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A SYSTEM OF NONLINEAR VARIATIONAL INCLUSIONS IN REAL BANACH SPACES

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ABSTRACT. In this paper, we introduce and study a system of nonlinear implicit variational inclusions (SNIVI) in real Banach spaces: determine elements $x^*,\ y^*,\ z^* \in E$ such that

$$\theta \in \alpha T(y^*) + g(x^*) - g(y^*) + A(g(x^*)) \quad \text{for } \alpha > 0,$$
$$\theta \in \beta T(z^*) + g(y^*) - g(z^*) + A(g(y^*)) \quad \text{for } \beta > 0,$$
$$\theta \in \gamma T(x^*) + g(z^*) - g(x^*) + A(g(z^*)) \quad \text{for } \gamma > 0,$$

where $T, g : E \to E$, θ is zero element in Banach space E, and $A : E \to 2^E$ be m-accretive mapping. By using resolvent operator technique for m-accretive mapping in real Banach spaces, we construct some new iterative algorithms for solving this system of nonlinear implicit variational inclusions. The convergence of iterative algorithms be proved in q-uniformly smooth Banach spaces and in real Banach spaces, respectively.

1. Introduction

In recent years, variational inequalities have been extended and generalized in different directions, using novel and innovative techniques. Useful and important generalizations of variational inequalities are variational inclusions which in the Hilbert spaces settings, we refer to [3]-[4], [6]-[7], and [9]. Recently, Huang [5] introduce and study a new class of generalized set-valued implicit variational inclusions in real Banach spaces.

Iterative algorithms have played a central role in the approximation-solvability, especially of nonlinear variational inequalities as well as of

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nonlinear equations in several fields such as applied mathematics, mathematics programming, mathematical finance, control theory and optimization, engineering sciences, and others. Very recently, Verma [10] introduced and investigated the approximation solvability of a new system of nonlinear variational inequalities involving strongly monotone mappings.

Inspired and motivated by the results in [5], [10], the purpose of this paper is to introduce and study a system of nonlinear implicit variational inclusions in real Banach spaces. By using the resolvent technique for the m-accretive mapping, we establish the equivalence between the system of nonlinear implicit variational inclusions and the system of resolvent equations in real Banach spaces.

The rest of this paper is organized as follows. In section 2 some preliminary results will be established. In section 3 we shall deal with the approximation solvability of a system of nonlinear of implicit variational inclusions in q-uniformly smooth Banach spaces. Finally in section 4 we give the proof of the convergence of iterative sequences generated by the algorithms in real Banach spaces for this system of nonlinear of implicit variational inclusions without compactness.

2. Preliminaries

Let E be an arbitrary real Banach space and let $J_p (p > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_p(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^p, \text{ and } \|f\| = \|x\|^{p-1}\},$$

where E^* denotes dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. It is well known (see, for example, [12]-[13]) that $J_p(x) = \|x\|^{p-2}J(x)$ if $x \neq 0$.

DEFINITION 2.1 [1]. Let $A : D(A) \subset E \to 2^E$ be a set-valued mapping.

(1) The mapping A is said to accretive if, for any $x, y \in D(A)$, $u \in Ax$, $v \in Ay$, there exists $j_p(x-y) \in J_p(x-y)$ such that

$$\langle u - v, j_p(x-y)\rangle \geq 0.$$

(2) The mapping A is said to be m-accretive if A is accretive and $(I + \rho A)(D(A)) = E$ for every (equivalently, for some) $\rho > 0$, where I is the identity mapping, (equivalently, if A is accretive and $(I+A)(D(A)) = E$).

REMARK 2.1. It is well known that, if $E = E^* = H$ is a Hilbert space, then $A : D(A) \subset H \to 2^H$ is an m-accretive mapping if and only if $A : D(A) \subset H \to 2^H$ is a maximal monotone mapping (see, for example, [2]).

