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SOME GENERAL CONVERGENCE PRINCIPLES WITH APPLICATIONS

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Abstract. In the present paper, some general convergence principles are established in metric spaces and then theses principles are applied to the convergence of the iterative sequences for approximating fixed points of certain classes of mappings. By virtue of our principles, most of the latest results obtained by several authors can be deduced easily.

1. Introduction and preliminaries

Let C be a nonempty subset of a normed linear space X, X^* denote the dual space of X and $J : X \to 2^{X^*}$ be the *normalized duality mapping* on X defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = ||x|| \cdot ||f||, ||f|| = ||x||\}, \quad x \in E.$$

Definition 1.1. Let $T : C \to C$ be a given mapping and $F(T)$ denote the set of fixed points of T.

(1) The mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in C.$$

(2) The mapping T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad x \in C, p \in F(T).$$

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(3) The mapping T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of nonnegative real numbers with $k_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|, \quad x, y \in C,\ n \geq 1.$$

(4) The mapping T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of nonnegative real numbers with $k_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \leq (1 + k_n)\|x - p\|, \quad x \in C,\ p \in F(T).$$

(5) The mapping T is said to be *of asymptotically nonexpansive type* if, for any $x \in C$,

$$\limsup_{n \to \infty}(\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}) \leq 0.$$

(6) The mapping T is said to be *of asymptotically quasi-nonexpansive type* if $F(T) \neq \emptyset$ and

$$\limsup_{n \to \infty}(\sup\{\|T^n x - p\| - \|x - p\| : p \in F(T)\}) \leq 0, \quad x \in C.$$

(7) The mapping T is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(x - Tx) - (y - Ty)\|^2, \quad x, y \in C.$$

(8) The mapping T is said to be *demicontractive* if $F(T) \neq \emptyset$ and there exists a constant $k \in [0, 1)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \quad x \in C,\ p \in F(T).$$

(9) The mapping T is said to be *hemicontractive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2, \quad x \in C,\ p \in F(T).$$

(10) The mapping T is said to be *asymptotically pseudocontractive* with a sequence $\{k_n\} \subset [1, \infty)$ if $k_n \to 1$ as $n \to \infty$ and, for all $n \geq 1$ and $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n\|x - y\|^2.$$

(11) The mapping T is called *asymptotically demicontractive* if $F(T) \neq \emptyset$ and there exist a sequence $\{k_n\}$ with $k_n \geq 1$ and $k_n \to 1$ and a constant $k \in [0, 1)$ such that

$$\|Tx - p\|^2 \leq {k_n}^2 \|x - p\|^2 + k\|x - Tx\|^2, \quad x \in C,\, p \in F(T).$$

(12) The mapping T is said to be *asymptotically hemicontractive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ with $k_n \geq 1$ and $k_n \to 1$ such that

$$\|Tx - p\|^2 \leq {k_n}^2 \|x - p\|^2 + \|x - Tx\|^2, \quad x \in C,\, p \in F(T).$$

(13) The mapping T is said to be *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad x, y \in C,\, n \geq 1.$$

(14) The mapping T is said to be *uniformly quasi-L-Lipschitzian* if $F(T) \neq \emptyset$ and there exists a positive constant L such that

$$\|T^n x - p\| \leq L\|x - p\|, \quad x \in C,\, p \in F(T),\, n \geq 1.$$

These classes of mappings above mentioned have been studied extensively by various authors (see [1]-[21]) and several important fixed point theorems have been established successfully for the classes of nonexpansive mappings and asymptotically nonexpansive mappings as well as the mappings of asymptotically nonexpansive type in nearly uniformly convex Banach spaces (see [5], [7], [18]). Also, various iterative schemes in connection with these classes of mappings have been introduced and used to approximate fixed points of these classes of mappings (see [1]-[3], [9], [13], [14], [19]).

Remark 1.1. (1) It is clear that the nonexpansive mappings with $F(T) \neq \emptyset$ are quasi-nonexpansive.

(2) The linear quasi-nonexpansive mappings are nonexpansive, but it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive, for example, define $Tx = \frac{x}{2} \sin \frac{1}{x}$ for all $x \neq 0$ and $T0 = 0$ in R.

(3) It is obvious that, if T is nonexpansive, then it is asymptotically nonexpansive with the constant sequence $\{1\}$.

