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Development of a Criterion for Efficient Numerical Calculation of Structural Vibration Responses

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The finite element method is one of the methods widely applied for predicting vibration in mechanical structures. In this paper, the effect of the mesh size of the finite element model on the accuracy of the numerical solutions of the structural vibration problems is investigated with particular focus on obtaining the optimal mesh size with respect to the solution accuracy and computational cost. The vibration response parameters of the natural frequency, modal density, and driving point mobility are discussed. For accurate driving point mobility calculation, the decay method is employed to experimentally determine the internal damping. A uniform plate simply supported at four corners is examined in detail, in which the response parameters are calculated by constructing finite element models with different mesh sizes. The accuracy of the finite element solutions of these parameters is evaluated by comparing with the analytical results as well as estimations based on the statistical energy analysis, or if not available, by testing the numerical convergence. As the mesh size becomes smaller than one quarter of the wavelength of the highest frequency of interest, the solution accuracy improvement is found to be negligible, while the computational cost rapidly increases. For mechanical structures, the finite element analysis with the mesh size of the order of quarter wavelength, combined with the use of the decay method for obtaining internal damping, is found to provide satisfactory predictions for vibration responses.

Key Words : Natural Frequency, Modal Density, Driving Point Mobility, Plate, Finite Element Method, Mesh Size, Damping

1. Introduction

The finite element method is one of the numerical simulation techniques widely used for predicting structural vibration. In the finite element analysis, the solution accuracy is significantly influenced by the element size. It is generally believed that the accuracy can be improved by generating successively finer element during the modeling process. However, the computational load increases proportionally as the element size is reduced, and the numerical round-off error can pose a serious problem (Cremer et al., 1973; Irwin and Graf, 1979; Fahy, 1985). Meanwhile, software solvers for noise and vibration applications that are commercially available generally recommend mesh sizes in the order of 1/3, 1/4, or 1/6 of the wavelength of interest, perhaps based on the engineering intuition of the code developers (see Labor, 1996 for a general discussion).

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Therefore, selecting a proper mesh size can be critical in obtaining accurate solutions with minimal numerical round-off errors and acceptable computational costs. Recent research works have touched on the issue of the mesh size in vibration analysis of mechanical structures. For instance, Jiang and Olson (1994) used different grid spacing to obtain free vibration responses of shell and curved beam structures. Ramesh and Ganesan (1994) investigated FEM modeling of damped plates for various boundary conditions, while Ahmadian, Gladwell and Ismail (1994) looked into the combined use of the FEM and modal data.

For cases in which the modal analysis cannot be readily carried out, the statistical energy analysis (SEA) provides another alternative for obtaining estimates of the structural vibration and noise responses. The statistical energy analysis method is based on the assumption that the level of vibration energy for a given frequency range is directly proportional to the number vibration modes existing in the frequency range. When sufficiently large number of modes is present, the SEA can furnish fairly accurate estimates of the vibration responses. In this paper, it will be used to provide estimates that can be compared with the numerical results of the finite element models. A general discussion of the theory can be found in Lyon (1975), and Burroughs and Fischer (1997), while experimental means of obtaining accurate modal density and damping for plates for use in the SEA was investigated by Clarkson and Pope (1981).

In this study, the effect of the mesh size on the numerical solution of mechanical structures for vibration analysis purpose is systematically investigated. The vibration response parameters of the natural frequency, modal density, and driving point mobility are discussed for uniform beams and plates. For the accurate calculation of the driving point mobility, accurate damping parameters are needed, and thus the decay method is used to experimentally determine the internal damping for use in the numerical simulations. For comparison with numerical results of the finite element models, the analytical expressions for the modal response parameters are obtained whenever possible, and if not possible, numerical estimates based on the SEA are calculated. Since the modal density and mobility calculations are closely related with sound radiation, the results obtained here applies to structure-bourn noise as well as vibration.