Let E be a real Banach space, θ is a zero element in E. Let $T, g : E \to E$ are two single-valued mappings. Suppose that $A : E \to 2^E$ is an m-accretive mapping. We consider a system of nonlinear implicit variational inclusion (abbreviated as SNIVI): determine elements $x^*, y^*, z^* \in E$ such that

$$\theta \in \alpha T(y^*) + g(x^*) - g(y^*) + A(g(x^*)) \quad \text{for } \alpha > 0, \tag{2.1}$$

$$\theta \in \beta T(z^*) + g(y^*) - g(z^*) + A(g(y^*)) \quad \text{for } \beta > 0, \tag{2.2}$$

and

$$\theta \in \gamma T(x^*) + g(z^*) - g(x^*) + A(g(z^*)) \quad \text{for } \gamma > 0. \tag{2.3}$$

Below are some special cases of SNIVI (2.1), (2.2) and (2.3).

(1) If $E = H$ is a Hilbert space and $A = \partial\varphi$, where $\varphi : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $\partial\varphi$ denotes the subdifferential of function φ, then SNIVI (2.1), (2.2) and (2.3) is equivalent to finding $x^*, y^*, z^* \in H$ such that

$$\begin{aligned} &\langle \alpha T(y^*) + g(x^*) - g(y^*),\ x - g(x^*)\rangle \\ &\geq\ \varphi(g(x^*)) - \varphi(x) \quad \text{for all } x \in H \text{ and for } \alpha > 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\langle \beta T(z^*) + g(y^*) - g(z^*),\ x - g(y^*)\rangle \\ &\geq\ \varphi(g(y^*)) - \varphi(x) \quad \text{for all } x \in H \text{ and for } \beta > 0, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &\langle \gamma T(x^*) + g(z^*) - g(x^*),\ x - g(z^*)\rangle \\ &\geq\ \varphi(g(z^*)) - \varphi(x) \quad \text{for all } x \in H \text{ and for } \gamma > 0. \end{aligned} \tag{2.6}$$

(2) If $E = H$ is a Hilbert space, $g = I$ is the identity mapping, and φ is the indicator function of a closed convex subset K in H, that is,

$$\varphi(u) = I_K(u) = \begin{cases} 0, & u \in K, \\ +\infty, & \text{other}, \end{cases}$$

then a system of nonlinear variational inclusion (2.4), (2.5) and (2.6) is equivalent to finding $x^*, y^*, z^* \in K$ such that

$$\langle \alpha T(y^*) + x^* - y^*,\ x - x^*\rangle \geq 0 \quad \text{for all } x \in K \text{ and for } \alpha > 0, \tag{2.7}$$

$$\langle \beta T(z^*) + y^* - z^*,\ x - y^*\rangle \geq 0 \quad \text{for all } x \in K \text{ and for } \beta > 0, \tag{2.8}$$

and

(2.9) $\langle \gamma T(x^*) + z^* - x^*,\ x - z^* \rangle \geq 0$ for all $x \in K$ and for $\gamma > 0$.

(3) For $x^* = y^* = z^*$ and $\alpha = \beta = \gamma = 1$, the system of nonlinear variational inequality (2.7)-(2.9) reduces to the following standard nonlinear variational inequality (NVI) problem : find an element $x^* \in K$ such that

$$\langle T(x^*),\ x - x^* \rangle \geq 0 \quad \text{for all } x \in K.$$

DEFINITION 2.2 [1]. Let $A : D(A) \subset E \to 2^E$ be an m-accretive mapping. Then, the resolvent operator associated with A defined by

$$R_A(u) = (I + A)^{-1}(u) \quad \text{for all } u \in E,$$

where I is the identity operator.

REMARK 2.2. It is well known that R_A is a single-valued and nonexpansive (see [1]).

LEMMA 2.1. *Elements $x^*, y^*, z^* \in E$ form a solution set of the SNIVI (2.1), (2.2) and (2.3) if and only if*

$$g(x^*) = R_A[g(y^*) - \alpha T(y^*)] \quad \text{for } \alpha > 0, \tag{2.10}$$

$$g(y^*) = R_A[g(z^*) - \beta T(z^*)] \quad \text{for } \beta > 0, \tag{2.11}$$

and

$$g(z^*) = R_A[g(x^*) - \gamma T(x^*)] \quad \text{for } \gamma > 0. \tag{2.12}$$

Proof. By using Definition 2.2, we can prove this lemma immediately. □

3. Convergence theorem in q-uniformly smooth Banach spaces

In this section, we always assume that E be a q-uniformly smooth Banach space $(1 < q \leq 2)$. It is easy to know that the generalized duality mapping $J_q : E \to 2^{E^*}$ is single-valued. We shall denote the single-valued duality map by j_q.

