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Test of Local Restriction on a Multinomial Parameter[1)]

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Abstract

If a restriction is imposed only to a (proper) subset of parameters of interest, we call it a local restriction. Statistical inference under a local restriction in multinomial setting is studied. The maximum likelihood estimation under a local restriction and likelihood ratio tests for and against a local restriction are discussed. A real data is analyzed for illustrative purpose.

Keywords : Chi-bar-square distribution, Likelihood ratio test, Local restriction, Multinomial Parameter, Simple order, Star-shaped, Stochastic ordering

1. Introduction

The use of multinomial model has long history since Karl Pearson introduced this model for goodness-of-fit test. The most frequently used setting up for null hypotheses is the equality of two multinomial parameters where one is known. In practice we frequently encounter the case that the real problem inherently impose an order restriction on parameters. To name a few, trend, tree, unimodal, and star-shape. There are vast number of literatures on this subject; Bhattacharya (1995,1997), Chacko (1966), Dykstra and Robertson (1982), Lee (1987), Robertson (1978), Shi (1989) are among others.

Let $X=\{1,2,\ldots,k\}$ be set of indices. A binary relation $\preceq$ on X is called a quasi-order if it is reflexive and transitive. A quasi-order is called if it is antisymmetric. Usually restriction is related to a partial order. In some situations the order restriction may be imposed locally on multinomial parameter. For example, increasing trend is imposed on the first k_0 $(<k)$ components of a multinomial parameter $p=(p_1,p_2,\ldots,p_k)$, i.e., $p_1\le p_2\le\cdots\le p_{k_0}$. Since a quasi-order may admit noncomparable elements, this restriction can be explained by introducing a suitable quasi-order. On the other hand, this quasi-order restricted to subset of comparable elements is a partial order. In this sense we call this restriction a local restriction.

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Without loss of generality let $X_{k_0} = \{1, 2, \ldots, k_0\}$ be a subset of $\boldsymbol{X}$ on which local restriction O_{k_0} is defined. We note that O_{k_0} may not be related to a partial order. For example consider a star-shape restriction $p_1 \geq (p_1 + p_2)/2 \geq \cdots \geq (p_1 + \cdots + p_{k_0})/k_0$. There is no way to connect this restriction to a certain partial order. Another example is stochastic ordering, which is $\sum_{j=1}^{i} p_j \geq \sum_{j=1}^{i} q_j$ for $i = 1, \ldots, k_0$. This restriction is not directly related to a partial order but the dual is.

Statistical inference under the local restriction may be found in many practical problems. Suppose we classify the freshmen in a university into k groups according to their high school ranking. Usually one is interested in comparing the relative frequencies (probability vector) to the previous year's. On the other hand, one might be interested in comparing only upper-half groups. For this case we need to employ a local restriction.

Let $\boldsymbol{q} = (q_1, \ldots, q_k)$ be a known probability vector. Define hypotheses H_i, $i = 0, 1, 2$ by

$$H_0 : \ \boldsymbol{p} = \boldsymbol{q},$$
$$H_1 : \ \boldsymbol{p} \text{ satisfies } O_{k_0} \text{ but } \boldsymbol{p} \neq \boldsymbol{q},$$

and H_2 places no restriction on $\boldsymbol{p}$ but $\sum_{i=1}^{k} p_i = 1$ and does not satisfy O_{k_0}. One is interested in testing H_0 vs H_1. In this paper we study the statistical inference under three types of local restrictions, which are restriction related to partial order, star-shape restriction, and stochastic ordering. In section 2, maximum likelihood (ML) estimation of $\boldsymbol{p}$ under local restriction is discussed. We have found that very simple transformation of parameter space enables us to use the existing inferential procedures for each types of restrictions. In section 3, likelihood ratio tests concerning local restriction is discussed and the asymptotic null distributions are derived. The asymptotic null distributions are chi-bar-square distributions as the full restriction cases. But we have found that each distribution does not involve the degenerated distribution, i.e., chi-square with zero degrees of freedom. In section 4, a real data is analyzed for illustrative purpose.