2. Vibration Response Parameters

2.1 Modal analysis of a plate

The vibrations in mechanical structures are usually generated by excitations from the external sources. For a two-dimensional plate structure, the equation governing the vertical motion w is given by

$$D\nabla^4 w+\rho h\frac{\partial^2 w}{\partial t^2}=0 \tag{1}$$

where $\nabla^4=\frac{\partial^2}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$ and D, E, h, ρ, ν denote the flexural rigidity, Young's modulus, plate thickness, density, and Poisson's ratio, respectively. For sinusoidal motion of the plate, the solution of Eq. (1) for the wave propagating in x-direction is given by

$$w=(Ae^{k_b x}+Be^{-k_b x}+Ce^{-jk_b x}+De^{jk_b x})\,e^{j\omega t} \tag{2a}$$

with

$$k_b=\left(\frac{\rho h\omega^2}{D}\right)^{\frac{1}{4}} \tag{2b}$$

where k_b denotes the wave number of the bending wave. The first two terms in Eq. (2a) represent a near-field motion that does not contribute to wave propagation, while the remaining two terms represent a bending wave propagating in space. The vibration energy is predominantly transmitted in the form of the bending wave, with the phase velocity given by (see Lyon, 1987 for a general discussion)

$$C_b=\sqrt{\omega\cdot\kappa\cdot C_l} \tag{3}$$

where C_l denotes the propagation speed of the longitudinal wave given by $(E/\rho(1-v^2))^{\frac{1}{2}}$, and κ is a parameter related to the structural stiffness for which a plate of uniform thickness h is given by $h/\sqrt{12}$. From the above equation, the phase

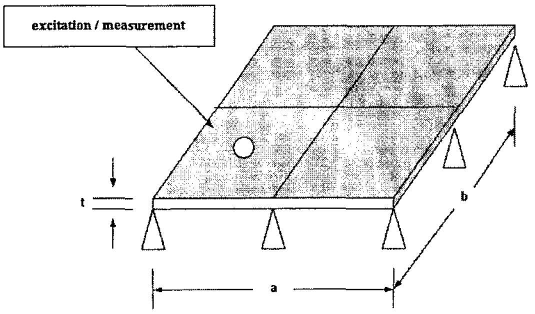

Fig. 1 Uniform plate simply supported at four corners

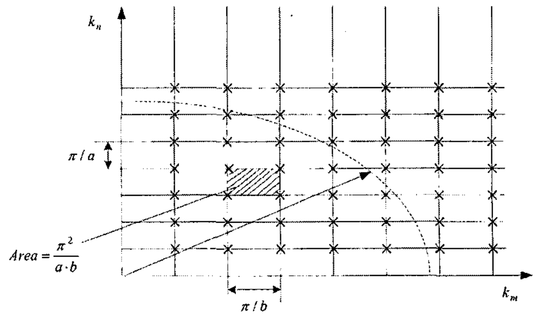

Fig. 2 Two-dimensional wave number lattice for a uniform plate

velocity is seen to be dependent on the frequency, and the propagation waveform is dispersed. The dispersion property greatly influences the natural frequency and the number of modes needed to store vibration energy.

The main parameters related to the vibration transmission are the modal density and mobility. For a rectangular plate of length a and width b, which is simply supported at four corners as shown in Fig. 1, the mode shape satisfying Eq. (1) can be given by (see Leissa, 1969)

$$\Psi_{mn}=A_{mn}\sin k_m \sin k_n y$$
$$k_m=\frac{mx}{a},\ k_n=\frac{n\pi}{b}\quad (m,\ n=1,\ 2,\ 3,\ \cdots) \tag{4}$$

where k_m and k_n denote the wave numbers in the x-and the y-directions, respectively. The natural frequencies are given by

$$\omega_{mn}=\sqrt{\frac{D}{\rho h}}(k_m^2+k_n^2)\cdot\kappa C_l(k_m^2+k_n^2) \tag{5}$$

Comparing Eqs. (3) and (5), the following relationship can be obtained

$$k^2=k_m^2+k_n^2 \tag{6}$$

Eq. (6) for the resonant modes can be represented graphically by a 2-dimensional lattice as shown in Fig. 2. In the figure, the interval between the adjacent modes for k_m and k_n are π/a and π/b, respectively, with each lattice point representing a resonance mode. The approximate number of the resonant modes up to a wave number k is given by dividing a quarter of the total area of the circle by the area per mode :

$$N(k)=\frac{k^2A}{4\pi} \tag{7}$$

where A denotes the surface area of the plate. The above equation can be expressed as a function of the frequency in the form of

$$N(f)=\frac{A\cdot f}{2\cdot\kappa\cdot C_l} \tag{8}$$

By taking the derivative of Eq. (8), the rate of change of the mode count with respect to the frequency can be expressed as

$$n(f)=\frac{dN}{df}=\frac{A}{2\cdot\kappa\cdot C_l} \tag{9}$$

Eq. (9) represents the modal density of the two-dimensional plate structure. Taking the inverse of Eq. (9) yields the mean frequency separation between the adjacent modes as given by

$$\overline{\delta f}=\frac{1}{n(f)}=\frac{t\cdot C_l}{\sqrt{3}\cdot A} \tag{10}$$

The above equation shows that the modal density for two-dimensional plates is independent of the frequency, is determined by the geometry (surface area and thickness), and exhibits a constant mean separation between the adjacent modes in the frequency domain.

