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LIMIT FUNCTIONS OF SKEW PRODUCT FOR CLASS M IN FATOU SET

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Abstract. In this paper, some limit functions of the iteration of the skew product in the stable sets are investigated, which is associated with finitely generated semigroup for the class M, under some additional conditions.

1. Introduction

Let E be a compact totally disconnected set in $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, and $f(z)$ be a function meromorphic in $E^c = \overline{\mathbf{C}} \backslash E$. For $z_0 \in E$, the cluster set $C(f, E^c, z_0)$ is defined as $\{w \in \overline{\mathbf{C}} : \lim_{n\to\infty} f(z_n) = w$, for some $z_n \in E^c$ with $z_n \to z_0$ as $n \to \infty\}$. E denotes by $E(f)$. Define

$$M = \{f : E(f) \neq \emptyset,\ f(z) \text{ is meromorphic in } E(f)^c \text{ and } C(f, E(f)^c, z_0) = \overline{\mathbf{C}} \text{ for all } z_0 \in E(f)\}.$$

For $f \in M$, denote by f^n the n-th iterate of f, i.e., $f^n = f(f^{n-1}), n = 1, 2, \cdots$. $E(f^n)$ is compact totally disconnected in $\overline{\mathbf{C}}$ for each n. If $E(f) = \emptyset$, we know that $f(z)$ is rational, we omit considering this case throughout. See [1].

Let $f \in M$ and $sing(f^{-t})$ be the set of singularities of f^{-t} and the limit values of these singularities for some integer $t \geq 1$.

Let m be a positive integer. We denote by Σ_m the one sided word space and denote by $\sigma : \Sigma_m \to \Sigma_m$ the shift map, i.e., for a word $w = (w_1, w_2, \cdots) \in \Sigma_m$, $\sigma w = (w_2, w_3, \cdots)$.

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Let $f_j \in M$, $j = 1, 2, \cdots, m, m \geq 1$. For a word $w = (w_1, w_2, \cdots) \in \Sigma_m$, define a map by

$$\tilde{f} \ : \ \Sigma_m \times \overline{\mathbf{C}} \to \Sigma_m \times \overline{\mathbf{C}}$$

$$(w, x) \to (\sigma w, f_{w_1} x).$$

$\tilde{f}$ is called the skew product associated with the generator system $\{f_1, \cdots, f_m\}$ or finitely generated semigroup with the semigroup operation being the composition of functions. Please see [5] for the case $f_j (j = 1, 2, \cdots, m)$ are rational and [7] for the case $f_j (j = 1, 2, \cdots, m)$ are meromorphic. We define another map as

$$\pi \circ \tilde{f} \ : \ \Sigma_m \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}$$

$$(w, x) \to f_{w_1}(x).$$

F_w is defined by

$$F_w = \{z \in \overline{\mathbf{C}} : \{f_{w_n} \circ \cdots \circ f_{w_1}(z)\}_n \text{ is well defined and normal in a neighborhood of } z\}.$$

$J_w = \overline{\mathbf{C}} \backslash F_w$. The Fatou set $\tilde{F}(\tilde{f})$ and the Julia set $\tilde{J}(\tilde{f})$ of the skew product $\tilde{f}$ are defined respectively by

$$\tilde{F}(\tilde{f}) = \overline{\mathbf{C}} \backslash \tilde{J}(\tilde{f}),$$

and

$$\tilde{J}(\tilde{f}) = \overline{\cup_{w \in \Sigma_m} \{w\} \times J_w}.$$

For each word $w = (w_1, \cdots, w_n, \cdots) \in \Sigma_m$, set $g_n(z) = f_{w_n} \circ \cdots \circ f_{w_1}(z)$ and $g_0(z) \equiv z$ throughout. We have $g_n \in M$ by Lemma 2 in [1]. Define

$$P_w = \cup_{n=0}^{\infty} g_n((\cup_{j=1}^{\infty} sing f_{w_j}^{-1}) \backslash E(g_n)).$$

P'_w is the derived set of P_w. The first result is stated below, which is discussed briefly in §3.