DEFINITION 3.1. Let $T, g : E \to E$ are two single-valued mappings.

(i) T is said to be r-strongly accretive with respect to g, $r \in (0, 1)$, if, for each $x, y \in E$, we have

$$\langle T(x) - T(y), j_q(g(x) - g(y)) \rangle \geq r\|g(x) - g(y)\|^q.$$

(ii) T is said to be s-Lipschitzian continuous with respect to g if there exists a constant $s \geq 1$ such that

$$\|T(x) - T(y)\| \leq s\|g(x) - g(y)\| \quad \text{for all } x, y \in E.$$

We also need the following lemma.

LEMMA 3.1 [12]. *If E is real q-uniformly smooth Banach space, then there exists a constant $c_q \geq 1$ such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q, \quad \forall x, y \in E.$$

Before we discuss the approximation-solvability of the SNIVI (2.1)-(2.3), we need to introduce the following iterative algorithm.

ALGORITHM 3.1. For an arbitrarily chosen initial point $x_0 \in E$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the following iterative algorithm:

$$g(x_{n+1}) = (1 - a_n)g(x_n) + a_n R_A[g(y_n) - \alpha T(y_n)] \quad \text{for } \alpha > 0,$$
$$g(y_n) = R_A[g(z_n) - \beta T(z_n)] \quad \text{for } \beta > 0,$$
$$g(z_n) = R_A[g(x_n) - \gamma T(x_n)] \quad \text{for } \gamma > 0,$$

$0 \leq a_n \leq 1$, $n = 0, 1, \ldots$, and $\sum_{n=0}^{\infty} a_n = \infty$.

If $E = H$ is a Hilbert space, $g = I$ is the identity mapping and $A = \partial\varphi$, where φ is the indicator function of a closed convex subset K in H, then algorithm 3.1 reduces to the following algorithm.

ALGORITHM 3.2. For an arbitrarily chosen initial point $x_0 \in E$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the following iterative algorithm:

$$x_{n+1} = (1 - a_n)x_n + a_n P_K[y_n - \alpha T(y_n)] \quad \text{for } \alpha > 0,$$
$$y_n = P_K[z_n - \beta T(z_n)] \quad \text{for } \beta > 0,$$
$$z_n = P_K[x_n - \gamma T(x_n)] \quad \text{for } \gamma > 0,$$

$0 \leq a_n \leq 1$, $n = 0, 1, \ldots$, and $\sum_{n=0}^{\infty} a_n = \infty$. Here, P_K is the projection of H onto K.

We now present, based on Algorithm 3.1, the approximation-solvability of the SNIVI (2.1), (2.2) and (2.3) involving r-strongly accretive and s-Lipschitzian mappings with respect to a single-valued map g, respectively, in a q-uniformly smooth Banach space setting.

THEOREM 3.2. *Let E ba a real q-uniformly smooth Banach space and $A : E \to 2^E$ be an m-accretive mapping. Let $T : E \to E$ are r-strongly accretive and s-Lipschitz continuous mappings with respect to*

g, respectively, and $g: E \to E$ be invertible. Let $x^*, y^*, z^* \in E$ form a solution set of the SNIVI (2.1)-(2.3) *and the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by Algorithm* 3.1. *Then, we have the following*

(1) *The estimates*

(i) $\|g(x_{n+1}) - g(x^*)\| \le (1-a_n)\|g(x_n) - g(x^*)\| + a_n\sigma\|g(y_n) - g(y^*)\|$,

(ii) $\|g(y_n) - g(y^*)\| \le \|g(z_n) - g(z^*)\|$ *for* $0 < \beta \le (qr/c_q s^q)^{1/(q-1)}$,

(iii) $\|g(z_n) - g(z^*)\| \le \|g(x_n) - g(x^*)\|$ *for* $0 < \gamma \le (qr/c_q s^q)^{1/(q-1)}$,

(iv) $\|g(x_{n+1}) - g(x^*)\| \le (1-a_n)\|g(x_n) - g(x^*)\| + a_n\sigma\|g(x_n) - g(x^*)\|$,

where $\sigma = (1 - q\alpha r + c_q\alpha^q s^q)^{1/q}$, and c_q is a constant appearing in Lemma 3.1.