On the other hand, for one-dimensional beam structures, the modal density is given by (Lyon, 1987)

$$n(f)=\frac{L}{C_b}=\frac{L}{\sqrt{2\pi\cdot\kappa\cdot f\cdot C_l}} \tag{11}$$

where L denotes the beam length. For the bending wave, the modal density in one-dimensional structures varies with the frequency, and the separation between any two adjacent modes in the frequency domain is proportional to $\sqrt{f}$, in contrast to a uniform frequency separation found in two-dimensional plate structures. The modal density of plate and beam are compared in Fig. 3.

2.2 Driving point mobility

The transfer function provides a measure of the system dynamic response to external excitation. The mobility is defined as the ratio of the system velocity response to the excitation force. In the case of a one-degree-of-freedom system, the mobility can be expressed by

$$M(\omega)=\frac{j\omega}{M}\frac{1}{(\omega_0^2-\omega^2)+j\omega_0\omega\eta} \tag{12}$$

where ω_0, η, and ω denote the natural frequency, the loss factor, and the excitation frequency, respectively.

Continuous structures such as plates are best modeled as multi-degree-of-freedom systems, and the responses can be expressed as linear superpositions of one-degree-of-freedom systems. The mobility can be expressed by (Lyon, 1987)

$$M(\omega)=\frac{j\omega}{M}\sum_n\frac{\phi_n(x_s)\,\phi_n(x_0)}{\omega_n^2-\omega^2+j\omega_n\omega\eta} \tag{13}$$

where $\phi_n(x_s)$ and $\phi_n(x_0)$ denote the mode shapes of the excitation point and measuring point, respectively. The driving point mobility refers to the

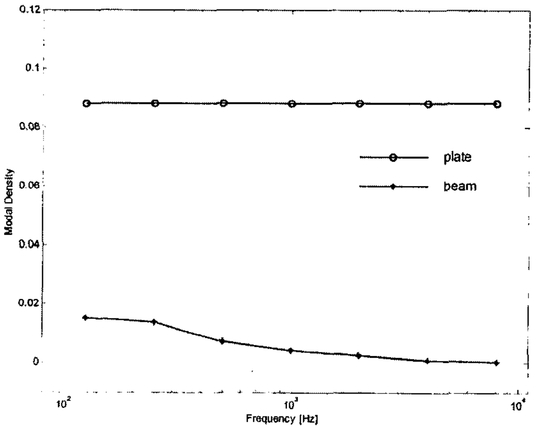

Fig. 3 Modal density of plate and beam by octave bands

mobility type in which the excitation point and measuring point coincide. For a uniform plate, the mean value of the real part of the driving point mobility within a given frequency band can be obtained by the following relationship between the excitation energy and the energy dissipated due to the internal damping of the plate (Lyon, 1987):

$$M_{dP}=\frac{n(f)}{4\cdot M}=\frac{1}{8\cdot\kappa\cdot C_l\cdot\rho_s} \tag{14}$$

where M and ρ_s denote the total mass and mass density per unit volume of the plate, respectively. Since the real part of the driving point mobility represents the magnitude of the energy input to the structure, it is always positive. The real part of the driving point mobility of the above equation is directly proportional to the modal density, which implies that it is related to the number of modes that can absorb the vibration energy from the excitation source.

3. Simulation Results

3.1 Plate model

For the present study, several finite element models with different element sizes are constructed for a homogeneous, isotropic plate. The primary focus is placed on obtaining the optimal element size for accurate estimation of the vibration response parameters while economizing on the computational effort. The rectangular plate is made of aluminum and simply supported at four corners, as shown in Fig. 1. The material properties and dimensions are specified in Table 1.

