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MULTIPLIER TRANSFORMATIONS AND STRONGLY CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. The purpose of the present paper is to introduce some new subclasses of strongly close-to-convex functions in the open unit disk defined by multiplier transformations and study their properties. Our results include several previous known results as special cases.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions defined in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(z) - 1 = 0$. If f and g are analytic in $\mathcal{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in $\mathcal{U}$ such that $f(z) = g(w(z))$. We denote by $\mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta)$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike and convex of order $\eta (0 \leq \eta < 1)$ in $\mathcal{U}$. (see, e.g., Srivastava and Owa [16]).

If $f \in \mathcal{A}$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \eta\right)\right| < \frac{\pi}{2}\beta \ \ (z \in \mathcal{U})$$

for some $\eta (0 \leq \eta < 1)$ and $\beta (0 < \beta \leq 1)$, then f is said to be strongly starlike of order β and type η in $\mathcal{U}$. If $f \in \mathcal{A}$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \eta\right)\right| < \frac{\pi}{2}\beta \ \ (z \in \mathcal{U})$$

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for some $\eta (0 \leq \eta < 1)$ and $\beta (0 < \beta \leq 1)$, then f is said to be strongly convex of order β and type η in $\mathcal{U}$. We denote by $\mathcal{S}^*(\beta, \eta)$ and $\mathcal{C}(\beta, \eta)$ [6], respectively, the subclasses of $\mathcal{A}$ consisting of all strongly starlike and strongly convex of order β and type η in $\mathcal{U}$. It is obvious that $f \in \mathcal{A}$ belongs to $\mathcal{C}(\beta, \eta)$ if and only if $zf' \in \mathcal{S}^*(\beta, \eta)$. We also note that $\mathcal{S}^*(1, \eta) = \mathcal{S}^*(\eta)$ and $\mathcal{C}(1, \eta) = \mathcal{C}(\eta)$. In particular, the classes $\mathcal{S}^*(\beta, 0)$ and $\mathcal{C}(\beta, 0)$ have been extensively studied by Mocanu [8] and Nunokawa [11].

For any integer n, we define the multiplier transformations I_n^λ of functions $f \in \mathcal{A}$ by

$$I_n^\lambda f(z) = z + \sum_{k=2}^{\infty} k \left(\frac{1+\lambda}{k+\lambda} \right)^n a_k z^k \quad (\lambda \geq 0). \tag{1.1}$$

Obviously, we have

$$I_n^\lambda (I_m^\lambda f(z)) = I_{n+m}^\lambda f(z)$$

for all integers m and n. The operators I_n^λ are closely related to the Komatu integral operators [5] and the differential and integral operators defined by Salagean [13]. We also note that $I_0^0 f(z) = zf'(z)$ and $I_1^0 f(z) = f(z)$. Now we define new classes of analytic functions by using the multiplier transformations I_n^λ defined by (1.1) as follows:

For any integer n, let $\mathcal{K}_n^\lambda(\gamma, \delta, \eta, A, B)$ be the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \arg \left(\frac{z(I_n^\lambda f(z))'}{I_n^\lambda g(z)} - \gamma \right) \right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < 1;\ 0 < \delta \leq 1;\ z \in \mathcal{U})$$

for some $g \in \mathcal{S}_n^\lambda(\eta, A, B)$, where

$$\mathcal{S}_n^\lambda(\eta, A, B) = \left\{ g \in \mathcal{A} : \frac{1}{1-\eta} \left(\frac{zI_n^\lambda g(z))'}{I_n^\lambda g(z)} - \eta \right) \prec \frac{1+Az}{1+Bz} \right\}$$

$$(0 \leq \eta < 1;\ -1 \leq B < A \leq 1;\ z \in \mathcal{U}).$$

We note that $\mathcal{K}_0^0(\gamma, 1, \eta, 1, -1)$ and $\mathcal{K}_1^0(\gamma, 1, \eta, 1, -1)$ are the classes of quasi-convex and close-to-convex functions of order γ and type η, respectively, introduced and studied by Noor and Alkhorasani [10] and Silverman [14]. Further, $\mathcal{K}_1^0(0, \delta, 0, 1, -1)$ is the class of strongly close-to-convex functions of order δ in the sense of Pommerenke [12].

In the present paper, we give some argument properties of analytic functions belonging to $\mathcal{A}$ which contain the basic inclusion relationships among the classes $\mathcal{K}_n^\lambda(\gamma, \delta, \eta, A, B)$. The integral preserving properties in connection with the operator I_n^λ defined by (1.1) are also considered. Furthermore, we obtain the previous results by Bernardi [1], Libera [4], Noor [9] and Noor and Alkhorasani [10] as special cases.