2. Maximum Likelihood Estimation

When a local restriction is imposed on a parameter the estimation of parameter seems to be intractable without transforming parameter space properly. In this section we are going to discuss the transformation of parameter space which enables us to use the existing estimation procedure developed for full restriction cases. We first consider the case that the restriction is related to a partial order. We assume that a random sample of size n is taken from a multinomial distribution with probability vector $\boldsymbol{p}$ and denote the corresponding vector of observed frequencies by $\boldsymbol{n}=(n_1,\dots,n_k)$. The ML estimate of $\boldsymbol{p}$ under local restriction can be obtained by the following reparametrization scheme.

Let $\preceq$ be a partial order defined on $\boldsymbol{X}$ and let $\preceq_{k_0}$, be a binary relation restricted to the subset $\boldsymbol{X}_{k_0}$ $\boldsymbol{X}_0$ of $\boldsymbol{X}$. We note that there is a nonempty subset of $\boldsymbol{X}$ whose element are incomparable since the restriction related to the partial order is local. It is easy to show that $\preceq_{k_0}$ is also a partial order on $\boldsymbol{X}_{k_0}$. Now we consider a simple one-to-one transformation of parameter space. Let $a_{k_0}=\sum_{i=1}^{k_0} p_i$, $\theta_i=p_i/a_{k_0}$ for $i=1,\dots,k_0$, and $a_i=p_i$ for $i=k_0+1,\dots,k$. Then the partial order $\preceq_{k_0}$ is preserved among θ_i for $i=1,\dots,k_0$.

The likelihood function becomes

$$\left[\prod_{i=1}^{k_0}\theta_i^{n_i}\right]\cdot\left[a_{k_0}^{\sum_{i=1}^{k_0}n_i}\cdot\prod_{i=k_0+1}^{k}a_i^{n_i}\right]. \tag{2.1}$$

The basic restrictions are $\sum_{i=1}^{k_0}\theta_i=1$, $\sum_{i=k_0}^{k}a_i=1$, and $0<a_i$, $\theta_i<1$. Since no restrictions relate a_i's and θ_i's to each other we can maximize the likelihood function (2.1) by maximizing two parts (each in bracket) separately.

Consider first the maximization of the latter part in (2.1). We see that no restriction other than the basic restriction is imposed. This is just a usual multinomial model. The ML estimate, $\hat{a}_i$, of a_i is given by

$$\hat{a}_{k_0}=\frac{\sum_{i=1}^{k_0}n_i}{n},\qquad \hat{a}_i=\frac{n_i}{n},\quad \text{for } i=k_0+1,\dots,k.$$

Next consider the first part in (2.1). The ML estimate of θ_i can be easily obtained by finding an isotonic regression of $\hat{\theta}$'s with respect to the restriction inherited from p_i's. Later we give the details of estimation procedures for the cases of simple order and star-shape restriction. Finally the evaluation of $\boldsymbol{p}$ at $\theta_i=\theta_i^*$ and $a_i=\hat{a}_i$ gives the ML estimate of $\boldsymbol{p}$ under the local restriction.

Example 2.1 Simple order Suppose that the local restriction is simple order. Let $A_{k_0}=\{\ \boldsymbol{x}\in \boldsymbol{R}^{k_0}: x_1\le x_2\le\cdots\le x_{k_0}\}$ and $\hat{\boldsymbol{\theta}}=(\hat{\theta}_1,\dots,\hat{\theta}_{k_0})$ where $\hat{\theta}_i=n_i/\sum_{j=1}^{k_0}n_j$ for $i=1,\dots,k_0$. Then the restricted ML estimate, θ_i^*, of θ_i is given by

$$\theta^{*i}=E(\hat{\boldsymbol{\theta}}|A_{k_0})_i \quad \text{for } i=1,\dots,k_0,$$

where $E(x|A_{k_0})$ is the projection of x onto a closed convex cone A_{k_0}. See Barlow, Bartholomew, Bremner and Brunk (1972) or Robertson, Wright and Dykstra (1988) for the details of projection theory. For a partial order other than simple order, we only need to substitute A_{k_0} with the corresponding closed convex cone.