Commercially available FEM software ANSYS is used for modeling and analysis. The plate is modeled by using four-node shell elements. The modal analysis and the frequency response analysis are performed. The highest frequency of interest is set by 1,414 Hz, which is the upper

Table 1 Specifications of a uniform plate

Young's Modulus (Gpa)	Density (kg/m³)	Poisson's ratio	a (mm)	b (mm)	t (mm)
71	2,700	0.33	1,000	600	6

Table 2 Natural frequencies (% error) for various mesh sizes
(λ: wavelength corresponding to the highest frequency of interest; frequency in hertz

Mode number \ Mesh Size	λ	$\lambda/2$	$\lambda/4$	$\lambda/6$	$\lambda/8$	$\lambda/10$	Exact Value
1(1, 1)	54.93 (1.6)	55.58 (0.44)	55.77 (0.11)	55.81 (0.04)	55.82 (0.02)	55.83 (0.01)	55.83
2(2, 1)	97.00 (3.17)	99.26 (0.92)	99.93 (0.24)	100.07 (0.06)	100.11 (0.06)	100.13 (0.04)	100.17
3(3, 1)	166.42 (4.40)	171.94 (1.22)	173.49 (0.33)	173.83 (0.09)	173.92 (0.09)	173.98 (0.05)	174.07
4(1, 2)	175.64 (1.88)	177.74 (0.71)	178.63 (0.21)	178.84 (0.09)	178.90 (0.06)	178.94 (0.03)	179.00

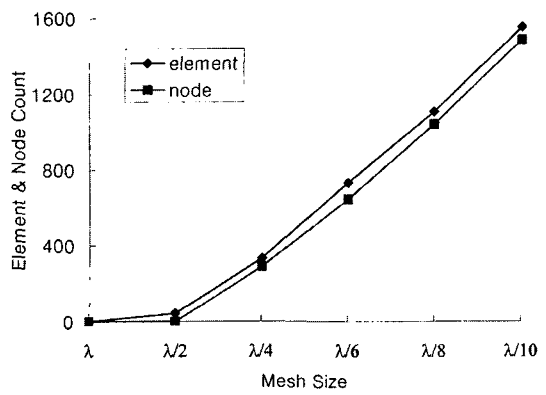

Fig. 4 Total element and node count for different mesh sizes (λ: wavelength corresponding to the highest frequency of interest)

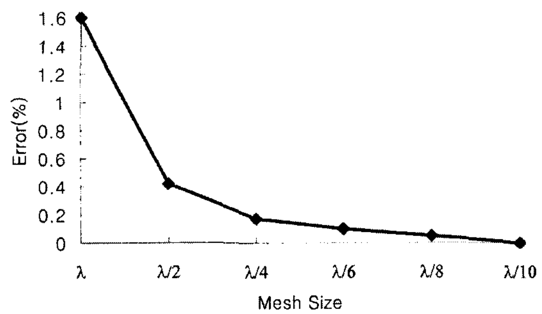

Fig. 5 Error in the first natural frequency (λ: wavelength corresponding to the highest frequency of interest)

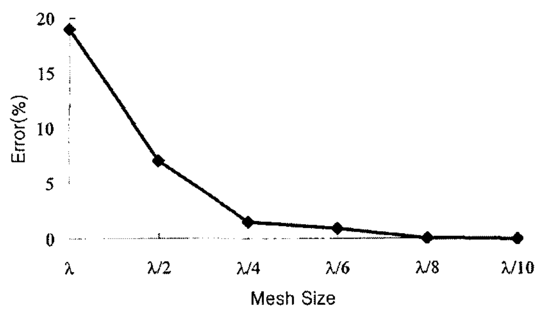

Fig. 6 Combined error in the first four natural frequencies (λ: wavelength corresponding to the highest frequency of interest)

limit for the octave band centered at 1,000 Hz. Once the highest frequency is known, the corresponding wavelength of the bending wave can be readily calculated according to Eq. (2b). For the plate at hand, the wavelength of the bending wave is calculated to be 0.2 m, and it will serve as the basis for selecting the mesh size. The total number of the elements and nodes for various mesh sizes are shown in Fig. 4. The element and node count increases geometrically with the mesh size reduction. It can be easily surmised that the computational load increases in a similar manner as the mesh size becomes smaller.

3.2 Effect on the natural frequency

For a plate simply supported at four corners, the natural frequencies can be analytically obtained. The exact natural frequencies for the first four modes calculated are 55.83 Hz, 100.17 Hz, 174.07 Hz, and 179.00 Hz. The corresponding modal indices are (1, 1), (2, 1) (3, 1), and (1, 2), respectively.