(2) *The sequence $\{x_n\}$ converges to x^* for $0 < \beta, \gamma \le (qr/c_q s^q)^{1/(q-1)}$, and $0 < \alpha < (qr/c_q s^q)^{1/(q-1)}$.*

Proof. Since (x^*, y^*, z^*) is a solution of the SNIVI (2.1), (2.2) and (2.3), it follows from Lemma 2.1 that

$$g(x^*) = R_A[g(y^*) - \alpha T(y^*)],$$

$$g(y^*) = R_A[g(z^*) - \beta T(z^*)],$$

$$g(z^*) = R_A[g(x^*) - \gamma T(x^*)].$$

Applying Algorithm 3.1 and Remark 2.2, we have

$$\begin{aligned} (3.1) \qquad & \|g(x_{n+1}) - g(x_*)\| \\ = \; & \|(1-a_n)g(x_n) + a_n R_A[g(y_n) - \alpha T(y_n)] \\ & - (1-a_n)g(x^*) - a_n R_A[g(y^*) - \alpha T(y^*)]\| \\ \le \; & (1-a_n)\|g(x_n) - g(x^*)\| \\ & + a_n\|R_A[g(y_n) - \alpha T(y_n)] - R_A[g(y^*) - \alpha T(y^*)]\| \\ \le \; & (1-a_n)\|g(x_n) - g(x^*)\| \\ & + a_n\|g(y_n) - g(y^*) - \alpha(T(y_n) - T(y^*))\|. \end{aligned}$$

Since T is r-strongly accretive and s-Lipschitzian continuous with respect to g, respectively, we have by Lemma 3.1 that

$$\begin{aligned} (3.2) \quad & \|g(y_n) - g(y^*) - \alpha(T(y_n) - T(y^*))\|^q \\ \le \; & \|g(y_n) - g(y^*)\|^q - \alpha q\langle T(y_n) - T(y^*),\ j_q(g(y_n) - g(y^*))\rangle \\ & + c_q\alpha^q\|T(y_n) - T(y^*)\|^q \\ \le \; & \|g(y_n) - g(y^*)\|^q - q\alpha r\|g(y_n) - g(y^*)\|^q + c_q\alpha^q s^q\|g(y_n) - g(y^*)\|^q \\ = \; & (1 - q\alpha r + c_q\alpha^q s^q)\|g(y_k) - g(y^*)\|^q. \end{aligned}$$

It follows that

(3.3) $\|g(x_{n+1})-g(x^*)\| \leq (1-a_n)\|g(x_n)-g(x^*)\|+a_n\sigma\|g(y_n)-g(y^*)\|,$

where $\sigma = (1 - q\alpha r + c_q\alpha^q s^q)^{1/q}$.

Next, we consider

$$\begin{aligned}\|g(y_n) - g(y_*)\| &= \|R_A[g(z_n) - \beta T(z_n)] - R_A[g(z^*) - \beta T(z^*)]\| \\ &\leq \|g(z_n) - g(z^*) - \beta(T(z_n) - T(z^*))\|.\end{aligned}$$

Similar to (3.2), we get

$$\|g(y_n) - g(y^*)\| \leq \delta_1\|g(z_n) - g(z^*)\|,$$

where $\delta_1 = (1 - q\beta r + c_q\beta^q s^q)^{1/q}$. This implies that

(3.4) $$\|g(y_n) - g(y^*)\| \leq \|g(z_n) - g(z^*)\| \quad \text{for } \delta_1 \leq 1.$$

Similarly, we have

(3.5) $\|g(z_n)-g(z^*)\| \leq \delta_2\|g(x_n)-g(x^*)\| \leq \|g(x_n)-g(x^*)\|$ for $\delta_2 \leq 1$,

where $\delta_2 = (1 - q\gamma r + c_q\gamma^q s^q)^{1/q}$.

