A FUBINI THEOREM FOR GENERALIZED ANALYTIC FEYNMAN INTEGRALS AND FOURIER-FEYNMAN TRANSFORMS ON FUNCTION SPACE

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Abstract. In this paper we use a generalized Brownian motion process to define a generalized analytic Feynman integral. We then establish a Fubini theorem for the function space integral and generalized analytic Feynman integral of a functional $F$ belonging to Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ and we proceed to obtain several integration formulas. Finally, we use this Fubini theorem to obtain several Feynman integration formulas involving analytic generalized Fourier-Feynman transforms. These results subsume similar known results obtained by Huffman, Skoug and Storvick for the standard Wiener process.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. The concept of $L_1$ analytic Fourier-Feynman transform (FFT) was introduced by Brue in [1]. In [2], Cameron and Storvick introduced an $L_2$ analytic FFT. In [14], Johnson and Skoug developed an $L_p$ analytic FFT theory for $1 \leq p \leq 2$ which extended the results in [1, 2] and gave various relationships between the $L_1$ and the $L_2$ theories. In [11, 12], Huffman, Skoug and Storvick established a Fubini theorem for various analytic Wiener and Feynman integrals.

In [3], Cameron and Storvick introduced a Banach algebra $\mathcal{S}$ of functionals on Wiener space which are a kind of stochastic Fourier transform.

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of complex Borel measures on $L_2[0, T]$. In [6], Chang and Chung use a
generalized Brownian motion process to define a function space integral.
In [9], Chang and Skoug studied the analytic generalized FFT (GFFT)
on function space.

In this paper we extend the results of [11, 12] to a very general
function space $C_{a,b}[0, T]$ and Banach algebra $S(L_2^{a,b}[0, T])$. Recall that the
Wiener process is free of drift and is stationary in time, while the
stochastic processes considered in this paper are subject to a drift $a(t)$ and
are nonstationary in time.

2. Definitions and preliminaries

Let $D = [0, T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A
real valued stochastic process $Y$ on $(\Omega, \mathcal{B}, P)$ and $D$ is called a
generalized Brownian motion process if $Y(0, \omega) = 0$ almost everywhere and
for $0 = t_0 < t_1 < \cdots < t_n \leq T$, the $n$-dimensional random vector
$(Y(t_1, \omega), \cdots, Y(t_n, \omega))$ is normally distributed with density function

$$K(\tilde{t}, \tilde{\eta}) = \left((2\pi)^n \prod_{j=1}^{n} (b(t_j) - b(t_{j-1}))\right)^{-1/2}$$

$$\cdot \exp\left\{-\frac{1}{2} \sum_{j=1}^{n} \frac{(\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1}))^2}{b(t_j) - b(t_{j-1})}\right\}$$

(2.1)

where $\tilde{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\tilde{t} = (t_1, \cdots, t_n)$, $a(t)$ is an absolutely
continuous real-valued function on $[0, T]$ with $a(0) = 0$, $a'(t) \in L^2[0, T]$, and
$b(t)$ is a strictly increasing, continuously differentiable real-valued
function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$.

As explained in [16, pp.18–20], $Y$ induces a probability measure $\mu$ on
the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued
functions $x(t), t \in D$, and $\mathcal{B}^D$ is the smallest $\sigma$-algebra of subsets of $\mathbb{R}^D$
with respect to which all the coordinate evaluation maps $e_x(x) = x(t)$
defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability
measure space. This measure space is called the function space induced
by the generalized Brownian motion process $Y$ determined by $a(\cdot)$ and
$b(\cdot)$.

We note that the generalized Brownian motion process $Y$ determined
by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and co-
variance function $r(s, t) = \min\{b(s), b(t)\}$. By theorem 14.2 [16, p.187],
the probability measure $\mu$ induced by $Y$, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions $x$ on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0,T], B(C_{a,b}[0,T]), \mu)$ is the function space induced by $Y$ where $B(C_{a,b}[0,T])$ is the Borel $\sigma$-algebra of $C_{a,b}[0,T]$.