In the present investigation, the modal analysis based on the finite element model is carried out while successively reducing the mesh size. The first four natural frequencies are computed for each mesh size, and the numerical solutions are compared with the exact values. The first four natural frequencies computed for different mesh sizes are listed in Table 2. The corresponding percentage errors are enclosed in parenthesis as well. Figure 5 plots the percentage error in the first natural frequency as a function of the mesh size. All of the other three modes show the similar trend, and the sum of the squared errors for the modes 1 through 4 for different mesh sizes are shown in Fig. 6. The errors of all four modes decrease in direct proportion to the mesh size reduction. For the mesh size at the quarter of the wavelength, the errors for the first four modes are reduced to 0.11%, 0.24%, 0.33%, and 0.21%, respectively. For the mesh sizes smaller than the quarter wavelength, reductions in the errors are found to be negligible.

3.3 Effect on the modal density

The effect of the mesh size on the mode count is investigated. For each mesh size, the total number of modes within the frequency range up to 1,414 Hz is computed and the results are compared with the analytically derived mode count of 38. The mode count for different mesh sizes are given in Fig. 7. The mode count should approach the correct number as the mesh size becomes successively smaller. In this case, the mode count reaches the correct number as the mesh size reaches the quarter wavelength. The mode count of 48 is obtained by applying the statistical energy analysis (SEA) for the same frequency range. The error in this case is due to the inclusion of a low frequency region for which the SEA cannot be properly applied. An experimentally derived frequency response function underestimated the mode count, perhaps due to overlapping of the modes, yielding the mode count of only 27. Total mode count for different methods are plotted in Fig. 8.

3.4 Effect on the driving point mobility

The harmonic response analysis is performed to estimate the mobility function at the driving point. The real part of the driving point mobility

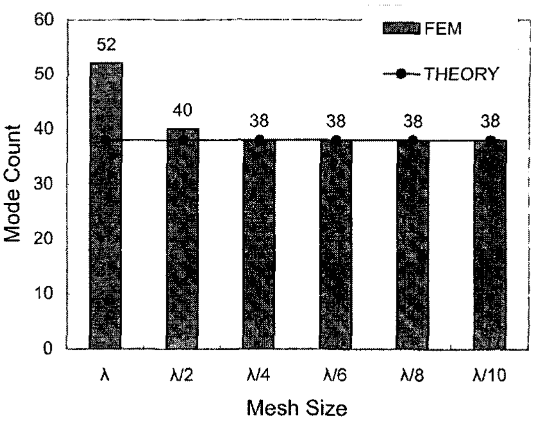

Fig. 7 Mode count for different mesh sizes (λ: wavelength corresponding to the highest frequency of interest)

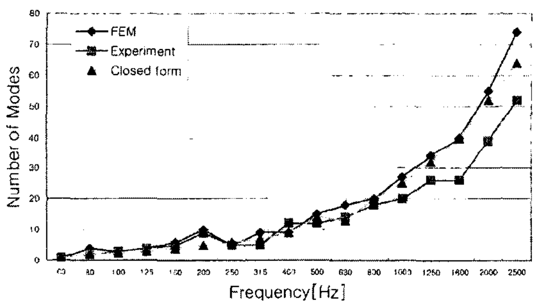

Fig. 8 Total mode count for different methods

is obtained by varying the frequency from 1 Hz to 1,414 Hz with 1 Hz increment. According to Eq. (14) for a uniform plate, the real part of the driving point mobility is independent of the excitation frequency. The mean mobility value for each octave band is calculated. The magnitude of the excitation is set by unity. The excitation/measurement point is indicated in Fig. 1.

The mobility function can be accurately calculated if and only if a reliable estimate of the structural damping is known. To obtain the internal damping, the decay method is applied here. An experimental setup is shown in Fig. 9. Following the impact on the plate by the impact hammer, band-pass filtering is performed on the response signal for each octave band, and the reverberation time required for the signal amplitude to decay by 60dB is measured. The loss factor can be calculated for each octave band by the following equation (Lyon, 1987) given by

$$\eta=\frac{2.2}{f_c \cdot T_R} \qquad (15)$$

where f_c and T_R denote the center frequency of the given octave band and the reverberation time, respectively. A typical response decay curve is shown in Fig. 10, and the averaged loss factor