It follows from (3.3), (3.4) and (3.5) that

(3.6) $$\begin{aligned}&\|g(x_{n+1}) - g(x^*)\| \\ \leq\ & (1 - a_n)\|g(x_n) - g(x^*)\| + a_n\sigma\|g(x_n) - g(x^*)\| \\ =\ & [1 - (1 - \sigma)a_n]\|g(x_n) - g(x^*)\| \\ \leq\ & \prod_{i=0}^{n}[1 - (1 - \sigma)a_i]\|g(x_0) - g(x^*)\|.\end{aligned}$$

Since $\sum_{n=0}^{\infty} a_k$ is divergent and $\sigma = (1 - q\alpha r + c_q\alpha^q s^q)^{1/q} < 1$ under the assumptions of the theorem, it implies by [11] that

$$\lim_{n\to\infty}\prod_{i=0}^{n}[1 - (1 - \sigma)a_i] = 0.$$

As a result, the sequence $\{g(x_n)\}$ converges to $g(x^*)$, by (3.5) the sequence $\{g(z_n)\}$ converges to $g(z^*)$, and by (3.4) the sequence $\{g(y_n)\}$ converges to $g(y^*)$ as well. Since g is invertible, thus

$$\lim_{n\to\infty} x_n = x^*, \quad \lim_{n\to\infty} y_n = y^*, \quad \lim_{n\to\infty} z_n = z^*.$$

This completes the proof. □

REMARK 3.1. If $a_n \equiv a$ ($a \in (0,1]$) for all $n \geq 1$, then we get from (3.6)

$$\begin{aligned} & \|g(x_{n+1}) - g(x^*)\| \\ \leq\ & (1-(1-\sigma)a)\|g(x_n) - g(x^*)\} \\ =\ & \rho\|g(x_n) - g(x^*)\| \leq \cdots \\ \leq\ & \rho^{n+1}\|g(x_0) - g(x^*)\|, \end{aligned}$$

where $\rho = 1-(1-\sigma)a \in (0,1)$. Thus, the algorithm 3.1 is effective.

Let $E = H$ is a Hilbert space, we know that E is 2-uniformly smooth and $c_2 = 1$ by [12]. And let $g = I$ (the identity mapping). Then, Theorem 3.1 reduces to the following corollary.

COROLLARY 3.1. *Let H ba a real Hilbert space and $T : K \to H$ an r-strongly accretive and s-Lipschitz continuous mappings from a nonempty closed convex subset K into H. Let $x^*, y^*, z^* \in K$ form a solution set for the SNIVI (2.7)-(2.9) and the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by Algorithm 3.2. Then, we have the following*

(1) *The estimates*
(i) $\|x_{n+1} - x^*\| \leq (1-a_n)\|x_n - x^*\| + a_n\sigma\|y_n - y^*\|$,
(ii) $\|y_n - y^*\| \leq \|z_n - z^*\|$ *for* $0 < \beta \leq 2r/s^2$,
(iii) $\|z_n - z^*\| \leq \|x_n - x^*\|$ *for* $0 < \gamma \leq 2r/s^2$,
(iv) $\|x_{n+1} - x^*\| \leq (1-a_n)\|x_n - x^*\| + a_n\sigma\|x_n - x^*\|$,
where $\sigma = (1-2\alpha r + \alpha^2 s^2)^{1/2}$.

(2) *The sequence $\{x_n\}$ converges to x^* for $0 < \beta, \gamma \leq 2r/s^2$, and $0 < \alpha < 2r/s^2$.*

REMARK 3.2. Corollary 3.1 can be considered as an extension of theorem 2.1 in [10] that from a system of nonlinear variational inequality with two variables to a system of nonlinear variational inequality with three variables.

4. Convergence theorem in Banach spaces

In this section, we always assume that E is a real Banach space. For $a_n \equiv 1 (n = 0, 1, \ldots)$, then algorithm 3.1 be reduced to the following algorithm.