It is easily seen that the reparametrization scheme used for partial order can be also used for star-shape restriction.

Example 2.2 Star-shape The local restriction of star-shape becomes

$$\theta_1 \geq \frac{\theta_1+\theta_2}{2} \geq \cdots \geq \frac{\theta_1+\cdots+\theta_{k_0-1}}{k_0-1} \geq \frac{1}{k_0}.$$

Now we can apply the estimation procedure given by Dykstra and Robertson (1982) directly. Let $\eta_i = \sum_{j=1}^{i}\theta_j / \sum_{j=1}^{i+1}\theta_j$ for $i=1,\ldots,k_0-1$. Then $\eta_i^* = \max\{\hat{\eta}_i,\ i/(i+1)\}$. Evaluation of θ at $\eta=\eta^*$ gives the ML estimate of θ under star-shape restriction.

It is easy to show that the above reparametrization scheme does not work for stochastic ordering problem. Now consider the following one-to-one transformation. Let $a_i = p_i$, $b_i = q_i$ for $i=1,\ldots,k_0$, $a_{k_0+1} = \sum_{j=k_0+1}^{k} p_j$, $b_{k_0+1} = \sum_{j=k_0+1}^{k} q_j$, $\phi_i = p_i/a_{k_0+1}$, $\tau_i = q_i/b_{k_0+1}$ for $i=k_0+1,\ldots,k$. Note that b_i's and τ_i are known. Then the basic restriction becomes

$$\sum_{i=1}^{k_0+1} a_i = 1,\quad \sum_{i=k_0+1}^{k}\phi_i = 1 \text{ and } 0 < a_i, \phi_i < 1.$$

Example 2.3 Stochastic ordering It is obvious that the restriction imposed on θ_i's is inherited from the restriction among p_i's, i.e., stochastic ordering. Then the local stochastic ordering restriction becomes

$$\sum_{j=1}^{i} a_j \geq \sum_{j=1}^{i} b_j \text{ for } i=1,\ldots,k_0, \text{ and } \sum_{j=1}^{k_0+1} a_j = \sum_{j=1}^{k_0+1} b_j = 1. \tag{2.2}$$

The likelihood function becomes

$$\left[\prod_{i=1}^{k_0} a_i^{n_i} \cdot a_{k_0+1}^{\sum_{j=k_0+1}^{k} n_j}\right] \cdot \left[\prod_{i=k_0+1}^{k} \phi_j^{n_j}\right]. \tag{2.3}$$

We observe that no restriction relate a_i and ϕ_i to each other. It follows from this fact that a_i and ϕ_i are statistically independent. And the maximum of (2.3) under (2.2) is obtained by maximizing the two parts separately.

First consider the latter part in (2.3). Since no restriction other than basic restriction is imposed it is just a usual multinomial model and then $\hat{\phi}_i = n_i / \sum_{j=k_0+1}^{k} n_j$, for

$i=k_0+1,\ldots,k$. For first part we can use estimation procedure in Robertson and Wright (1981). The restricted ML estimate, $\boldsymbol{a}^*$, of $\boldsymbol{a}$ is given by

$$\boldsymbol{a}^* = \widehat{\boldsymbol{a}}E_{\widehat{\boldsymbol{a}}}\left(\frac{\boldsymbol{b}}{\widehat{\boldsymbol{a}}}\,|A'_{k_0}\right)$$

where $\boldsymbol{a}=(a_1,\ldots,a_{k_0+1})$, $\boldsymbol{b}=(b_1,\ldots,b_{k_0+1})$, $\widehat{\boldsymbol{a}}$ is unrestricted ML estimate of $\boldsymbol{a}$, and $A'_{k_0}=\{\boldsymbol{x}\in \boldsymbol{R}^{k_0+1}:\ x_1\geq x_2\geq\cdots\geq x_{k_0+1}\}$. All vector operations are componentwise.

Finally, it is straightforward to show that all the ML estimates given above are strongly consistent since all transformation is continuous mapping.

So far we have discussed two types of transformations of parameter space. In both transformations we collapse a part of cells into a cell and find kind of conditional probability vector within collapsed cell. The difference is; the first one imposes restriction on conditional probability vector while the other imposes the restriction on original cells including collapsed one.