2. Main results

In proving our main results, we need the following lemmas.

LEMMA 2.1 [2]. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* Re $(\beta h(z) + \gamma) > 0(\beta, \gamma \in \mathbb{C})$. *If p is analytic in $\mathcal{U}$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathcal{U}).$$

LEMMA 2.2 [7]. *Let h be convex univalent in $\mathcal{U}$ and ω be analytic in $\mathcal{U}$ with* Re $\omega(z) \geq 0$. *If p is analytic in $\mathcal{U}$ and $p(0) = h(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathcal{U}).$$

LEMMA 2.3 [11]. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$\left|\arg\, p(z)\right| < \frac{\pi}{2}\alpha \text{ for } |z| < |z_0| \tag{2.1}$$

and

$$\left|\arg\, p(z_0)\right| = \frac{\pi}{2}\alpha \quad (0 < \alpha \leq 1). \tag{2.2}$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{2.3}$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg\, p(z_0) = \frac{\pi}{2}\alpha \tag{2.4}$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg\, p(z_0) = -\frac{\pi}{2}\alpha \tag{2.5}$$

and

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia \;\; (a > 0). \tag{2.6}$$

At first, with the help of Lemma 2.1, we obtain the following

PROPOSITION 2.1. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* Re $h(z) > 0$. *If a function $f \in \mathcal{A}$ satisfies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(I_n^\lambda f(z))'}{I_n^\lambda f(z)} - \eta\right) \prec h(z) \;\; (0 \leq \eta < 1;\; z \in \mathcal{U}),$$

then

$$\frac{1}{1-\eta}\left(\frac{z(I_{n+1}^\lambda f(z))'}{I_{n+1}^\lambda f(z)} - \eta\right) \prec h(z) \;\; (0 \leq \eta < 1;\; z \in \mathcal{U}).$$

Proof. Let

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(I_{n+1}^\lambda f(z))'}{I_{n+1}^\lambda f(z)} - \eta\right),$$

where p is analytic function with $p(0) = 1$. By using the equation

$$z(I_{n+1}^\lambda f(z))' = (\lambda + 1)I_n^\lambda f(z) - \lambda I_{n+1}^\lambda f(z), \tag{2.7}$$

we get

$$\lambda + \eta + (1-\eta)p(z) = (\lambda + 1)\frac{I_n^\lambda f(z)}{I_{n+1}^\lambda f(z)}. \tag{2.8}$$

Taking logarithmic derivatives in both sides of (2.8) and multiplying by z, we have

$$p(z)+\frac{zp'(z)}{\lambda+\eta+(1-\eta)p(z)}=\frac{1}{1-\eta}\left(\frac{z(I_n^\lambda f(z))'}{I_n^\lambda f(z)}-\eta\right)\quad (z\in\mathcal{U}).$$

Applying Lemma 2.1, it follows that $p\prec h$, that is,

$$\frac{1}{1-\eta}\left(\frac{z(I_{n+1}^\lambda f(z))'}{I_{n+1}^\lambda f(z)}-\eta\right)\prec h(z)\ \ (z\in\mathcal{U}).$$

□

Taking $h(z)=(1+Az)/(1+Bz)(-1\le B<A\le 1)$ in Proposition 2.1, we have

COROLLARY 2.1. *The inclusion relation,* $\mathcal{S}_n^\lambda(\eta,A,B)\subset\mathcal{S}_{n+1}^\lambda(\eta,A,B)$, *holds for any integer* n.

Letting $n=\lambda=0$ and $h(z)=((1+z)/(1-z))^\beta$ $(0<\beta\le 1)$ in Proposition 2.1, we have the following inclusion relation.

COROLLARY 2.2. $\mathcal{C}(\beta,\eta)\subset\mathcal{S}^*(\beta,\eta)$.