ALGORITHM 4.1. For an arbitrarily chosen initial point $x_0 \in E$, compute the sequences $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ by the following iterative algorithm:

$$g(x_{n+1}) = R_A[g(y_n) - \alpha T(y_n)] \quad \text{for } \alpha > 0,$$
$$g(y_n) = R_A[g(z_n) - \beta T(z_n)] \quad \text{for } \beta > 0,$$
$$g(z_n) = R_A[g(x_n) - \gamma T(x_n)] \quad \text{for } \gamma > 0.$$

In the following, we shall investigate iterative sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by the algorithm 4.1, which converge to the solution (x^*, y^*, z^*) of SNIVI (2.1)-(2.3).

Let us recall the following definition.

DEFINITION 4.1. An operator $g : E \to E$ is called:

(i) λ-strongly accretive, $\lambda \in (0,1)$, if, for any $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle g(x) - g(y), j(x-y)\rangle \geq \lambda \|x-y\|^2.$$

(ii) μ-Lipschitzian continuous, if there exists a constant $\mu \geq 1$ such that

$$\|g(x) - g(y)\| \leq \mu \|x-y\|, \quad x, y \in E.$$

In the sequel we need the following lemma.

LEMMA 4.1. (See [5, 8].) *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \quad \text{for all } j(x+y) \in J(x+y). \tag{4.1}$$

Now we prove the following theorem.

THEOREM 4.2. *Let E be a real Banach space and $A : E \to 2^E$ be an m-accretive mapping. Let $T : E \to E$ is s-Lipschitzian continuous mapping, and $g : E \to E$ be μ-Lipschitzian continuous and $(g - I)$ is k-strongly accretive, where $\mu \geq 1$ and $k \in (0,1)$ both are constants. Let (x^*, y^*, z^*) is a solution of the SNIVI (2.1)-(2.3) and the sequence $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by Algorithm 4.1. If the following conditions are satisfied,*

$$1 \leq \mu < \sqrt{2k+1}, \qquad 0 < \alpha,\ \beta,\ \gamma < \min\left\{\frac{1}{s}, \frac{2k+1-\mu^2}{2(k+1)s}\right\}, \tag{4.2}$$

then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to x^*, y^*, and z^*, respectively.

Proof. For convenience, letting

$$u_{n+1} = g(y_n) - \alpha T(y_n), \qquad u^* = g(y^*) - \alpha T(y^*),$$
$$v_{n+1} = g(z_n) - \beta T(z_n), \qquad v^* = g(z^*) - \beta T(z^*),$$

$$w_{n+1} = g(x_n) - \gamma T(x_n), \qquad w^* = g(x^*) - \gamma T(x^*).$$

Since (x^*, y^*, z^*) is a solution of the SNIVI (2.1)-(2.3), it follows from Lemma 2.1 that

$$g(x^*) = R_A(u^*), \quad g(y^*) = R_A(v^*), \quad g(z^*) = R_A(w^*).$$

Since T is s-Lipschitzian continuous and g is μ-Lipschitzian continuous, it follows from Algorithm 4.1 and Lemma 4.1 that, for any $j(u_{n+1}-u^*) \in J(u_{n+1} - u^*)$,

$$\begin{aligned}
&\|u_{n+1} - u^*\|^2 \\
= \;& \|g(y_n) - g(y^*) - \alpha(T(y_n) - T(y^*))\|^2 \\
\leq \;& \|g(y_n) - g(y^*)\|^2 - 2\alpha\langle T(y_n) - T(y^*),\ j(u_{n+1} - u^*)\rangle \\
\leq \;& \mu^2\|y_n - y^*\|^2 + 2\alpha\|T(y_n) - T(y^*)\| \cdot \|u_{n+1} - u^*\| \\
\leq \;& \mu^2\|y_n - y^*\|^2 + 2\alpha s\|y_n - y^*\| \cdot \|u_{n+1} - u^*\| \\
\leq \;& \mu^2\|y_n - y^*\|^2 + \alpha s[\|y_n - y^*\|^2 + \|u_{n+1} - u^*\|^2] \\
= \;& (\mu^2 + \alpha s)\|y_n - y^*\|^2 + \alpha s\|u_{n+1} - u^*\|^2,
\end{aligned}$$

which implies that

$$\|u_{n+1} - u^*\|^2 \leq \frac{\mu^2 + \alpha s}{1 - \alpha s}\|y_n - y^*\|^2. \tag{4.3}$$

