Commun. Korean Math. Soc. 18 (2003), No. 3, pp. 533–540

SEMI-CONVERGENCE OF p-STACKS

WON KEUN MIN

ABSTRACT. We introduce the notion of semi-convergence of p-stacks and by using that notion we characterize the semi-interior, semi-closure, separation axioms and semi-continuity on a topological space. Also we introduce a new notion of p-semicompactness and investigate its properties in terms of semi-convergence of p-stacks.

1. Introduction

Levin introduced the concepts of semi-open sets and semi-continuity on a topological space and obtained many significant properties in [5]. Crossley and Hilderbrnd [2] introduced the concept of irresolute function on a topological space and investigated some its properties. Maheshwari and Prasad [6] studied the new separation axioms defined by semi-open and semi-closed sets. In 1993, Latif characterized the semi-continuity and irresolute function in terms of semi-neighborhoods. Also he studied semi-convergence of filters and tried to characterize semi-continuity and irresolute function in terms of semi-convergence of filters. Recently in [1], D. C. Kent and author introduced the notion of neighborhood structures and neighborhood spaces. In order to describe a convergence theory in neighborhood spaces, we introduced "p-stacks" which are more general than filters and showed a satisfactory convergence theory can be obtained for neighborhood spaces using the more general notion of a "p-stack".

In this paper, by using the notion of semi-convergence of p-stacks, we will characterize the semi-interior, semi-closure, separation axioms,

Received November 8, 2002.
2000 Mathematics Subject Classification: 54A20, 54C08, 54D10.
Key words and phrases: p-stacks, semi-convergence of p-stacks, ultrapstacks, p-semicompact.
This work was supported by a grant from Research Institute for Basic Science at Kangwon National University.

semi-continuity on topological spaces. Also we will introduce a new notion of p-semicompactness and investigate its properties.

2. Preliminaries

Let X, Y and Z be topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure (resp. interior) of S will be denoted by clS (resp. $intS$). A subset S of X is called semi-open set [2] if $S \subset cl(int(S))$. The complement of a semi-open set is called semi-closed set. The family of all semi-open sets in X will be denoted by $SO(X)$. A function $f : X \to Y$ is called irresolute [2] if $f^{-1}(V) \in SO(X)$ for each semi-open set V of Y. A subset $M(x)$ of a space X is called a semi-neighborhood of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subset M(x)$ [3].

Given a set X, a collection $\mathbf{C}$ of subsets of X is called a stack if $A \in \mathbf{C}$ whenever $B \in \mathbf{C}$ and $B \subset A$. A stack $\mathbf{H}$ on a set X is called a p-stack [1] if it satisfies the following condition:

(p) $A, B \in \mathbf{H}$ implies $A \cap B \neq \emptyset$.

Condition (p) is called the pairwise intersection property(P.I.P). A collection B of subsets of X with the P.I.P is called a p-stack base. For any collection $\mathbf{B}$, we denote by $\langle \mathbf{B} \rangle = \{A \subset X :$ there exists $B \in \mathbf{B}$ such that $B \subset A\}$ the stack generated by $\mathbf{B}$, and if $\{B\}$ is a p-stack base, then $\langle \{B\} \rangle$ is a p-stack. We will denote simply $\langle \{B\} \rangle = \langle B \rangle$. In case $x \in X$ and $B = \{x\}$, $\langle x \rangle$ is usually denoted by $\dot{x}$. Let $pS(X)$ denote the collection of all p-stacks on X, partially ordered by inclusion. The maximal elements in $pS(X)$ are called ultrapstacks. It is obvious that every ultrafilter is an ultrapstack, and that every p-stack is contained in an ultrapstack. For a function $f : X \to Y$ and $\mathbf{H} \in pS(X)$, the image stack $f(\mathbf{H})$ in $pS(Y)$ has p-stack base $\{f(H) : H \in \mathbf{H}\}$. Likewise, if $\mathbf{G} \in pS(Y)$, $f^{-1}(\mathbf{G})$ denotes the p-stack on X generated by $\{f^{-1}(G) : G \in \mathbf{G}\}$.

DEFINITION 2.1 ([3], [6]). Let (X, τ) be a topological space and $A \subset X$.