(4) If T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian with $L = \sup\{k_n : n \geq 1\}$ and asymptotically pseudocontractive. However, the converses of these claims are not true.

(5) It is easily seen that, if C is a bounded subset of X and $T : C \to C$ is asymptotically nonexpansive, then it must be of asymptotically nonexpansive type. However, the converse is not true.

(6) If T is asymptotically non-expansive with $F(T) \neq \emptyset$, then it is uniformly quasi-L-Lipschitzian. For more comparisions of these mappings, readers may consult Chang et al. [1] and the references therein.

REMARK 1.2. We note here that the fixed point set $F(T)$ of a uniformly quasi-L-Lipschitzian mapping T is a closed subset of D provided that D is closed. To see this, assume that $\{p_n\} \subset F(T)$ with $p_n \to p$ as $n \to \infty$. Then $p \in D$ since D is closed. By the definition of T, we have

$$\|Tp - p_n\| \leq L\|p - p_n\|,$$

which implies that $Tp = p$.

In 1991, Schu [13], [14] introduced the following iterative schemes:

Let X be a normed linear space, C a nonempty convex subset of X and $T : C \to C$ a given mapping. Then, for arbitrary $x_1 \in C$, the *modified Ishikawa iterative scheme* $\{x_n\}$ is defined by

$$\text{(MIS)} \qquad \begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, & n \geq 1, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, & n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are some suitable sequences in $[0, 1]$. With X, C, $\{\alpha_n\}$ and x_1 as above, the *modified Mann iterative scheme* $\{x_n\}$ is defined by

$$\text{(MIM)} \qquad \begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, & n \geq 1. \end{cases}$$

In 1998, Xu [19] introduced the following iterative scheme with errors:

Let X be a normed linear space, C be a nonempty convex subset of X and $T : C \to C$ be a given mapping. Then, for arbitrary $x_1 \in C$, the *Ishikawa iterative scheme* $\{x_n\}$ *with errors* is defined by

$$\text{(XISE)} \qquad \begin{cases} y_n = \bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n v_n, & n \geq 1, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, & n \geq 1, \end{cases}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a}_n\}$, $\{\bar{b}_n\}$, $\{\bar{c}_n\}$ are sequences in $[0,1]$ with $a_n+b_n+c_n = \bar{a}_n+\bar{b}_n+\bar{c}_n = 1$. With X, C, $\{u_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and x_1 as above, the *modified Mann iterative scheme* $\{x_n\}$ *with errors* is defined by

$$\text{(XMSE)} \qquad \begin{cases} x_1 \in C, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 1. \end{cases}$$

Based on the iterative scheme with errors introduced by Xu [19], the following iterative schemes have been used and studied by several authors (see [1], [2], [9], [11]):

Let X be a normed linear space, C be a nonempty convex subset of X and $T : C \to C$ a given mapping. Then, for arbitrary $x_1 \in C$, the *modified Ishikawa iteration scheme* $\{x_n\}$ *with errors* is defined by

$$\text{(MISE)} \qquad \begin{cases} y_n = \bar{a}_n x_n + \bar{b}_n T^n x_n + \bar{c}_n v_n, \quad n \geq 1, \\ x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n, \quad n \geq 1, \end{cases}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a}_n\}$, $\{\bar{b}_n\}$, $\{\bar{c}_n\}$ are sequences in $[0,1]$ with $a_n+b_n+c_n = \bar{a}_n+\bar{b}_n+\bar{c}_n = 1$. With X, C, $\{u_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and x_1 as above, the *modified Mann iteration scheme* $\{x_n\}$ *with errors* is defined by

$$\text{(MME)} \qquad \begin{cases} x_1 \in C, \\ x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n, \quad n \geq 1. \end{cases}$$

On the other hand, Petryshyn and Williamson [12] in 1973 presented a sufficient and necessary condition for the Mann iterative sequence to converge to fixed points for quasi-nonexpansive mappings. In 1996, Zhou and Jia [21] gave some sufficient conditions for the Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings in a real uniformly convex Banach space. In 1997, Ghosh and Debnath [4] extended the results of Petryshyn and Williamson [12] and proved the sufficient and necessary condition for the Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings. Recently, Liu [8], [9] has extended the above results to the class of asymptotically quasi-nonexpansive mappings and gave some sufficient and necessary conditions to converge to fixed points for the modified Ishikawa iterative sequence of asymptotically quasi-nonexpansive mappings.