A subset $B$ of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided $\rho B$ is $B(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set $N$ is said to be scale-invariant null provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set to hold scale-invariant almost everywhere (s.a.e.) [4, 10, 15].

Let $L_{a,b}^2[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L_{a,b}^2[0,T] = \left\{ v : \int_0^T v^2(s) db(s) < \infty \text{ and } \int_0^T v^2(s) da(s) < \infty \right\}$$

where $|a(t)|$ denotes the total variation of the function $a$ on the interval $[0,t]$.

For $u, v \in L_{a,b}^2[0,T]$, let

$$\langle u, v \rangle_{a,b} = \int_0^T u(t) v(t) d[b(t)] + |a(t)|.$$

Then $(\cdot, \cdot)_{a,b}$ is an inner product on $L_{a,b}^2[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L_{a,b}^2[0,T]$. In particular note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore $(L_{a,b}^2[0,T], \| \cdot \|_{a,b})$ is a separable Hilbert space.

Let $\{\phi_j\}_{j=1}^\infty$ be a complete orthogonal set of real-valued functions of bounded variation on $[0,T]$ such that

$$\langle \phi_j, \phi_k \rangle_{a,b} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

and for each $v \in L_{a,b}^2[0,T]$, let

$$v_n(t) = \sum_{j=1}^n \langle v, \phi_j \rangle_{a,b} \phi_j(t)$$

(2.4)
for $n = 1, 2, \cdots$. Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund (PWZ) stochastic integral $\langle v, x \rangle$ is defined by the formula

$$\langle v, x \rangle = \lim_{n \to \infty} \int_0^T v_n(t)dx(t)$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists; one can show that for each $v \in L^2_{a,b}[0,T]$, the PWZ stochastic integral $\langle v, x \rangle$ exists for $\mu$-a.e. $x \in C_{a,b}[0,T]$.

We denote the function space integral of a $B(C_{a,b}[0,T])$-measurable functional $F$ by

$$\int_{C_{a,b}[0,T]} F(x) d\mu(x)$$

whenever the integral exists.

We are now ready to state the definition of the generalized analytic Feynman integral.

**Definition 2.1.** Let $\mathbb{C}$ denote the complex numbers and let $\mathbb{C}_+ = \{ \lambda \in \mathbb{C} : \text{Re}\lambda > 0 \}$. Let $F : C_{a,b}[0,T] \to \mathbb{C}$ be such that the function space integral

$$J(\lambda) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2} x) d\mu(x)$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of $F$ over $C_{a,b}[0,T]$ with parameter $\lambda$, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_{a,b}[0,T]}^{an,\lambda} F(x) d\mu(x) = J^*(\lambda).$$

Let $q \neq 0$ be a real number and let $F$ be a functional such that $\int_{C_{a,b}[0,T]}^{an,\lambda} F(x) d\mu(x)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of $F$ with parameter $q$ and we write

$$\int_{C_{a,b}[0,T]}^{an,t} F(x) d\mu(x) = \lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]}^{an,\lambda} F(x) d\mu(x)$$

where $\lambda$ approaches $-iq$ through $\mathbb{C}_+$.

Now, we give the definition of the Banach algebra $S(L^2_{a,b}[0,T])$. 
Definition 2.2. Let $M(L^2_{a,b}[0,T])$ be the space of complex-valued, countably additive Borel measures on $L^2_{a,b}[0,T]$. The Banach algebra $S(L^2_{a,b}[0,T])$ consists of those functionals $F$ on $C_{a,b}[0,T]$ expressible in the form

$$F(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, x \rangle\} df(v)$$

for s.a.e. $x \in C_{a,b}[0,T]$ where the associated measure $f$ is an element of $M(L^2_{a,b}[0,T])$.

Remark 2.3. (i) When $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $S(L^2_{a,b}[0,T])$ reduces to the Banach algebra $S$ introduced by Cameron and Storvick in [3]. For further work on $S$, see the references referred to in Section 20.1 of [13].