THEOREM 1. *Let $f_j \in M$, $j = 1, 2, \cdots, m$, $m \geq 1$, and $\tilde{f}$ be the skew product associated with the generator system $\{f_1, \cdots, f_m\}$. Given a word $w = (w_1, \cdots, w_n, \cdots) \in \Sigma_m$, if there exist a $u \in \mathbf{C}$, a $R > 0$ such that $|g_n(u)| < R$, $n = 0, 1, 2, \cdots$, and an $e \in \cap_{j=1}^{\infty} E(f_{w_j}) \backslash P'_w$, then*

$$\pi \circ \tilde{f}^n((w, z)) \not\to e, \ \forall (w, z) \in \tilde{F}(\tilde{f}), \ n \to \infty.$$

$V \times U \subset \tilde{F}(\tilde{f})$ is called a component of $\tilde{F}(\tilde{f})$ if U is the largest connected open set in $\mathbf{C}$ such that $\{\pi \circ \tilde{f}^n\}$ is normal in $V \times U$, i.e., for each word $w = (w_1, w_2, \cdots, w_n, \cdots) \in V$, $\{f_{w_n} \circ \cdots \circ w_1\}_n$ is a normal family in U in the sense of Montel. Furthermore, if $\pi \circ \tilde{f}^p(V \times U) \cap \pi \circ \tilde{f}^q(V \times U) = \emptyset$ for all $p \neq q$, $V \times U$ is said to be wandering, and so is each point $(w, z) \in V \times U$. Motivated by Proposition A.1 in [4], we have a more general result stated below.

THEOREM 2. *Let $f_j \in M$, $j = 1, 2, \cdots, m$, $m \geq 1$, and $\tilde{f}$ be the skew product associated with the generator system $\{f_1, \cdots, f_m\}$. If $(w, z) \in \tilde{F}(\tilde{f})$ is not wandering, then*

$$\log^+ \log^+ |\pi \circ \tilde{f}^n((w, z))| = O(n), n \to \infty.$$

In addition, if $\pi(\tilde{J}(\tilde{f}))$ has an unbounded component, then

$$\log^+ |\pi \circ \tilde{f}^n((w, z))| = O(n), n \to \infty.$$

2. Proof of theorems

In order to prove Theorem 1, we need the following results.

LEMMA 1. [1] *Let $f \in M$. If ψ is a Möbius transformation and $f_\psi = \psi \circ f \circ \psi^{-1}$, then $f_\psi \in M$, $F(f_\psi) = \psi(F(f))$ and $J(f_\psi) = \psi(J(f))$.*

$F(f)$ stands for the Fatou set of $f \in M$, that is

$$F(f) = \{ \text{ the largest open set in which all } f^n, n \in \mathbf{N} \text{ are meromorphic and } \{f^n\} \text{ is normal } \}.$$

$J(f)$ is the complement of $F(f)$ in $\overline{\mathbf{C}}$. By Lemma 1, we have $f_\psi^n = \psi \circ f^n \circ \psi^{-1}$ for any integer $n \geq 1$. If U is a component of $F(f)$, then $\psi(U)$ is the corresponding component of $F(f_\psi)$. If $\{f^n\}$ has a limit function ∞ on U, by a suitable ψ chosen, some finite constant c is the corresponding limit function of $\{f_\psi\}$ on $\psi(U)$, and vice versa.

Define

$$S_p(f) = \cup_{k=0}^{p-1} f^k(sing(f^{-1}) \backslash E(f^k)),$$

and

$$P(f) = \cup_{p=1}^{\infty} S_p(f).$$

Clearly,

$$sing(f^{-p}) \subseteq S_p(f) \subseteq S_{p+1}(f).$$

LEMMA 2. [6] *Let f be a transcendental meromorphic function. If $S_p(f) \subseteq D(0,R)$ and $|f^p(c)| < R$ for some constants $R > 0$ and $c \in \mathbf{C}$, then for any analytic point $z(\neq c)$ of f^p, we have*

$$|(f^p)'(z)| \geq \frac{|f^p(z)|(\log|f^p(z)| - \log R)}{4|z-c|}.$$

Let Ω be a hyperbolic domain in $\mathbf{C}$ and $\lambda_\Omega(z)$ be the hyperbolic density on Ω. If $D(a,r) = \{z : |z-a| < r\}$, then $\lambda_{D(a,r)}(z) = \frac{r}{r^2 - |z-a|^2}$. If $D_r = \{z : |z| > r\}$, then $\lambda_{D_r \setminus \{\infty\}}(z) = \frac{1}{|z|(\log|z| - \log r)}$.

LEMMA 3. (Schwarz-Pick Lemma) [6] *Let U and Ω be both hyperbolic domains and f be analytic in U such that $f(U) \subseteq \Omega$. Then*

$$\lambda_\Omega(f(z))|f'(z)| \leq \lambda_U(z), \ z \in U,$$

with equality if and only if f is a covering map of Ω from U.