PROPOSITION 2.2. *Let h be convex univalent in $\mathcal{U}$ with $h(0)=1$ and* Re $h(z)>0$. *If a function $f\in\mathcal{A}$ satisfies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(I_n^\lambda f(z))'}{I_n^\lambda f(z)}-\eta\right)\prec h(z)\ \ (0\le\eta<1;\ z\in\mathcal{U}),$$

then

$$\frac{1}{1-\eta}\left(\frac{z(I_n^\lambda F_c(f)(z))'}{I_n^\lambda F_c(f)(z)}-\eta\right)\prec h(z)\ \ (0\le\eta<1;\ z\in\mathcal{U}),$$

where F be the integral operator defined by

$$F_c(f):=F_c(f)(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt\quad (c\ge 0). \tag{2.9}$$

Proof. From (2.9), we have

$$z(I_n^\lambda F_c(f)(z))'=(c+1)I_n^\lambda f(z)-cI_n^\lambda F_c(f)(z). \tag{2.10}$$

Let

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(I_n^\lambda F_c(f)(z))'}{I_n^\lambda F_c(f)(z)} - \eta\right),$$

where p is analytic function with $p(0) = 1$. Then, by using (2.10), we get

$$c + \eta + (1-\eta)p(z) = (c+1)\frac{I_n^\lambda f(z)}{I_n^\lambda F_c(f)(z)}. \tag{2.11}$$

Taking logarithmic derivatives in both sides of (2.11) and multiplying by z, we have

$$p(z) + \frac{zp'(z)}{c+\eta+(1-\eta)p(z)} = \frac{1}{1-\eta}\left(\frac{z(I_n^\lambda f(z))'}{I_n^\lambda f(z)} - \eta\right) \quad (z \in \mathcal{U}).$$

Therefore, by Lemma 2.1, we have

$$\frac{1}{1-\eta}\left(\frac{z(I_n^\lambda F_c(f)(z))'}{I_n^\lambda F_c(f)(z)} - \eta\right) \prec h(z) \quad (z \in \mathcal{U}).$$

□

Letting $h(z) = (1+Az)/(1+Bz)(-1 \le B < A \le 1)$ in Proposition 2.2, we have immediately

COROLLARY 2.3. *If $f \in \mathcal{S}_n^\lambda(\eta, A, B)$, then $F_c(f) \in \mathcal{S}_n^\lambda(\eta, A, B)$, where F_c is the integral operator defined by* (2.6).

REMARK 2.1. If we take $h(z) = ((1+z)/(1-z))^\beta$ $(0 < \beta \le 1)$ in Proposition 2.2, we see immediately that all functions belonging to the classes $\mathcal{S}^*(\beta, \eta)$ and $\mathcal{C}^*(\beta, \eta)$, respectively, preserve the angles under the integral operator defined by (2.9).

Now, we derive

THEOREM 2.1. *Let $f \in \mathcal{A}$ and $0 < \delta \le 1, 0 \le \gamma < 1$. If*

$$\left|\arg\left(\frac{z(I_n^\lambda f(z))'}{I_n^\lambda g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g \in \mathcal{S}_n^\lambda(\eta, A, B)$, then

$$\left|\arg\left(\frac{z(I_{n+1}^\lambda f(z))'}{I_{n+1}^\lambda g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation* :

$$(2.12)\quad \delta = \begin{cases} \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t_1}{\frac{(1-\eta)(1+A)}{1+B}+\eta+\lambda+\alpha\sin\frac{\pi}{2}t_1}\right) & \textit{for } B \neq -1, \\ \alpha & \textit{for } B = -1, \end{cases}$$

and

$$(2.13)\qquad t_1 = \frac{2}{\pi}\sin^{-1}\left(\frac{(1-\eta)(A-B)}{(1-\eta)(1-AB)+(\eta+\lambda)(1-B^2)}\right).$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(I_{n+1}^{\lambda}f(z))'}{I_{n+1}^{\lambda}g(z)} - \gamma\right).$$

Using (2.7) and simplifying, we have

$$(2.14)\qquad ((1-\gamma)p(z)+\gamma)I_{n+1}^{\lambda}g(z) = (\lambda+1)I_n^{\lambda}f(z) - \lambda I_{n+1}^{\lambda}f(z).$$

Differentiating (2.14) and multiplying by z, we obtain

$$(2.15)\qquad \begin{aligned} &(1-\gamma)zp'(z)I_{n+1}^{\lambda}g(z) + ((1-\gamma)p(z)+\gamma)z(I_{n+1}^{\lambda}g(z))' \\ &= (\lambda+1)z(I_n^{\lambda}f(z))' - \lambda z(I_{n+1}^{\lambda}f(z))'. \end{aligned}$$

Since $g \in \mathcal{S}_n^{\lambda}(\eta, A, B)$, by Corollary 2.1, we know that $g \in \mathcal{S}_{n+1}^{\lambda}(\eta, A, B)$. Let