(ii) $M(L^2_{a,b}[0,T])$ is a Banach algebra under the total variation norm where convolution is taken as the multiplication.

(iii) One can show that the correspondence $f \rightarrow F$ is injective, carries convolution into pointwise multiplication and that $S(L^2_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{L^2_{a,b}[0,T]} |df(v)|.$$

In [3], Cameron and Storvick carry out these arguments in detail for the Banach algebra $S$.

The following function space integral and generalized analytic Feynman integral formulas are used several times in this paper [5, 9].

$$\int_{C_{a,b}[0,T]} \exp\{i\alpha\langle v, x \rangle\} d\mu(x) = \exp\left\{-\frac{\alpha^2\langle v^2, b' \rangle}{2} + i\alpha\langle v, a' \rangle\right\}$$

for all $\alpha > 0$, and

$$\int_{C_{a,b}[0,T]}^{anf_a} \exp\{i\langle v, x \rangle\} d\mu(x) = \exp\left\{-\frac{i\langle v^2, b' \rangle}{2q} + i\left(\frac{i}{q}\right)^{1/2} \langle v, a' \rangle\right\}$$

for all real $q \neq 0$, $(i/q)^{1/2}$ is always chosen to have positive real part and $v \in L^2_{a,b}[0,T]$ where

$$(v, a') = \int_0^T v(t)a'(t)dt = \int_0^T v(t)da(t)$$

and

$$(v^2, b') = \int_0^T v^2(t)b'(t)dt = \int_0^T v^2(t)db(t).$$
REMARK 2.4. If \( a(t) \equiv 0 \) on \([0, T]\), then for all \( F \in \mathcal{S}(L^2_{a,b}[0, T])\) with associated measure \( f \), the generalized analytic Feynman integral of \( F \) will always exist for all real \( q \neq 0 \) and be given by the formula

\[
\int_{C_{a,b}[0, T]}^\text{anf}_q F(x) d\mu(x) = \int_{L^2_{a,b}[0, T]} \exp\left\{ -\frac{i(v^2, b')}{2q} \right\} df(v).
\]

However for \( a(t) \) as in this section, and proceeding formally using equations (2.9) and (2.11), we see that \( \int_{C_{a,b}[0, T]}^\text{anf}_q F(x) d\mu(x) \) will be given by the formula

\[
\int_{C_{a,b}[0, T]}^\text{anf}_q F(x) d\mu(x) = \int_{L^2_{a,b}[0, T]} \exp\left\{ -\frac{i(v^2, b')}{2q} + i \left( \frac{v}{q} \right)^\frac{1}{2} (v, a') \right\} df(v)
\]

if it exists. But the integral on the right hand-side of (2.15) might not exist if the real part of

\[
\exp\left\{ -\frac{i(v^2, b')}{2q} + i \left( \frac{v}{q} \right)^\frac{1}{2} (v, a') \right\}
\]

is positive. However

\[
\left| \exp\left\{ -\frac{i(v^2, b')}{2q} + i \left( \frac{v}{q} \right)^\frac{1}{2} (v, a') \right\} \right| = \left\{ \begin{array}{ll}
\exp\left\{ -(2q)^{-1/2}(v, a') \right\}, & q > 0 \\
\exp\left\{ -(2q)^{-1/2}(v, a') \right\}, & q < 0
\end{array} \right.
\]

and so the generalized analytic Feynman integral of \( F \) will certainly exist provided the associated measure \( f \) satisfies the condition

\[
\int_{L^2_{a,b}[0, T]} \exp\left\{ |2q|^{-1/2} \int_0^T |v(s)|d|a|(s) \right\} df(v) < \infty.
\]

3. Generalized Feynman integrals

In this section we establish a Fubini theorem for the function space integral and the generalized analytic Feynman integral for a functional \( F \) in a Banach algebra \( \mathcal{S}(L^2_{a,b}[0, T]) \). We also use this Fubini theorem to establish several generalized analytic Feynman integration formulas.