Similarly, we have

$$\|v_{n+1} - v^*\|^2 \leq \frac{\mu^2 + \beta s}{1 - \beta s}\|z_n - z^*\|^2. \tag{4.4}$$

$$\|w_{n+1} - w^*\|^2 \leq \frac{\mu^2 + \gamma s}{1 - \gamma s}\|x_n - x^*\|^2. \tag{4.5}$$

Since $g - I$ is k-strongly accretive, from Algorithm 4.1 and Lemma 4.1, it follows that, for any $j(y_n - y^*) \in J(y_n - y^*)$,

$$\begin{aligned}
&\|y_n - y^*\|^2 \\
= \;& \|R_A(v_{n+1}) - R_A(v^*) - [g(y_n) - y_n - (g(y^*) - y^*)]\|^2 \\
\leq \;& \|R_A(v_{n+1}) - R_A(v^*)\|^2 - 2\langle (g - I)(y_n) - (g - I)(y^*),\ j(y_n - y^*)\rangle \\
\leq \;& \|v_{n+1} - v^*\|^2 - 2k\|y_n - y^*\|^2,
\end{aligned}$$

which implies that

$$\|y_n - y^*\|^2 \leq \frac{1}{2k + 1}\|v_{n+1} - v^*\|^2. \tag{4.6}$$

Similarly, we have

$$\|z_n - z^*\|^2 \leq \frac{1}{2k+1}\|w_{n+1} - w^*\|^2, \tag{4.7}$$

and

$$\|x_{n+1} - x^*\|^2 \leq \frac{1}{2k+1}\|u_{n+1} - u^*\|^2. \tag{4.8}$$

From (4.3)-(4.8), we get

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
\leq\ & \frac{\mu^2 + \alpha s}{(2k+1)(1-\alpha s)}\|y_n - y^*\|^2 \\
\leq\ & h_1 \frac{1}{2k+1}\|v_{n+1} - v^*\|^2 \qquad \left(h_1 = \frac{\mu^2 + \alpha s}{(2k+1)(1-\alpha s)}\right) \\
\leq\ & h_1 \frac{\mu^2 + \beta s}{(2k+1)(1-\beta s)}\|z_n - z^*\|^2 \\
\leq\ & h_1 h_2 \frac{1}{2k+1}\|w_{n+1} - w^*\|^2 \qquad \left(h_2 = \frac{\mu^2 + \beta s}{(2k+1)(1-\beta s)}\right) \\
\leq\ & h_1 h_2 h_3 \|x_n - x^*\|^2 \qquad \left(h_3 = \frac{\mu^2 + \gamma s}{(2k+1)(1-\gamma s)}\right) \\
=\ & h\|x_n - x^*\|^2,
\end{aligned}$$

where $h = h_1 h_2 h_3$. Now we prove that $0 < h < 1$. In fact, from the condition (4.2), it follows that

$$0 < \alpha < \frac{1}{s}, \qquad 0 < 2k + 1 - \mu^2,$$

and

$$2(k+1)s\alpha < 2k + 1 - \mu^2,$$

which implies that

$$0 < h_1 = \frac{\mu^2 + \alpha s}{(2k+1)(1-\alpha s)} < 1.$$

Similarly, we have

$$0 < h_2 < 1, \qquad \text{and} \qquad 0 < h_3 < 1.$$

Thus $0 < h < 1$. As a result, the sequence $\{x_n\}$ converges to x^*, by (4.7) and (4.5) the sequence $\{z_n\}$ converges to z^*, and by (4.4)-(4.7) the sequence $\{y_n\}$ converges to y^* as well. This completes the proof. □

REMARK 4.1. It is easy to know that if $f : E \to E$ is a continuous and k-strongly accretive mapping, then f maps E onto E (see, for examples, [5, 14]).

REMARK 4.2. In [5], Huang studied a new class of generalized set-valued implicit variational inclusions in real Banach spaces. In Theorem 4.1, we investigate a system of single-valued implicit variational inclusions in real Banach spaces, and the meaning of parameters of α, β, γ are different from that the parameter ρ in [5].

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