This paper is motivated by the following results of Ghosh and Debnath [4] and of Liu [8], [9] concerning the convergence conditions of the iterative schemes for quasi-nonexpansive and asymptotically quasi-nonexpansive mappings in Banach spaces.

Let C be a nonempty subset of a normed linear space X and $T : C \to C$ be a self-mapping. For any $x_1 \in C$, define a sequence $\{x_n\}$ in C by

$$x_n = T^n_{\lambda,\mu} x_1, \quad T_{\lambda,\mu} = (1-\lambda)I + \lambda T T_\mu,$$

where $\lambda \in (0,1)$, $\mu \in [0,1)$ and $T_\mu = (1-\mu)I + \mu T$.

In [4], Ghosh and Debnath proved the following result:

THEOREM GD. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a continuous mapping such that*

(i) $F(T) \neq \emptyset$,

(ii) *T is quasi-nonexpansive, i.e., for all $x \in C$ and $p \in F(T)$,*

$$\|Tx - p\| \le \|x - p\|.$$

Then, for any $x_1 \in C$, the sequence $\{x_n\}$ with $x_n = T^n_{\lambda,\mu} x_1$ converges to a fixed point of T in D if and only if $d(x_n, F(T)) \to 0$ as $n \to \infty$.

REMARK 1.3. We noted that, in Theorem GD, the only use of continuity assumption imposed on T is to ensure the closedness of $F(T)$. Note that $F(T)$ is indeed a closed subset of D and we see that the continuity assumption imposed on T is unnecessary.

Very recently, Liu [8], [9] extended the results of Ghosh and Debnath [4] and proved the following results:

THEOREM L1. *Let E be a nonempty closed convex subset of a Banach space and $T : E \to E$ be an asymptotically quasi-nonexpansive mapping with the nonempty fixed point set $F(T)$. Let a sequence $\{x_n\}$ be defined by (MIS) with the restriction that $\sum_{n=1}^{\infty} k_n < \infty$. Then the sequence $\{x_n\}$ converges to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

THEOREM L2. *Let E be a nonempty closed convex subset of a Banach space and $T : E \to E$ be an asymptotically quasi-nonexpansive mapping with the nonempty fixed point set $F(T)$. Let a sequence $\{x_n\}$ be defined* *by (MISE) with the restrictions that* $\sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ *and* $\sum_{n=1}^{\infty} \bar{c}_n < \infty$. *Then the sequence* $\{x_n\}$ *converges to a fixed point of* T *if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

REMARK 1.4. We note that the proofs of Ghosh and Debnath [4, Theorem 1] and Liu [8, Theorem 1], [9, Theorem 1] have some common ingredients. This fact enlightens us to explore the possibility to establish more general results.

It is our purpose in this paper to generalize and unify the above results. We establish, in a more abstract framework, much more general convergence principles and, as applications of our principles, the results of Ghosh and Debnath [4], Liu [8], [9] and others can be deduced from our principles.

The following lemma is easily deduced from Tan and Xu [16, Lemma 1]. Also, see Cho et al. [3, Lemma 1.1] and references therein.

LEMMA 1.1. *Suppose that* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are sequences of nonnegative real numbers satisfying the following inequality:*

$$a_{n+1} \le (1 + b_n)a_n + c_n,$$

where $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$. *Then* $\lim_{n\to\infty} a_n$ *exists and, moreover, if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Proof. For any $m, n \ge 1$, by induction, we have

$$\begin{aligned} a_{n+m} &\le (1 + b_{n+m-1})a_{n+m-1} + c_{n+m-1} \\ &\le \cdots \\ &\le \exp\Big\{\sum_{j=n}^{n+m-1} b_j\Big\}\Big(a_n + \sum_{j=n}^{n+m-1} c_j\Big), \end{aligned}$$

which shows that $\{a_n\}$ is bounded and hence $\sum_{n=1}^{\infty} a_n b_n < \infty$ since $\sum_{n=1}^{\infty} b_n < \infty$ by the assumption. Thus the conclusion of the lemma follows from Tan and Xu [16, Lemma 1]. This completes the proof. □

Let (X, d) be a metric space, C be a nonempty subset of X and $T : C \to C$ be a given mapping with the nonempty fixed point set $F(T)$. Suppose that the iterative scheme $\gamma : x_{n+1} = \gamma(T, x_n)$ for all $n \ge 1$ is well-defined with respect to the chosen iterative parameters. The iterative scheme γ or iterative sequence $\{x_n\}$ is said to be *of monotone type* if there exist sequences $\{r_n\}$ and $\{s_n\}$of nonnegative real numbers such that $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and the following inequality holds:

$$(@) \qquad d(x_{n+1}, p) \leq (1 + r_n) d(x_n, p) + s_n, \quad p \in F(T),\ n \geq 1.$$

REMARK 1.5. Several known results on the convergence theorems have been shown that the monotone type iterative schemes are rich in examples. In this connection, readers refer to Chang et al. [1], Chidume [2], Osilike and Igbokwe [11] and the references therein.