(1) $sint(A) = \cup\{U \in SO(X) : U \subset A\}$;
(2) $scl(A) = \cap\{F \subset X : A \subset F$ and $X - F \in SO(X)\}$;
(3) X is $semi-T_1$ if for every two distinct points x and y in X, there exist two semi-open sets U and V such that $x \in U, y \notin U$ and $y \in V, x \notin V$;
(4) X is $semi-T_2$ if for every two distinct points x and y in X,

there exist two disjoint semi-open sets U and V such that $x \in U$ and $y \in V$;

(5) X is *semi*-regular if for semi-closed set H and $x \notin H$, there exist two disjoint semi-open sets U and V such that $H \subset U$ and $x \in V$;

(6) X is semi-compact if each cover of X by semi-open sets has a finite subcover.

LEMMA 2.2 [1]. *For* $\mathbf{H} \in pS(X)$, *the following are equivalent:*

(1) $\mathbf{H}$ *is an ultrapstack;*
(2) *If* $A \cap H \neq \emptyset$ *for all* $H \in \mathbf{H}$, *then* $A \in \mathbf{H}$;
(3) $B \in \mathbf{H}$ *implies* $X - B \in \mathbf{H}$.

THEOREM 2.3 [1]. *Let* $f : X \rightarrow Y$ *be a function and* $\mathbf{H} \in pS(X)$. *Then*

(1) *If* $\mathbf{H}$ *is a filter, so is* $f(\mathbf{H})$;
(2) *If* $\mathbf{H}$ *is an ultrafilter, so is* $f(\mathbf{H})$;
(3) *If* $\mathbf{H}$ *is an ultrapstack, so is* $f(\mathbf{H})$.

3. Main results

DEFINITION 3.1. Let (X, μ) be a topological space, $x \in X$ and let $\mathbf{S}_\mu(x) = \{V \subset X : V \text{ is a semi-neighborhood of } x\}$. Then we call the family $\mathbf{S}_\mu(x)$ the semi-neighborhood stack at x.

DEFINITION 3.2. Let (X, μ) be a topological space, $x \in X$, and $\mathbf{F} \in pS(X)$. A p-stack $\mathbf{F}$ on X semi-converges to x if $\mathbf{S}_\mu(x) \subset F$.

From Definition 3.2, we get the following theorem.

THEOREM 3.3. *Let* (X, μ) *be a topological space.*

(1) $\dot{x}$ *semi-converges to* x, *for all* $x \in X$;
(2) *If* $\mathbf{F}$ *semi-converges to* x *and* $\mathbf{F} \subset \mathbf{G}$ *for* $\mathbf{F}$, $\mathbf{G} \in pS(X)$, *then* $\mathbf{G}$ *semi-converges to* x;
(3) *If both* $\mathbf{F}$ *and* $\mathbf{G}$ *are p-stacks semi-converging to* x, *then* $\mathbf{F} \cap \mathbf{G}$ *semi-converges to* x;
(4) *If p-stacks* $\mathbf{F}_i$ *semi-converge to* x *for all* $i \in J$, *then* $\cap \mathbf{F}_i$ *semi-converges to* x.

THEOREM 3.4. *Let (X, μ) be a topological space and $A \subset X$. Then the following are equivalent:*

(1) $x \in scl(A)$;
(2) *There is* $\mathbf{F} \in pS(X)$ *such that* $A \in \mathbf{F}$ *and* $\mathbf{F}$ *semi-converges to* x;
(3) *For all* $V \in \mathbf{S}_\mu(x)$, $A \cap V \neq \emptyset$.

PROOF. (1) $\Rightarrow$ (2) Let x be an element in $scl(A)$, then $U(x) \cap A \neq \emptyset$ for each semi-open $U(x)$ of x. Let $\mathbf{F} = \mathbf{S}_\mu(x) \cup \langle A \rangle$. Then the p-stack $\mathbf{F}$ semi-converges to x and $A \in \mathbf{F}$.

(2) $\Rightarrow$ (3) Let $\mathbf{F}$ be a p-stack and $A \in \mathbf{F}$ and p-stack $\mathbf{F}$ semi-converge to x. Then $\mathbf{S}_\mu(x) \subset \mathbf{F}$. Thus since $\mathbf{S}_\mu(x)$ is a p-stack, we get $U \cap A \neq \emptyset$ for all $U \in \mathbf{S}_\mu(x)$.

