and
\[
c_2 = -\frac{\kappa_4}{\kappa_1}, \quad c_3 = -\frac{\kappa_5}{\kappa_1}
\]
in the fourth and the fifth equations of Lemma 4, we obtain \( \kappa_4 = \kappa_5 = 0 \) which completes the proof. \( \square \)

At this point we assume that the transformations (45) are diagonal transformations, that is, they satisfy \( B_i^j = B_j^i = 0 \) (\( i \neq j \)). For this case, it follows from the sixth equation of Lemma 4 that \( c_4 = 0 \). Using this we obtain the following theorem.

**Theorem 6.3.** Let \( C \) be a null curve of \( \text{M}_4 \) with Frenet frame of Type 1 such that \( \kappa_1 \neq 0 \). Then, there exist a lightlike 2-surface which is invariant with respect to both the parameter transformations on \( C \) and the diagonal screen vector bundle transformations.

**Proof.** Let \( C^* \) be an integral curve of the vector field \( W_4 \). Since, by Lemma 4, \( c_4 = 0 \) for a diagonal screen vector bundle transformation, the 2-surface \( S = C \times C^* \) is always invariant with respect to this particular class of screen vector bundle transformations. \( S \) can neither be Lorentz nor definite because its two base vectors \( \{ \ell, W_4 \} \) contain a single null vector \( \ell \). Therefore, \( S \) must be lightlike. This completes the proof. \( \square \)

**Remark 5.** Theorem 6.3 can be used to define null sectional curvature of a null vector in \( \text{M}_4 \) in a similar way as introduced by Beem-Ehrlich [1, p. 571] for a Lorentzian manifold. Also see O’Neill [10, pp. 152–153 & p. 163] on null geodesic in surfaces and lightlike particles and Harris [7] on triangle comparison theorem for Lorentz manifolds.
Now we consider the case when $F$ and $\overline{F}$ are both of Type 2. Using the equations (10) and the method of the proposition 6.2, we have the following general result for $M_q$.

\[
\begin{align*}
B_1^1 &= B_2^2, \\
B_2^1 &= B_1^2, \\
B_3^1 &= B_3^2 = B_4^1 = B_4^2 = B_5^1 = B_5^2 = B_6^1 = B_6^2 = 0, \\
L_1 &= D \left\{ L_1 - C_2 \frac{dt}{d\lambda} \right\}, \\
L_2 &= E \left\{ L_2 - C_1 \frac{dt}{d\lambda} \right\},
\end{align*}
\]  

(50)

where

\[
D = B_1^1 - B_2^1, \quad E = B_1^1 + B_2^1, \quad C_1 = \frac{1}{\sqrt{2}}(c_2 - c_1) \text{ and } C_2 = \frac{1}{\sqrt{2}}(c_2 + c_1).
\]

**Lemma 5.** Let $C$ be a null curve of $M_q$ such that $\kappa_1 \neq 0$, and two Frenet frames $F$ and $\overline{F}$ of Type 2. Then their curvature functions are related by

\[
\begin{align*}
K_1 &= K_1 E \left( \frac{dt}{d\lambda} \right)^2, \\
K_2 &= E \left\{ K_2 + \frac{\hbar C_1}{dt} \frac{dt}{d\lambda} \right\} + \frac{1}{2} \left( \frac{dt}{d\lambda} \right)^2 \sum \epsilon_i (c_i)^2 - (C_1 k_4 + c_3 K_7 + c_4 K_8) \frac{dt}{d\lambda}, \\
K_3 &= D \left\{ K_3 + \frac{\hbar C_2}{dt} \frac{dt}{d\lambda} + (C_2 k_4 - c_3 K_5 - c_4 K_6) \frac{dt}{d\lambda} \right\}, \\
K_4 &= \left\{ k_4 + K_1 C_2 \frac{dt}{d\lambda} - E \frac{dD}{d\lambda} \right\} \frac{dt}{d\lambda}, \\
K_5 &= D \left\{ K_5 B_3^1 + K_6 B_3^2 \right\} \frac{dt}{d\lambda}, \\
K_6 &= D \left\{ K_5 B_4^1 + K_6 B_4^2 \right\} \frac{dt}{d\lambda}, \\
K_7 &= E \left\{ (K_7 + K_1 \frac{dt}{d\lambda} c_3) B_5^3 + (K_8 + K_1 \frac{dt}{d\lambda} c_4) B_5^4 \right\} \frac{dt}{d\lambda}, \\
K_8 &= E \left\{ (K_7 + K_1 \frac{dt}{d\lambda} c_3) B_6^3 + (K_8 + K_1 \frac{dt}{d\lambda} c_4) B_6^4 \right\} \frac{dt}{d\lambda}.
\end{align*}
\]  

(51)

**Proof.** The matrix $(B_j^j(x))$, in the relations (50), is made up of two $2 \times 2$ matrices (a Lorentz and an orthogonal). Therefore, using (50), the general transformations
are given by
\[
\begin{align*}
\bar{\lambda} &= \frac{dt}{dt} \lambda, \\
\bar{N} &= -\frac{1}{2} \frac{dt}{dt} \sum_{i=1}^{4} \varepsilon_i(c_i)^2 \lambda + \frac{dt}{dt} N + C_1 L_1 + C_2 L_2 + c_3 W_3 + c_4 W_4, \\
L_1 &= D \left\{ L_1 - \frac{dt}{dt} C_2 \bar{\lambda} \right\}, \\
L_2 &= E \left\{ L_2 - \frac{dt}{dt} C_1 \bar{\lambda} \right\}, \\
W_\alpha &= \sum_{\beta=3}^{4} B_\alpha^\beta \left( W_\beta - \frac{dt}{dt} c_\beta \lambda \right).
\end{align*}
\] (52)

Straightforward calculations from above relations and the use of (15) implies (51), which proves this lemma.