2. The main results

Now, we prove the main results in this paper.

THEOREM 2.1. *Let C be a nonempty subset of a complete metric space (X, d) and $T : C \to C$ be a mapping with the nonempty closed fixed point set $F(T)$. Suppose that the iterative scheme γ : $x_{n+1} = \gamma(T, x_n)$ is of monotone type. Then the sequence $\{x_n\}$ converges strongly to a point $p \in F(T)$ if and only if*

$$\liminf_{n \to \infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$.

Proof. The necessity is obvious and so it is omitted. Now we will prove the sufficiency.

Suppose that

$$\liminf_{n \to \infty} d(x_n, F(T)) = 0.$$

We will prove that the sequence $\{x_n\}$ converges strongly to a point $p \in F(T)$. It follows from (@) that

$$d(x_{n+1}, F(T)) \leq (1 + r_n) d(x_n, F(T)) + s_n$$

for all $n \geq 1$. By Lemma 1.1, we assert that $d(x_n, F(T))$ exists and hence $d(x_n, F(T)) \to 0$ as $n \to \infty$. Setting

$$A_n = \exp\Big\{\sum_{j=n}^{\infty} r_j\Big\}, \quad B_n = \sum_{j=n}^{\infty} s_j,$$

then $A_n \to 1$ and $B_n \to 0$ as $n \to \infty$ by the assumptions that $\sum_{j=1}^{\infty} r_j < \infty$ and $\sum_{j=1}^{\infty} s_j < \infty$. For $m, n \geq 1$ and $p \in F(T)$, by using (@), we have

$$\begin{aligned} d(x_{n+m}, p) &\leq (1 + r_{n+m-1}) d(x_{n+m-1}, p) + s_{n+m-1} \\ &\leq \exp\Big\{\sum_{j=0}^{m-1} r_{n+j}\Big\}\Big(d(x_n, p) + \sum_{j=0}^{m-1} s_{n+j}\Big) \\ &\leq \exp\Big\{\sum_{j=n}^{\infty} r_j\Big\}\Big(d(x_n, p) + \sum_{j=n}^{\infty} s_j\Big) \\ &\leq A_n(d(x_n, p) + B_n) \end{aligned}$$

and so

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p) + d(x_n, p) \\ &\leq (1 + A_n) d(x_n, p) + A_n B_n \end{aligned}$$

for all $p \in F(T)$. Therefore, we have

$$d(x_{n+m}, x_n) \leq (1 + A_n) d(x_n, F(T)) + A_n B_n$$

and hence $d(x_{n+m}, x_n) \to 0$ as $n \to \infty$ for all $m \geq 1$, which shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, we may assume that $x_n \to p$ as $n \to \infty$. Thus we have $d(p, F(T)) = 0$ because of the facts that $d(x_n, F(T)) \to 0$ and $x_n \to p$ as $n \to \infty$.

Finally, we have $p \in F(T)$ since $F(T)$ is closed by the assumption of the theorem. This completes the proof. □

Note that

$$\liminf_{n \to \infty} d(x_n, F(T)) = 0$$

if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges to a point $p \in F(T)$.

Thus we have the following:

THEOREM 2.2. *Let C be a nonempty subset of a complete metric space (X, d) and $T : C \to C$ be a mapping with the nonempty closed fixed point set $F(T)$. Suppose that the iterative sequence $\{x_n\}$ is of monotone type. Then the sequence $\{x_n\}$ converges strongly to some $p \in F(T)$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges to a point $p \in F(T)$.*

REMARK 2.1. In Banach spaces, Theorems 2.1 and 2.2 are also still true. They generalize and unify the results of Liu [8], [9] and Ghosh and Debnath [7] from Banach spaces to abstract metric spaces. As applications of Theorems 2.1 and 2.2, we can deduce the corresponding results of Liu [8], [9], Ghosh and Denath [7] and others.