Proof of Theorem 1. We use the argument of Zheng's in [6] to prove Theorem 1. Suppose that there is a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\pi \circ \tilde{f}^{n_k}((w,z)) \to e \ (k \to \infty)$$

for some $(w,z) \in \tilde{F}(\tilde{f})$, by contradiction. Without loss of a generality, we may assume that $e = \infty$. Since if $e \neq \infty$, by Lemma 1, there is a Möbius transformation ψ such that

$$h^{n_k}(z) := \psi \circ f_{w_{n_k}} \circ \cdots \circ f_{w_1} \circ \psi^{-1}(z)$$

and $h^{n_k}(z) \to \infty$ as $k \to \infty$. So there is a neighborhood U of z such that

$$\pi \circ \tilde{f}^{n_k}(\{w\} \times U) \to \infty \ (k \to \infty).$$

Take $(w,v) \in \{w\} \times U(v \neq u)$ and a $r > 0$ such that $D(v,r) \subset U$. By the hypotheses of Theorem 1, there is a $k_0 > 0$, for all $k > k_0$, we have

$$|\pi \circ \tilde{f}^{n_k}((w,v))| > R, \ |\pi \circ \tilde{f}^{n_k}((w,u))| < R.$$

By Lemma 2, we have

$$\begin{aligned} &|(\pi \circ \tilde{f}^{n_{k+1}})'((w,v))| \\ &= |(\pi \circ \tilde{f}^{n_{k+1}-n_k})'(\pi \circ \tilde{f}^{n_k}((w,v)))||(\pi \circ \tilde{f}^{n_k})'((w,v))|, \end{aligned}$$

and

$$\begin{aligned} |(\pi \circ \tilde{f}^{n_{k+1}})'((w,v))| \ \geq \ & \tfrac{|\pi\circ\tilde{f}^{n_{k+1}}((w,v))| \log|\pi\circ\tilde{f}^{n_{k+1}}((w,v))|/R}{4|\pi\circ\tilde{f}^{n_k}((w,v))-u|} \\ & \times \tfrac{|\pi\circ\tilde{f}^{n_k}((w,v))| \log|\pi\circ\tilde{f}^{n_k}((w,v))|/R}{4|v-u|}. \end{aligned}$$

On the other hand, according to Lemma 3, from

$$\pi \circ \tilde{f}^{n_{k+1}} : \{w\} \times D(v,r) \to D_R\backslash\{\infty\},$$

we have

$$\lambda_{D_R\backslash\{\infty\}}(\pi \circ \tilde{f}^{n_{k+1}}((w,v)))|(\pi \circ \tilde{f}^{n_{k+1}})'((w,v))| \le \lambda_{D(v,r)}(v),$$

and then, combining the above, we obtain

$$\frac{|\pi \circ \tilde{f}^{n_k}((w,v))|(\log|\pi \circ \tilde{f}^{n_k}((w,v))| - \log R)}{16|(\pi \circ \tilde{f}^{n_k}((w,v)) - u)(v-u)|} \le \frac{1}{r}.$$

This inequality derives from a contradiction, i.e., the left approaches ∞ as $k \to \infty$, since $\pi \circ \tilde{f}^{n_k}((w,v)) \to \infty$ as $k \to \infty$.

Theorem 1 follows. □

In order to prove Theorem 2, we need to recall some known properties on hyperbolic sets. Let W be a hyperbolic domain in $\mathbf{C}$. $\rho_W(z_1,z_2)$ stands for the hyperbolic distance between z_1 and z_2 on W, i.e.,

$$\rho_W(z_1,z_2) = \inf_{\gamma\in W} \int_\gamma \lambda_W(z)|dz|,$$

where γ are all Jordan curve connecting z_1 to z_2 in W. If W is simply-connected and $d(z,\partial W)$ is the euclidean distance between $z \in W$ and ∂W, then for any $z \in W$,

$$\frac{1}{2d(z,\partial W)} \le \lambda_W(z) \le \frac{2}{d(z,\partial W)}.$$

Let $f : W \to Y$ be analytic, where W and Y are hyperbolic domains. By the Principle of Hyperbolic Metric, we have

$$\rho_Y(f(z_1),f(z_2)) \le \rho_W(z_1,z_2),\ \forall z_1,z_2 \in W.$$

Proof of Theorem 2. We can obtain the first result by the method of the proof of Proposition A.1 in [4]. We only need to prove the second result in detail.