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(I_{n+1}^{\lambda}g(z))'}{I_{n+1}^{\lambda}g(z)} - \eta\right).$$

Then, using (2.7) once again, we have

$$(2.16)\qquad (1-\eta)q(z)+\eta+\lambda = (\lambda+1)\frac{I_n^{\lambda}g(z)}{I_{n+1}^{\lambda}g(z)}.$$

From (2.15) and (2.16), we obtain

$$\frac{1}{1-\gamma}\left(\frac{z(I_n^{\lambda}f(z))'}{I_n^{\lambda}g(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{(1-\eta)q(z)+\eta+\lambda}.$$

While, by using the result of Silverman and Silvia [15], we have

$$\left|q(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \qquad (z \in \mathcal{U}\ ;\ B \neq -1) \tag{2.17}$$

and

$$\operatorname{Re}\{q(z)\} > \frac{1-A}{2} \qquad (z \in \mathcal{U}\ ;\ B = -1). \tag{2.18}$$

Then, from (2.17) and (2.18), we obtain

$$(1-\eta)q(z) + \eta + \lambda = \rho e^{i\frac{\pi\phi}{2}},$$

where

$$\begin{cases} \frac{(1-\eta)(1-A)}{1-B} + \eta + \lambda \ < \ \rho \ < \ \frac{(1-\eta)(1+A)}{1+B} + \eta + \lambda \\ -t_1 \ < \ \phi \ < \ t_1 \text{ for } B \neq -1, \end{cases}$$

when t_1 is given by (2.11), and

$$\begin{cases} \frac{(1-\eta)(1-A)}{2} + \eta + \lambda \ < \ \rho \ < \ \infty \\ -1 \ < \ \phi \ < \ 1 \text{ for } B = -1. \end{cases}$$

We note that p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ in $\mathcal{U}$ by applying the assumption and Lemma 2.2 with $\omega(z) = 1/((1-\eta)q(z) + \eta + \lambda)$. Hence $p(z) \neq 0$ in $\mathcal{U}$.

If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.1) and (2.2) are satisfied, then(by Lemma 2.3) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

At first, suppose that $p(z_0)^{\frac{1}{\alpha}} = ia$ $(a > 0)$. Then we obtain

$$\begin{aligned} &\arg\left(p(z_0) + \frac{z_1 p'(z_0)}{(1-\eta)q(z_0) + \eta + \lambda}\right) \\ &= \frac{\pi}{2}\alpha + \arg\left(1 + i\alpha k(\rho e^{i\frac{\pi\phi}{2}})^{-1}\right) \\ &\geq \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha k \sin\frac{\pi}{2}(1-\phi)}{\rho + \alpha k \cos\frac{\pi}{2}(1-\phi)}\right) \\ &\geq \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t_1}{\frac{(1-\eta)(1+A)}{1+B} + \eta + \lambda + \alpha\sin\frac{\pi}{2}t_1}\right) \\ &= \frac{\pi}{2}\delta, \end{aligned}$$

where δ and t_1 are given by (2.12) and (2.13), respectively. Similarly, for the case $B = -1$, we have

$$\arg\left(p(z_0) + \frac{z_0 p'(z_0)}{(1-\eta)q(z_0) + \eta + \lambda}\right) \geq \frac{\pi}{2}\alpha.$$

These evidently contradict the assumption of Theorem 2.1.

Next, suppose that $p(z_0)^{\frac{1}{\alpha}} = -ia$ $(a > 0)$. Applying the same method as the above, we have

$$\begin{aligned}
&\arg\left(p(z_0) + \frac{z_0 p'(z_0)}{(1-\eta)q(z_0) + \eta + \lambda}\right) \\
\leq & -\frac{\pi}{2}\alpha - \tan^{-1}\left(\frac{\alpha \cos\frac{\pi}{2}t_1}{\frac{(1-\eta)(1+A)}{1+B} + \eta + \lambda + \alpha \sin\frac{\pi}{2}t_1}\right) \\
= & -\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and t_1 are given by (2.12) and (2.13), respectively. Similarly, for the case $B = -1$, we have

$$\arg\left(p(z_0) + \frac{z_1 p'(z_0)}{(1-\eta)q(z_0) + \eta + \lambda}\right) \leq -\frac{\pi}{2}\alpha.$$

These also are contradiction to the assumption of Theorem 2.1. Therefore we complete the proof of Theorem 2.1. □

From Theorem 2.1, we see easily the following

Corollary 2.4. *The inclusion relation,* $\mathcal{K}_n^{\lambda}(\gamma, \delta, \eta, A, B) \subset \mathcal{K}_{n+1}^{\lambda}(\gamma, \delta, \eta, A, B)$, *holds for any integer* n.