REMARK 2.2. If the completeness of X is replaced by the compactness of subset C, then the conclusions of Theorems 2.1 and 2.2 still hold. In fact, when C is a compact convex subset of (X, d), then C is complete.

From Theorem 2.1, we have the following:

COROLLARY 2.3. *Let X, C, T and $\{x_n\}$ be same as in Theorem 2.1. Suppose, furthermore, that the following conditions hold:*

(i) *T is asymptotically regular in x_0, i.e.,*

$$\liminf_{n\to\infty} d(x_n, Tx_n) = 0,$$

(ii) $\liminf_{n\to\infty} d(x_n, Tx_n) = 0$ *implies that*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

COROLLARY 2.4. *Let X, C, T and $\{x_n\}$ be same as in Theorem 2.1. Assume that T is asymptotically regular in x_0 and satisfies Condition (A), i.e., there is an increasing function $f : R^+ \to R^+$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that*

$$d(x_n, Tx_n) \geq f(d(x_n, F(T))$$

for all $n \geq 1$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

REMARK 2.3. Corollaries 2.1 and 2.2 extend and improve Theorems 2 and 3 of Liu [8] by weakening the asymptotic regular on T and removing some redundant restrictions on the iterative parameters $\{\alpha_n\}$ and $\{\beta_n\}$.

3. Applications

In this section, we will apply our general principles in the section 2 to some specific settings.

THEOREM 3.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be an asymptotically quasi-nonexpansive* *mapping such that* $\sum_{n=1}^{\infty} k_n < \infty$. *Let a sequence* $\{x_n\}$ *be defined by* $(MISE)$ *with the restrictions that* $\sum_{n=1}^{\infty} c_n < \infty$ *and* $\sum_{n=1}^{\infty} \bar{c}_n < \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $p \in F(T)$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Proof. Since C is a closed subset of X and $T : C \to C$ is asymptotically quasi-nonexpansive, by Remark 1.2, we see that $F(T)$ is a closed subset of C. From Liu [9, Lemma 1 (a)], we know that the sequence $\{x_n\}$ is of monotone type. Therefore, Theorem 2.1 is applicable to deduce Theorem 3.1. This completes the proof. □

THEOREM 3.2. *Let* C *be a nonempty closed convex subset of a Banach space* X *and* $T : C \to C$ *be a quasi-nonexpansive mapping. For arbitrary initial value* $x_1 \in C$, *let a sequence* $\{x_n\}$ *be defined by* $(XISE)$ *with the restrictions* $\sum_{n=1}^{\infty} c_n < \infty$ *and* $\sum_{n=1}^{\infty} \bar{c}_n < \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $p \in F(T)$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Proof. By Remark 1.2, we see that $F(T)$ is closed and so it is sufficient to show that the sequence $\{x_n\}$ is of monotone type.

Setting

$$M = \max\{\sup\{\|u_n - p\| : n \geq 1\}, \sup\{\|v_n - p\| : n \geq 1\}\}$$

and using (XISE), then, for any $n \geq 1$ and $p \in F(T)$, we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|a_n(x_n - p) + b_n(Ty_n - p) + c_n(u_n - p)\| \\
&\leq a_n\|x_n - p\| + b_n\|y_n - p\| + c_n M \\
&\leq a_n\|x_n - p\| + b_n(\bar{a}_n\|x_n - p\| + \bar{b}_n\|Tx_n - p\| \\
&\quad + \ \bar{c}_n\|v_n - p\|) + c_n M \\
&\leq (a_n + b_n(\bar{a}_n + \bar{b}_n))\|x_n - p\| + (\bar{c}_n + c_n)M \\
&\leq (1 - b_n - c_n + b_n(1 - \bar{c}_n))\|x_n - p\| + (\bar{c}_n + c_n)M \\
&\leq \|x_n - p\| + (\bar{c}_n + c_n)M,
\end{aligned}$$

which shows that the sequence $\{x_n\}$ is of monotone type. Therefore, the conclusion of the theorem follows from Theorem 2.1. This completes the proof. □

REMARK 3.1. Theorem 3.1 indeed provides a simple proof for Liu [8, Theorem 1], [9, Theorem 1], while Theorem 3.2 extends and improves Ghosh and Debnath [4, Theorem 1].

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