By a procedure same as for the Theorem 6.2, one can prove the following:

**Theorem 6.4.** Let $C$ be a null curve of $M_q$ with screen bundle space $S(TC^\perp)$ and a Frenet frame $F$ of Type 2 such that $\kappa_1 \neq 0$ on $\mathcal{U}$. Then there exists a screen vector bundle $\bar{S}(TC^\perp)$ which induces another Frenet frame $\bar{F}$ on $\mathcal{U}$ such that $\bar{K}_7 = \bar{K}_8 = 0$ on $\mathcal{U}$.

Next we consider the case when $F$ and $\bar{F}$ are both of Type 3. Using the equations (24), we have the following general result for $M_q$.

\[
\begin{align*}
\bar{\lambda} &= \frac{dt}{dt} \lambda, \\
\bar{N} &= -\frac{1}{2} \frac{dt}{dt} \sum_{i=1}^{4} \varepsilon_i(c_i)^2 \lambda + \frac{dt}{dt} N + \sum_{\alpha=1}^{4} C_\alpha L_\alpha, \\
L_1 &= \left\{ L_1 - C_2 \frac{dt}{dt} \bar{\lambda} \right\}, \\
L_2 &= E \left\{ L_2 - C_1 \frac{dt}{dt} \bar{\lambda} \right\}, \\
L_3 &= G \left\{ L_3 - C_4 \frac{dt}{dt} \bar{\lambda} \right\}, \\
L_4 &= H \left\{ L_4 - C_3 \frac{dt}{dt} \bar{\lambda} \right\},
\end{align*}
\] (53)

where
\[
\begin{align*}
D &= B_1^1 - B_2^1, & E &= B_1^1 + B_2^1, & G &= B_3^3 - B_4^3, & H &= B_3^3 + B_4^3, \\
C_1 &= \frac{1}{\sqrt{2}}(c_2 - c_1), & C_2 &= \frac{1}{\sqrt{2}}(c_2 + c_1), & C_3 &= \frac{1}{\sqrt{2}}(c_4 - c_3), & C_4 &= \frac{1}{\sqrt{2}}(c_4 + c_3).
\end{align*}
\]
By the procedure same as for the Lemma 5, one can prove the following:

**Lemma 6.** Let $C$ be a null curve of $M_{q}$ such that $\kappa_{1} \neq 0$, and two Frenet frames $F$ and $\mathcal{F}$ of Type 3. Then the functions $G$ and $H$ are constant and their curvature functions are related by

$$
\begin{align*}
K_{1} &= K_{1}E\left(\frac{dt}{dt}\right)^{2}, \\
\bar{K}_{2} &= E\left\{K_{2} + \bar{h}C_{1} + \frac{dC_{1}}{dt} - \frac{K_{1}}{2} \left(\frac{dt}{dt}\right)^{2} \sum \epsilon_{i}(c_{i})^{2} - (C_{1}K_{4} + C_{3}K_{7} + C_{4}K_{6})\frac{dt}{dt}\right\}, \\
\bar{K}_{3} &= D\left\{K_{3} + \bar{h}C_{2} + \frac{dC_{2}}{dt} + (C_{2}K_{4} - C_{3}T_{5})\frac{dt}{dt}\right\}, \\
\bar{T}_{3} &= H\left\{T_{3} + \bar{h}C_{3} + \frac{dC_{3}}{dt} + (C_{4}T_{4} + C_{2}T_{6} + C_{3}T_{8})\frac{dt}{dt}\right\}, \\
0 &= \bar{h}C_{4} + \frac{dC_{4}}{dt} + (C_{1}T_{5} + C_{2}T_{7} - C_{4}T_{8})\frac{dt}{dt}, \\
K_{4} &= \left\{K_{4} - K_{1}C_{2}\frac{dt}{dt} + E\frac{dD}{dt}\right\}\frac{dt}{dt}, \\
\bar{T}_{4} &= DHT_{4}\frac{dt}{dt}, \\
\bar{T}_{5} &= DG \frac{dt}{dt}, \\
\bar{T}_{6} &= E\left\{T_{6} + K_{1}C_{3}\frac{dt}{dt}\right\}\frac{dt}{dt}, \\
\bar{T}_{7} &= E\left\{T_{7} - K_{1}C_{4}\frac{dt}{dt}\right\}\frac{dt}{dt}, \\
\bar{T}_{8} &= T_{8}\frac{dt}{dt}.
\end{align*}
$$

(54)

By the method of Theorem 6.2 and 6.4, one can prove the following:

**Theorem 6.5.** Let $C$ be a null curve of $M_{q}$ with screen bundle space $S(TC^{\perp})$ and a Frenet frame $F$ of Type 3 such that $\kappa_{1} \neq 0$ on $\mathcal{U}$. Then there exists a screen vector bundle $S(TC^{\perp})$ which induces another Frenet frame $F$ on $\mathcal{U}$ such that $\bar{T}_{6} = \bar{T}_{7} = 0$ on $\mathcal{U}$.

7. **Concluding remark**

In this paper we have shown that it is possible to find general compound Frenet equations (24), with a variety of Frenet frames of Types 1, 2 and 3, for a null curve in a 6-dimensional semi-Riemannian manifold of index $q$. This is only a step
further of the earlier work of Duggal & Bejancu [4] on null curves of Lorentzian manifolds and of Duggal & Jin [5] on null curves of semi-Riemannian manifolds of index 2. However, the general case of null curves in semi-Riemannian manifolds of arbitrary dimension is still an open problem. We guess that this case is much more complicated and the null curve have \( \min\{q, m + 2 - q\} \)-type Frenet equations. We hope that the publication of this paper will help in solving the general case.

REFERENCES


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