Taking $n = \lambda = 0$ in Theorem 2.1, we have

Corollary 2.5. *Let* $f \in \mathcal{A}$. *If*

$$\left|\arg\left(\frac{(zf'(z))'}{g'(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < 1;\ 0 < \delta \leq 1)$$

for some $g \in \mathcal{S}_0^0(\eta, A, B)$, *then*

$$\left|\arg\left(\frac{zf'(z)}{g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where $\alpha (0 < \alpha \leq 1)$ *is the solution of the equation given by* (2.12).

REMARK 2.2. If we put $A = 1$, $B = -1$ and $\delta = 1$ in Corollary 2.5, then we see that every quasi-convex function of order γ and type η is close-to-convex function of order γ and type η, which reduces the result obtained by Noor [9].

Letting $n = \lambda = \gamma = 0$, $B \to A (A < 1)$ and $g(z) = z$ in Theorem 2.1, we obtain

COROLLARY 2.6. *Let* $f \in \mathcal{A}$ *and* $0 < \delta \leq 1$. *If*

$$| \arg \, (f'(z) + zf''(z)) \, | < \frac{\pi}{2} \delta,$$

then

$$| \arg \, f'(z) \, | < \frac{\pi}{2} \alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation* :

$$\delta = \alpha + \frac{2}{\pi} \tan^{-1} \alpha.$$

Next, we prove

THEOREM 2.2. *Let* $f \in \mathcal{A}$ *and* $0 < \delta \leq 1$, $0 \leq \gamma < 1$. *If*

$$\left| \arg \left(\frac{z(I_n^\lambda f(z))'}{I_n^\lambda g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta$$

for some $g \in \mathcal{S}_n^\lambda (\eta, A, B)$, *then*

$$\left| \arg \left(\frac{z(I_n^\lambda F_c(f)(z))'}{I_n^\lambda F_c(g)(z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha,$$

where F_c *is defined by* (2.9), *and* α $(0 < \alpha \leq 1)$ *is the solution of the equation given by* (2.12).

Proof. Let

$$p(z) = \frac{1}{1 - \gamma} \left(\frac{z(I_n^\lambda F_c(f)(z))'}{I_n^\lambda F_c(g)(z)} - \gamma \right).$$

Since $g \in \mathcal{S}_n^\lambda(\eta, A, B)$, we have from Proposition 2.2 that $F_c(g) \in \mathcal{S}_n^\lambda(\eta, A, B)$. Using (2.7) we have

$$((1-\gamma)p(z)+\gamma)I_n^\lambda F_c(g)(z) = (c+1)I_n^\lambda f(z) - cI_n^\lambda F_c(f)(z).$$

Then, by a simple calculation, we get

$$(1-\gamma)zp'(z)+((1-\gamma)p(z)+\gamma)((1-\eta)q(z)+c+\eta) = (c+1)\frac{z(I_n^\lambda f(z))'}{I_n^\lambda F_c(g)(z)},$$

where

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(I_n^\lambda F_c(g)(z))'}{I_n^\lambda F_c(g)(z)} - \gamma\right).$$

Hence we have

$$\frac{1}{1-\gamma}\left(\frac{z(I_n^\lambda f(z))'}{I_n^\lambda g(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{(1-\eta)q(z)+\eta+c}.$$

The remaining part of the proof in Theorem 2.2 is similar to that of Theorem 2.1 and so we omit it. □

From Theorem 2.2, we see easily the following

COROLLARY 2.7. *If* $f \in \mathcal{K}_n^\lambda(\gamma, \delta, \eta, A, B)$, *then* $F_c(f) \in \mathcal{K}_n^\lambda(\gamma, \delta, \eta, A, B)$, *where* F_c *is the integral operator defined by* (2.9).

REMARK 2.3. If we take $n = \lambda = 0$ and $n = 1, \lambda = 0$ with $\delta = 1$, $A = 1$ and $B = -1$ in Corollary 2.7, respectively, then we have the corresponding results obtained by Noor and Alkhorasani [10]. Furthermore, taking $n = 1$, $\gamma = \lambda = 0$, $A = 1$, $B = -1$ and $\delta = 1$ in Corollary 2.7, we obtain the classical result by Bernardi [1], which implies the result studied by Libera [4].

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