In our first Lemma we obtain a Fubini theorem for function space integrals of a functional \( F \in \mathcal{S}(L^2_{a,b}[0, T]) \).
LEMMA 3.1. Let $F$ be an element of $\mathcal{S}(L^2_{\alpha,\beta}[0,T])$ given by (2.9). Then for all $\alpha, \beta > 0$,

\begin{equation}
\int_{C_{\alpha,\beta}[0,T]} \left[ \int_{C_{\alpha,\beta}[0,T]} F(\alpha y + \beta z) d\mu(y) \right] d\mu(z)
= \int_{C_{\alpha,\beta}[0,T]} \left[ \int_{C_{\alpha,\beta}[0,T]} F(\alpha y + \beta z) d\mu(z) \right] d\mu(y).
\end{equation}

In addition, both expressions in (3.1) are given by the expression

\begin{equation}
\int_{L^2_{\alpha,\beta}[0,T]} \exp \left\{ -\frac{1}{2} (\alpha^2 + \beta^2)(v^2, b') + i(\alpha + \beta)(v, a') \right\} df(v).
\end{equation}

Proof. Since $F$ is an element of $\mathcal{S}(L^2_{\alpha,\beta}[0,T])$, we have

\begin{equation}
\int_{C_{\alpha,\beta}[0,T]} |F(\rho x)| d\mu(x) < \infty
\end{equation}

for each $\rho > 0$. Hence by the usual Fubini theorem, we have the equation (3.1) above. Further, by using (2.10), we have for all $\alpha, \beta > 0$,

\begin{equation}
\int_{C_{\alpha,\beta}[0,T]} \left[ \int_{C_{\alpha,\beta}[0,T]} F(\alpha y + \beta z) d\mu(y) \right] d\mu(z)
= \int_{C_{\alpha,\beta}[0,T]} \left[ \int_{C_{\alpha,\beta}[0,T]} \int_{L^2_{\alpha,\beta}[0,T]} \exp\{i \langle v, \alpha y \rangle + i \langle v, \beta z \rangle\} df(v) d\mu(y) \right] d\mu(z)
= \int_{L^2_{\alpha,\beta}[0,T]} \int_{C_{\alpha,\beta}[0,T]} \exp \{ i \langle v, \beta z \rangle \}
\cdot \left[ \int_{C_{\alpha,\beta}[0,T]} \exp \{ i \langle v, \alpha y \rangle \} d\mu(y) \right] d\mu(z) df(v)
= \int_{L^2_{\alpha,\beta}[0,T]} \exp \left\{ -\frac{\alpha^2}{2} (v^2, b') + i\alpha (v, a') \right\}
\cdot \left[ \int_{C_{\alpha,\beta}[0,T]} \exp \{ i \langle v, \beta z \rangle \} d\mu(z) \right] df(v)
= \int_{L^2_{\alpha,\beta}[0,T]} \exp \left\{ -\frac{1}{2} (\alpha^2 + \beta^2)(v^2, b') + i(\alpha + \beta)(v, a') \right\} df(v).
\end{equation}
THEOREM 3.2. Let \( q_0 \) be a nonzero real number and let \( F \) be an element of \( \mathcal{S}(L^2_{a,s}[0,T]) \) given by (2.9) whose associated measure \( f \) satisfies the condition

\[
\int_{L^2_{a,s}[0,T]} \exp\left\{ 4|2q_0|^{-1/2} \int_0^T |v(s)|^2 |d|a|(s)\right\} |df(v)| < \infty.
\]

Then for all nonzero real numbers \( q_1 \) and \( q_2 \) with \( |q_1| \geq |q_0|, |q_2| \geq |q_0| \) and \( q_1 + q_2 \neq 0 \),

\[
\int_{C_{a,s}[0,T]} \left[ \int_{C_{a,s}[0,T]} F(y + z) d\mu(y) \right] d\mu(z)
\]

\[
= \int_{C_{a,s}[0,T]} \left[ \int_{C_{a,s}[0,T]} F_{q_1,q_2}(x) d\mu(x) \right] d\mu(y)
\]

\[
= \int_{C_{a,s}[0,T]} \left[ \int_{C_{a,s}[0,T]} F(y + z) d\mu(z) \right] d\mu(y)
\]

where \( F_{q_1,q_2} \) is given by (3.11) below.