(3) $\Rightarrow$ (1) It is obvious. □

THEOREM 3.5. *Let (X, μ) be a topological space, $A \subset X$. Then the following are equivalent:*

(1) $x \in sint(A)$;
(2) *For every p-stack* $\mathbf{F}$ *semi-converging to* x, $A \in \mathbf{F}$;
(3) $A \in \mathbf{S}_\mu(x)$.

PROOF. (1) $\Rightarrow$ (2) Let x be an element in $sint(A)$ and let $\mathbf{F}$ be a p-stack semi-converging to x. Since $x \in sint(A)$, there is a semi-open subset U such that $x \in U \subset A$, so $A \in \mathbf{S}_\mu$. Thus by the definition of semi-convergence of p-stack, we can say $A \in \mathbf{F}$

(2) $\Rightarrow$ (3) The semi-neighborhood stack $\mathbf{S}_\mu(x)$ is always semi-converges to x. Thus by the condition (2), $A \in \mathbf{S}_\mu(x)$.

(3) $\Rightarrow$ (1) It is obvious. □

Now by using semi-convergence of p-stacks, we characterize the properties of $semi-T_1$, $semi-T_2$ and *semi*-regular induced by semi-open subsets on a topological space.

THEOREM 3.6. *Let (X, μ) be a topological space. Then the following are equivalent:*

(1) (X, μ) *is* $semi-T_1$;
(2) $\cap \mathbf{S}_\mu(x) = \{x\}$ *for* $x \in X$;
(3) *If* $\dot{x}$ *semi-converges to* y, *then* $x = y$.

PROOF. (1) $\Rightarrow$ (2) Let y be an element in $\cap\mathbf{S}_\mu(x)$, then $y \in U$ for each semi-open neighborhood U of x. Since X is $semi - T_1$, we get $y = x$.

(2) $\Rightarrow$ (3) Let $\dot{x}$ semi-converge to y. Since $\mathbf{S}_\mu(y) \subset \dot{x}$, x is an element in $\cap\mathbf{S}_\mu(y)$. Thus $x = y$.

(3) $\Rightarrow$ (1) Suppose that X is not $semi - T_1$, then there are distinct x and y such that every semi-open neighborhood of x contains y. Thus $\mathbf{S}_\mu(x) \subset \dot{y}$ and $\dot{y}$ semi-converges to x. This contradicts the hypothesis.□

THEOREM 3.7. *Let (X, μ) be a topological space. Then the following are equivalent:*

(1) *(X, μ) is $semi - T_2$;*
(2) *Every semi-convergent p-stack* $\mathbf{F}$ *on X semi-converges to exactly one point;*
(3) *Every semi-convergent ultrapstack* $\mathbf{F}$ *on X semi-converges to exactly one point.*

PROOF. (1) $\Rightarrow$ (2) Suppose that X is $semi - T_2$ and a p-stack $\mathbf{F}$ semi-converges to x. For any $y \neq x$, there are disjoint semi-open sets $U(x)$ and $U(y)$ containing x and y, respectively. Since $\mathbf{S}_\mu(x) \subset \mathbf{F}$ and $\mathbf{F}$ is a p-stack, both $U(x)$ and $X - U(y)$ are elements of $\mathbf{F}$. Thus $\mathbf{F}$ is not finer than $\mathbf{S}_\mu(y)$, so $\mathbf{F}$ doesn't semi-converge to y.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Suppose that X is not $semi - T_2$. Then there must exist x, y such that $U(x) \cap U(y) \neq \emptyset$ for every semi-open sets $U(x)$ and $U(y)$ of x and y, respectively. Let $\mathbf{F}$ be a ultrapstak finer than a p-stack $\mathbf{S}_\mu(x) \cup \mathbf{S}_\mu(y)$. Then $\mathbf{F}$ is finer than $\mathbf{S}_\mu(x)$ and $\mathbf{S}_\mu(y)$, so the ultrapstack $\mathbf{F}$ semi-converges to both x and y. This contradicts (2). □

If (X, μ) is a topological space and $\mathbf{F} \in pS(X)$, then $\mathbf{B} = \{scl(F) : F \in \mathbf{F}\}$ is a p-stack base on X, and the semi-closure p-stack generated by $\mathbf{B}$ is denoted by $scl(\mathbf{F})$.