3. Likelihood Ratio Test

First we consider the first type transformation for which the restriction is related to a partial order. The likelihood ratio statistic is given by

$$\Lambda_{01}=\frac{\prod_{i=1}^{k_0}(\theta_i^{\cdot})^{n_i}\cdot(a_{k_0}^{\cdot})^{\sum_{j=1}^{k_0}n_j}\cdot\sum_{i=k_0+1}^{k}(a_i^{\cdot})^{n_i}}{\prod_{i=1}^{k_0}(\theta_i^{*})^{n_i}\cdot(\widehat{a}_{k_0})^{\sum_{j=1}^{k_0}n_j}\cdot\sum_{i=k_0+1}^{k}(\widehat{a}_i)^{n_i}},$$

where $\theta_i^{\cdot}=q_i/\sum_{j=1}^{k_0}q_j$ for $i=1,\ldots,k_0$, $a_{k_0}^{\cdot}=\sum_{j=1}^{k_0}q_j$ and $a_i^{\cdot}=q_i$ for $i=k_0+1,\ldots,k$. The test rejects H_0 for large value of $T_{01}=-2\ln\Lambda_{01}$

$$2\sum_{i=1}^{k_0}n_i(\ln\theta_i^*-\ln\theta_i^{\cdot})+2(\sum_{i=1}^{k_0}n_i)(\ln\ \widehat{a}_{k_0}-\ln a_{k_0}^{\cdot})+2\sum_{i=k_0+1}^{k}n_i(\ln\ \widehat{a}_i-\ln a_i^{\cdot}).\quad(3.1)$$

We now derive the asymptotic null distribution of T_{01}. Before stating the theorem we need to consider a new partial order $\preceq_{\theta^{\cdot}}$, induced by $\boldsymbol{\theta}^{\cdot}$ and original partial order $\preceq$ on X_{k_0} which requires that $i\preceq_{\theta^{\cdot}}j$ only when $i\preceq j$ and $\theta_i^{\cdot}=\theta_j^{\cdot}$. That is, the partial order $\preceq_{k_0}$ is preserved only on the subset of indices where the corresponding θ_i's are the same.

Theorem 1 If H_0 is true, then for each t

$$\lim_{n\to\infty} \Pr[T_{01}\geq t] = \sum_{\ell=1}^{k_0} P_{\theta^\cdot}(\ell, k_0; \leq_{k_0}) \Pr[\chi^2_{k-k_0+\ell-1}\geq t],$$

where $P_{\theta^\cdot}(\ell, k_0; \leq_{k_0})$ are the level probabilities with respect to weight $\theta^\cdot$ and the induced partial order $\leq_{\theta^\cdot}$. Note that $\sum_{i=1}^{k_0} n_i$ must go to infinity as well as n.

Proof: The first term in (3.1), denoted by T^1_{01}, is the one-sample test statistic for testing equality of multinomial parameter against an order restriction corresponding to a certain partial order. Hence by Theorem 5.2.1 of Robertson *et al.* (1988), for every $t>0$,

$$\lim_{n\to\infty} \Pr[T^1_{01}\geq t] = \sum_{\ell=1}^{k_0} P_{\theta^\cdot}(\ell, k_0; \leq_{k_0}) \Pr[\chi^2_{\ell-1}\geq t],$$

On the other hand, the last two terms in (3.1), denoted by T^2_{01}, is the one-sample test statistic for testing equality of multinomial parameter against all alternatives. Hence T^2_{01} is asymptotically chi-square distributed with degrees of freedom $k-k_0$. Moreover T^1_{01} and T^2_{01} are independent. We have the result from the convolution formula.

As we have mentioned earlier it is of interest to observe that no degenerated distribution is involved in the asymptotic null distribution. This is one of the main differences between inferences for local restriction and for full restriction. For most full restriction cases the asymptotic null distribution of test statistic contains a degenerated random variable which is distributed as chi-square with zero degrees of freedom. Hence the distribution function has a jump at zero. The jump size is the value of corresponding level probability. Unlike the full restriction case the asymptotic null distribution for local restriction has no degenerated variables.