Also, all expressions in (3.6) below are given by the expression

\[
\int_{L^2_{a,s}[0,T]} \exp\left\{ -\frac{i}{2} \left( \frac{1}{q_1} + \frac{1}{q_2} \right) (v^2, b') + i \left( \left( \frac{i}{q_1} \right)^{\frac{1}{2}} + \left( \frac{i}{q_2} \right)^{\frac{1}{2}} \right) (v, a') \right\} df(v).
\]

Proof. Using the usual Fubini theorem, (2.15), and (2.10), we have that for all \( \lambda_2 > 0 \),

\[
\int_{C_{a,s}[0,T]} \left[ \int_{C_{a,s}[0,T]} F(y + \lambda_2^{-1/2} z) d\mu(y) \right] d\mu(z)
\]

\[
= \int_{L^2_{a,s}[0,T]} \left[ \int_{C_{a,s}[0,T]} \exp \{ i\langle v, y \rangle \} d\mu(y) \right] \cdot \exp \{ i\lambda_2^{-1/2} \langle v, z \rangle \} d\mu(z) df(v)
\]

\[
= \int_{L^2_{a,s}[0,T]} \exp\left\{ -\frac{i}{2} (v^2, b') + i \left( \left( \frac{i}{q_1} \right)^{\frac{1}{2}} \right) (v, a') \right\}
\]
\[
\int_{C_{a,b}[0,T]} \exp\{i \lambda_2^{-1/2} \langle v, z \rangle\} d\mu(z) d\nu(v)
\]
\[
= \int_{L_{a,b}^2[0,T]} \exp\left\{ -\frac{i (v^2, b')}{2q_1} + i \left( \frac{i}{q_1} \right)^{1/2} (v, a') - \frac{(v^2, b')}{2\lambda_2} + i \lambda_2^{-1/2} (v, a') \right\} d\mu(v).
\]

But the last expression above is an analytic function of \( \mathbb{C}_+ \) and is a continuous function of \( \lambda_2 \) in \( \hat{\mathbb{C}}_+ = \{ \lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \text{Re}\lambda \geq 0 \} \), and so setting \( \lambda_2 = -iq_2 \) yields (3.7).

Also, using (2.15) with \( q \) replaced with \( q_2 \), we obtain that for all \( \lambda_1 > 0 \)

\[
\int_{C_{a,b}[0,T]} \left[ \int_{C_{a,b}[0,T]} F(\lambda_1^{-1/2} y + z) d\mu(z) \right] d\mu(y)
\]
\[
= \int_{L_{a,b}^2[0,T]} \exp\left\{ -\frac{(v^2, b')}{2\lambda_1} + i \lambda_1^{-1/2} (v, a') - \frac{i (v^2, b')}{2q_2} + i \left( \frac{i}{q_2} \right)^{1/2} (v, a') \right\} d\mu(v).
\]

By the same argument with \( \lambda_1 = -iq_1 \), we have the expression (3.7) above. Moreover, the expression (3.7) is equal to

\[
\int_{L_{a,b}^2[0,T]} \exp\left\{ -\frac{i}{2} \left( \frac{1}{q_1} + \frac{1}{q_2} \right) (v^2, b') + i \left( \frac{i}{q_1} + \frac{i}{q_2} \right)^{1/2} (v, a') \right\} d\mu(q_1, q_2)(v)
\]
\[
= \int_{C_{a,b}[0,T]} \left[ \int_{C_{a,b}[0,T]} \exp\left\{ -\frac{i}{2} \left( \frac{q_1 q_2}{q_1 + q_2} \right) (v^2, b') + i \left( \frac{i}{q_1 + q_2} \right)^{1/2} (v, a') \right\} d\mu(q_1, q_2)(v) \right] d\mu(x)
\]
where