THEOREM 3.8. *Let (X, μ) be a topological space. Then the following are equivalent:*

(1) *(X, μ) is semi-regular;*
(2) *For every x in X, $\mathbf{S}_\mu(x) = scl(\mathbf{S}_\mu(x))$;*
(3) *If a p-stack* $\mathbf{F}$ *semi-converges to x, then the semi-closure p-stack $scl(\mathbf{F})$ semi-converges to x.*

PROOF. (1) $\Rightarrow$ (2) Let F be an element in $\mathbf{S}_\mu(x)$. There exists a semi-open neighborhood $U(x)$ such that $U(x) \subset F$. Since X is semi-regular, there is a semi-open neighborhood $W(x)$ of x such that $W(x) \subset scl(W(x)) \subset U(x) \subset F$. Since $scl(W(x)) \in scl(\mathbf{S}_\mu(x))$ and $scl(\mathbf{S}_\mu(x))$ is a p-stack, $F \in scl(\mathbf{S}_\mu(x))$.

(2) $\Rightarrow$ (3) Let a p-stack $\mathbf{F}$ semi-converge to x. Then $\mathbf{S}_\mu(x) \subset \mathbf{F}$, and so $scl(\mathbf{S}_\mu(x)) \subset scl(\mathbf{F})$. By the condition (2), we get that $scl(\mathbf{F})$ semi-converges to x.

(3) $\Rightarrow$ (1) Let U be a semi-open set containing $x \in X$. Since $\mathbf{S}_\mu(x)$ semi-converges to x, by (3) $scl(\mathbf{S}_\mu(x))$ semi-converges to x , and so $U \in scl(\mathbf{S}_\mu(x))$. Then by the definition of the semi-closure of p-stacks, we can get a semi-open neighborhood V of x such that $V \subset scl(V) \subset U$. □

Latif [4] showed the following : Let X and Y be topological spaces. Then a function $f : X \to Y$ is irresolute if and only if for each x in X and each semi-neighborhood U of $f(x)$, there is a semi-neighborhood V of x such that $f(V) \subset U$.

Now we get another characterization of the irresolute function on a topological space using the notion of p-stacks.

THEOREM 3.9. *Let (X, μ) and (Y, ν) be topological spaces. If $f : (X, \mu) \to (Y, \nu)$ is a function, then the following statements are equivalent*:

(1) *f is irresolute*;
(2) *$\mathbf{S}_\nu(f(x)) \subset f(\mathbf{S}_\mu(x))$ for all $x \in X$*;
(3) *If a p-stack $\mathbf{F}$ semi-converges to x, then the image p-stack $f(\mathbf{F})$ semi-converges to $f(x)$.*

PROOF. (1) $\Rightarrow$ (2) Let V be any member of $\mathbf{S}_\nu(f(x))$ in Y. Then there is a semi-open set W such that $W \subset V$. Since f is irresolute, there exists a semi-open neighborhood $U \in \mathbf{S}_\mu(x)$ such that $f(U) \subset W \subset V$, thus $V \in f(\mathbf{S}_\mu(x))$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) If f is not irresolute, then for some $x \in X$, there is a semi-open neighborhood $V \in \mathbf{S}_\nu(f(x))$ such that for all semi-open neighborhood $U \in \mathbf{S}_\mu(x)$, $f(U)$ is not included in V. For all $U \in \mathbf{S}_\mu(x)$, since $f(U) \cap (Y - V) \neq \emptyset$, we get a p-stack $\mathbf{F} = f(\mathbf{S}_\mu(x)) \cup \langle Y - V \rangle$. And since $U \cap f^{-1}(Y - V) \neq \emptyset$, also we get a p-stack $\mathbf{G} = \mathbf{S}_\mu(x) \cup \langle f^{-1}(Y - V) \rangle$ which semi-converges to x. But since $f(\mathbf{G})$ is a finer p-stack than $\mathbf{F}$ and $Y - V \in \mathbf{F}$, $f(\mathbf{G})$ can't semi-converge to $f(x)$, contradicting to (3). □

Now we introduce a new notion of p-semicompactness by p-stacks and investigate the related properties.