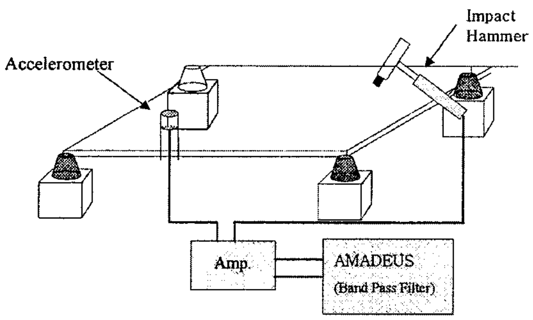

Fig. 9 Experimental setup for measuring internal damping

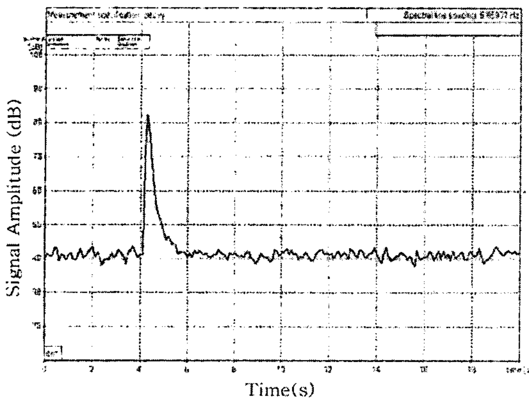

Fig. 10 Response signal amplitude decay curve

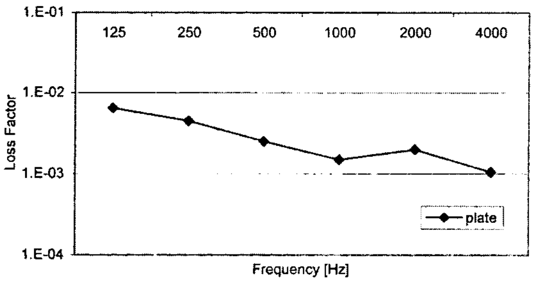

Fig. 11 Loss factor by octave bands

value for each octave band is given in Fig. 11. It can be seen that the damping factor diminishes with the frequency increase.

Although the analytical solution of the driving point mobility cannot be obtained, the numerical solutions, for instance, by the SEA method can be computed. As we have seen in the case of the modal density, however, the SEA results will contain significant errors, that is, strictly speaking, the exact mobility results which can be used to verify the FEM results are not available. Therefore, the numerical convergence to a steady-state value as the mesh size is successively reduced will be used as a criterion for accepting the results.

The results are plotted in Fig 12. According to Eq. (14), the driving point mobility of the plate is characterized by a constant value that is proportional to the modal density and independent of the excitation frequency. Therefore, the region in Fig. 12 exhibiting steady-state values, i.e., horizontal line segments, should correspond to the region of acceptable numerical solutions. For each octave band, the mean values of the real parts of the mobility functions for different mesh sizes are shown in Fig. 12. For all cases, the mean value for each octave band converges to a constant value as the mesh size is reduced. Once the mesh size has reached the quarter wavelength, the change in the mean value due to additional reductions in the mesh size becomes negligible.

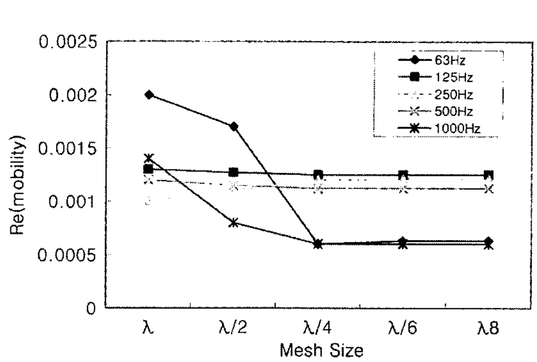

Fig. 12 Driving point mobility by octave bands for different mesh sizes (λ: wavelength corresponding to the highest frequency of interest)

4. Conclusions

The purpose of the present investigation is to find an optimal mesh size in the finite element modeling of structures for predicting vibrational responses. The three vibration response parameters of the natural frequency, modal density, and driving point mobility function of a two-dimensional plate structure are considered. While reducing the mesh size, the finite element solutions of the natural frequency and mode count are compared with the theoretical solutions. Since the theoretical results are not available, the mean value of the driving point mobility functions for each octave band is tested for the numerical convergence to a steady-state value.

For the natural frequency and modal density, the mesh size reduction generally leads to higher accuracy. As the mesh size gets smaller than one quarter of the wavelength of the highest frequency of interest, however, the gain in the accuracy is found to be either negligible or non-existent. The mean driving point mobility also achieves convergence to a constant steady-state value as the mesh size reaches the quarter wavelength for all octave bands.

For the mesh sizes smaller than the quarter wavelength, the improvement in the solution accuracy is negligible, while computational cost rapidly increases. In order to predict the vibration characteristics of mechanical structures by applying the finite element method, therefore, it can be concluded that the mesh size equal to one quarter of the wavelength of the highest frequency of interest is optimal from the points of the solution accuracy and computational cost.

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