$(w,z) \in \tilde{F}(\tilde{f})$ implies that there is a component $V \times U$ of $\tilde{F}(\tilde{f})$ such that $(w,z) \in V \times U$. Without loss of generality, we may assume $\pi \circ \tilde{f}^n(V \times U) \subset U$ for all n. Let Γ be an unbounded component of $\pi(\tilde{J}(\tilde{f}))$. Then $\mathbf{C}\backslash\Gamma$ is simply-connected and

$$\pi \circ \tilde{f}^n : V \times U \to \mathbf{C}\backslash\Gamma,\ n = 1,2,\cdots.$$

Take $a \in \Gamma$, for any $z \in \mathbf{C}\backslash\Gamma$, we have $\lambda_{\mathbf{C}\backslash\Gamma}(z) \ge \frac{1}{2(|z|+|a|)}$. For any $(w,z) \in V \times U$, draw a Jordan arc γ connecting z to $\pi \circ \tilde{f}((w,z))$ in U, then

$$\int_{|\pi\circ\tilde{f}^{n}((w,z))|}^{|\pi\circ\tilde{f}^{n+1}((w,z))|}\lambda_{\mathbf{C}\backslash\Gamma}(z)|dz|\leq\int_{|z|}^{|\pi\circ\tilde{f}((w,z))|}\lambda_{U}(z)|dz|\leq A<\infty,$$

for some constant $A>0$. So

$$|\pi\circ\tilde{f}^{n+1}((w,z))|\leq e^{2A}(|\pi\circ\tilde{f}^{n}((w,z))|+|a|).$$

Inductively, we have

$$|\pi\circ\tilde{f}^{n+1}((w,z))|\leq ne^{2nA}M_{1},$$

where $M_1=|\pi\circ\tilde{f}((w,z))|+|a|$. And then for all sufficiently large n,

$$\log^{+}|\pi\circ\tilde{f}^{n+1}((w,z))|\leq 2\times A\times n+o(n).$$

Theorem 2 follows. □

3. A remark for Theorem 1

When $m=1$, the dynamical behavior of $\tilde{f}$ is as the same as f's, since in this case $\tilde{f}^{n}((w,z))=(w,f_{w_1}^{n})$, $n=1,2,\cdots$. According to that and Theorem 1, the following holds.

COROLLARY 1. *Let $f\in M$. Suppose that there exist constants $a\in\mathbf{C}$ and $R>0$ such that $|f^{n}(a)|<R$, $n=0,1,\cdots$. If there exists an $e\in E(f)\backslash P'(f)$, then there is no point $z\in F(f)$ satisfying*

$$|f^{n}(z)|\to e\ (n\to\infty).$$

Baker, et al proved Corollary 1 (see Theorem F in [1]) by the method from [3] when z is in an invariant component of $F(f)$, without our assuming that there exist constants $a\in\mathbf{C}$ and $R>0$ such that $|f^{n}(a)|<R$, $n=0,1,\cdots$.

For the case $f:\mathbf{C}\to\overline{\mathbf{C}}$ is meromorphic and transcendental, since it is well known that the Julia set of f is the closure of the set of all repelling fixed points of f, (in general, we don't know whether $f\in M$ always has such property, this is the reason why we assume some additional conditions in Theorem 1 and Corollary 1), it is easy to verify that the assumption of Corollary 1 is satisfied, and the following holds.

COROLLARY 2. *Let $f:\mathbf{C}\to\overline{\mathbf{C}}$ be meromorphic and transcendental. If $\infty\notin P'(f)$, then there is no point $z\in F(f)$ satisfying*

$$|f^{n}(z)|\to\infty\ (n\to\infty).$$

For Corollary 2, we don't know whether f has a wandering domain. But, if we add another conditions: $\#J(f) \cap P'(f) < \infty$ and $P'(f) \cap J_\infty \backslash \{\infty\} = \emptyset$, Zheng in [6] proved that f has no wandering domains, where $J_\infty = \cup_{n=0}^{\infty} f^{-n}(\infty)$.

If f is entire or meromorphic in $\mathbf{C}$ and has a wandering domain U, then $f^n(z) \to p \in P'(f)$ for each $z \in U$, $p \neq \infty$, see [2, 6]. For $f \in M$, if f has a wandering domain U, I don't know whether one of the limit functions of $\{f^n(z)\}$ in U must be an essential singularity of f^k for some k?

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