Next we consider star-shape restriction. Though the restriction is not related to a partial order we can use the above arguments to find the asymptotic null distribution.

Example 3.1 Star-shape revisited First of all we need to set $q_1=\cdots=q_k=1/k$. Dykstra and Robertson (1982) showed that under H_0

$$\lim_{n\to\infty} \Pr[T^1_{01}\geq t] = \sum_{\ell=1}^{k_0} \binom{k_0-1}{\ell-1}\left(\frac{1}{2}\right)^{k_0-1} \Pr[\chi^2_{\ell-1}\geq t].$$

Again by convolution formula we can show that

$$\lim_{n\to\infty} \Pr[T_{01}\geq t] = \sum_{\ell=1}^{k_0} \binom{k_0-1}{\ell-1}\left(\frac{1}{2}\right)^{k_0-1} \Pr[\chi^2_{k-k_0+\ell-1}\geq t].$$

Next we consider the second type of transformation which is used for local stochastic ordering. We use the same notations used in section 2. The likelihood ratio statistic becomes

$$\Lambda_{01} = \left[\frac{\prod_{i=1}^{k_0} b_i^{n_i} \cdot b_{k_0+1}^{\sum_{j=k_0+1}^{k} n_j}}{\prod_{i=1}^{k_0} (a_i^*)^{n_i} \cdot (a_{k_0+1}^*)^{\sum_{j=k_0+1}^{k} n_j}} \right] \cdot \left[\frac{\prod_{i=k_0+1}^{k} \tau_j^{n_j}}{\prod_{i=k_0+1}^{k} \hat{\phi}_j^{n_j}} \right].$$

The test rejects H_0 for large value of

$$S_{01} = -2\ln\Lambda_{01} = 2\sum_{i=1}^{k_0} n_i(\ln a_i^* - \ln b_i) + 2(\sum_{j=k_0+1}^{k} n_j)(\ln a_{k_0+1}^* - \ln b_{k_0+1})$$
$$+2\sum_{i=k_0+1}^{k} n_i(\ln \hat{\phi}_i - \ln\tau_i).$$

Let the first two terms be S_{01}^1 and the rest S_{01}^2. It is clear that S_{01}^1 is of form one-sample test statistic for testing equality of multinomial parameter against an order restriction and S_{01}^2 is an usual chi-square test statistic. Hence we can apply Theorem 1 here with minor modifications. The asymptotic null distribution of S_{01}^1 is a chi-bar-square distribution with appropriate level probability associated to the specific local restriction. And S_{01}^2 is asymptotically chi-square distributed with degrees of freedom $k-k_0$.

Example 3.2 Stochastic Ordering revisited It follows from Robertson and Wright (1981) that for all t>0,

$$\lim_{n\to\infty} \Pr[S_{01}^1 \geq t] = \sum_{\ell=1}^{k_0+1} P(\ell, k_0+1;\ \boldsymbol{b})\Pr[\chi^2_{k_0+1-\ell} \geq t],$$

where $P(\ell, k_0;\ \boldsymbol{b})$ is the level probability with respect to simple order and weights $\boldsymbol{b}$. Using convolution formula, we can show easily that

$$\lim_{n\to\infty} \Pr[S_{01} \geq t] = \sum_{\ell=1}^{k_0+1} P(\ell, k_0+1;\ \boldsymbol{b})\Pr[\chi^2_{k-\ell+1} \geq t].$$

Finally we discuss about fining level probabilities. In Theorem 1, the computation of $P_{\theta^*}(\ell, k_0; \preceq_{k_0})$ is relatively easy since partial order is restricted to the subsets of equal parameters. The index set $\boldsymbol{X}$ can be decomposed into several subsets in which q_i's are the same. We call each subset a block. First we compute level probabilities within each block and combine them. Since the dimension of each block is usually small and weights are equal within block, it is easy to compute level probabilities. The recursive formula for well-known ordering such as simple order, simple tree ordering are available Robertson *et al.* (1988). For the second type of transformation it is, however, difficult to compute the level probability $P(\ell, k_0+1;\ \boldsymbol{b})$ if $k_0>4$ even though $\boldsymbol{b}$ is known. The approximation of level probability due to Robertson and Wright (1983) is useful to find a critical value. The pattern approximation is recommended instead of equal-weights approximation because $b_{k_0+1} = \sum_{i=k_0+1}^{k} q_i$ and hence is likely to have a bigger value than other b_i's.