DEFINITION 3.10. Let (X, μ) be a topological space and A be a subset of X. A subset A of a topological space (X, μ) is p-semicompact if every ultrapstack containing A semi-converges to a point in A. A space (X, μ) is p-semicompact if X is p-semicompact.

EXAMPLE 3.11. Let $X = \{a, b, c\}$. In case τ is the discrete topology, let $\mathbf{H}$ be an ultrapstack containing a p-stack $\mathbf{F}$ generated by $\{\{a, b\}, \{b, c\}, \{a, c\}\}$. Then it doesn't semi-converge to any point in X. Thus the topological space (X, τ) is not p-semicompact. But in case $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$, the topological space (X, τ) is p-semicompact.

THEOREM 3.12. *If a topological space (X, μ) is p-semicompact and $A \subset X$ is semi-closed, then A is p-semicompact.*

PROOF. Let $\mathbf{F}$ be an ultrapstack containing A. From Definition 3.10, there is $x \in X$ such that $\mathbf{F}$ semi-converges to x. Thus $\mathbf{S}_\mu(x) \subset \mathbf{F}$, and since $A \in \mathbf{F}$ and $\mathbf{F}$ is a p-stack, $A \cap V \neq \emptyset$ for all $V \in \mathbf{S}_\mu(x)$. So by Theorem 3.4, we can say $x \in scl(A) = A$. □

THEOREM 3.13. *The irresolute image of a p-semicompact set is p-semicompact.*

PROOF. Let a function $f : (X, \mu) \to (Y, \nu)$ be irresolute, let $A \subset X$ be p-semicompact, and let $\mathbf{H}$ be an ultrapstack containing $f(A)$. If $\mathbf{G}$ is an ultrapstack containing the p-stack base $\{f^{-1}(H) : H \in \mathbf{H}\} \cup \langle A \rangle$, then for some $x \in A$, $\mathbf{G}$ semi-converges to x, and $\mathbf{H} = f(\mathbf{G})$ semi-converges to $f(x)$. Thus $f(A)$ is p-semicompact. □

THEOREM 3.14. *A topological space (X, μ) is p-semicompact if and only if each semi-open cover of X has a two-element subcover.*

PROOF. Suppose $\mathbf{H}$ is an ultrapstack in X such that it doesn't semi-converge to any point in X. Then for each $x \in X$, there is a semi-open subset $U_x \in \mathbf{S}_\mu(x)$ such that $U_x \notin \mathbf{H}$. By Lemma 2.2(3), $X - U_x \in \mathbf{H}$, for all $x \in X$. Thus $\mathbf{U} = \{U_x : x \in X\}$ is a semi-open cover of X. But $\mathbf{U}$ has no two-element subcover of X, for if $U, V \in \mathbf{U}$ and $X \subset U \cup V$, then $(X - U) \cap (X - V) = X - (U \cup V) = \emptyset$, contradicting the assumption that $\mathbf{H}$ is a p-stack.

Conversely, let $\mathbf{U}$ be a semi-open cover of X with no two-element subcover of X. Then $\mathbf{B} = \{X - U : U \in \mathbf{U}\}$ is p-stack base, and any ultrapstack containing $\mathbf{B}$ cannot semi-converge to any point in X. □

References

[1] D. C. Kent and W. K. Min, *Neighborhood Spaces*, International Journal of Mathematics and Mathematical Sciences **32** (2002), no. 7, 387–399.

[2] S. G. Crossley and S. K. Hildebrnd, *Semi-closed sets and semi-continuity in topological spaces*, Texas J. Sci. **22** (1971), 123–126.

[3] R. M. Latif, *Semi-convergence of filters and nets*, Soochow Journal of Mathematics(to appear).

[4] ———, *On characterizations of mappings*, Soochow Journal of Mathematics **19** (1993), no. 4, 475–495.

[5] N. Levin, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[6] S. N. Maheshwari and R. Prasad, *Some new separation axioms*, Annales de la societe scientifique de Bruxelles, T. **89(III)** (1975), 395–402.

[7] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, *A note on semi-continuity and precontinuity*, Indian J. pure apple. Math. **13** (1982), no. 10, 1119–1123.

Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea
E-mail: wkmin@cc.kangwon.ac.kr