4. An Example

We analyze a real data to illustrate the inferential procedure discussed in sections 2 and 3. Here we give only the example for local stochastic ordering. since the inferential procedures for other restrictions are quite similar. The 131 prospective students are admitted initially to the College of Engineering, Pusan University of Foreign Studies on 1999 regular admission program and tabulated in Table 1 according to their high school ranks. There are originally 15 scales in high school ranks but top 8 scales are used. The smaller the number, the higher the school rank. The anticipated proportion of enrolled students for each high school rank group is q_i. A school official claims that large portion of the newly admitted student with high-school rank 5 or higher tends not to attend the university. This assertion can be hypothesized as a local stochastic ordering, i.e., $\sum_{j=1}^{i} p_j \geq \sum_{j=1}^{i} q_j$ for $i=1,\ldots,5$, where p_i denote observed relative frequency of ith group and q_i expected proportion of enrolled students. Note that $k_0=5$.

	n_i	p_i	q_i	a_i	b_i	ϕ_i	τ_i	p_i^*
1	7	0.0534	0.10	0.0534	0.10			0.1000
2	8	0.0611	0.10	0.0611	0.10			0.1000
3	13	0.0992	0.10	0.0992	0.10			0.1000
4	18	0.1374	0.10	0.1374	0.10			0.1223
5	29	0.2214	0.20	0.2214	0.20			0.1971
6	30	0.2290	0.20	0.4275	0.40	0.5357	0.500	0.2039
7	24	0.1832	0.15			0.4286	0.375	0.1631
8	2	0.0153	0.05			0.0357	0.125	0.0135
Total	131							

Table 1. Computational Details

Note that the observed relative frequency $\hat{p}_i$ does not satisfy the restriction, the restricted ML estimate p_i^* is given in Table 1. Since $k_0+1>5$ the exact level probability can not be computed. As we discussed earlier, equal-weight approximation is not appropriate since the ratio of maximum weight to minimum weight is 0.5/0.05=10, which is quite larger than the recommended value 3. A pattern approximation is used for computing level probability. See Robertson and Wright (1983) or Robertson *et al.* (1988). A FORTRAN program for computing level probability is given in Pillers, Robertson and Wright (1984). See also Cran (1981). The level probabilities, $P(\ell, 6; \boldsymbol{b}), \ell = 1, \ldots, 6$ for the example are 0.19292, 0.40015, 0.29098, 0.09883, 0.01610, 0.00102, respectively. The computed value of test statistic is 6.3698 and hence the p-value 0.4571. We can conclude that the school official claim has no evidence. On the other hand we consider the test of equality against no restriction for the comparison sake. The value of test statistic for testing H_0 against all alternative is 13.4445 with p-value 0.0974. The null hypothesis can be rejected at significance level 0.1 and hence the opposite result.

5. Concluding Remarks

To our knowledge no statistical inference concerning local restriction has been studied rigorously using order restricted statistical inferential approach. As we pointed earlier when restriction is related to partial order some local restriction can be viewed as full restriction related to a certain quasi-order. There is, however, no way to connect this to a certain partial order for some restrictions such as star-shape restriction. Hence we have proposed unified methods for inferential procedure concerning local restriction. Fortunately we were able to find that very simple transformation of parameter space enables us to use the existing inferential method under order restriction for some well-known restrictions.

We also have found that asymptotic null distributions of each test statistic do not contain chi-square variable with zero degrees of freedom.

In this paper we consider only one-sample problem. The inferential procedure proposed here can be extended to two-sample problem with minor modifications for most type of restrictions. For some problems, however, the extension is not trivial. An example is uniform stochastic ordering problem which was not discussed in this paper. The two-sample problem for local restriction will appear elsewhere.

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