\[
F_{q_1, q_2}(x) = \int_{L_{a,b}^2[0,T]} \exp\{i \langle v, x \rangle\} d\mu(q_1, q_2)(v)
\]
and
\[ f_{q_1,q_2}(E) = \int_E \exp \left\{ i \left( \left( \frac{i}{q_1} \right)^{\frac{1}{2}} + \left( \frac{i}{q_2} \right)^{\frac{1}{2}} \right) (v, a') - i \left( \frac{i}{q_1} + \frac{i}{q_2} \right)^{\frac{1}{2}} (v, a') \right\} df(v) \]
for every \( E \in \mathcal{B}(L^{2}_{a,b}[0,T]) \). Finally, we have that
\[ \| f_{q_1,q_2} \| = \int_{L^{2}_{a,b}[0,T]} | df_{q_1,q_2} (v) | \]
\[ \leq \int_{L^{2}_{a,b}[0,T]} \exp \left\{ |2q_1|^{-1/2} \int_0^T |v(s)|d|a|(s)\right\} \]
\[ \cdot \exp \left\{ |2q_2|^{-1/2} \int_0^T |v(s)|d|a|(s)\right\} \]
\[ \cdot \exp \left\{ \left| \frac{2q_1q_2}{q_1 + q_2} \right|^{-1/2} \int_0^T |v(s)|d|a|(s)\right\} \]
\[ \leq \int_{L^{2}_{a,b}[0,T]} \exp \left\{ 4|q_0|^{-1/2} \int_0^T |v(s)|d|a|(s)\right\} | df(v) | < \infty. \]

Hence \( f_{q_1,q_2} \) is an element of \( M(L^{2}_{a,b}[0,T]) \) and so \( F_{q_1,q_2} \) is in \( S(L^{2}_{a,b}[0,T]) \).
Thus we have the desired results. \( \Box \)

**Corollary 3.3.** Let \( q_0 \) and \( F \) be as in Theorem 3.2. Then for all real \( q \neq 0 \) with \( |q| \geq |q_0| \),
\[ \int_{C_{a,b}[0,T]}^{\text{an} f_{q_2}} \left[ \int_{C_{a,b}[0,T]}^{\text{an} f_{q_2}} F(y + z) d\mu(y) \right] d\mu(z) = \int_{C_{a,b}[0,T]}^{\text{an} f_{q_2}/2} F_{q,q}(x) d\mu(x) \]
where
\[ F_{q,q}(x) = \int_{L^{2}_{a,b}[0,T]} \exp \left\{ i \langle v, x \rangle \right\} df_{q,q} (v) \]
and
\[ f_{q,q}(E) = \int_E \exp \left\{ 2i \left( \frac{1}{q} \right)^{\frac{1}{2}} (v, a') - i \left( \frac{2i}{q} \right)^{\frac{1}{2}} (v, a') \right\} df(v) \]
for every \( E \in \mathcal{B}(L^{2}_{a,b}[0,T]) \).
A Fubini theorem for generalized analytic Feynman integrals

Theorem 3.4. Let $q_0$ be a nonzero real number and let $F$ be an element of $S(L^2_a([0,T]))$ given by (2.9). Let $q_1, \cdots, q_n-1$ and $q_n$ be nonzero real numbers satisfying the following:

i) $|q_j| \geq |q_0|$ for all $j = 1, \cdots, n$;

ii) for all $j, l = 1, \cdots, n$, $q_j + q_l \neq 0$.

iii) for all $k = 2, \cdots, n$, $\sum_{j=1}^{k} \frac{q_{k-j}}{q_j} \neq 0$.

Suppose that the associated measure $f$ of $F$ satisfies the condition

$$
(3.17) \quad \int_{L^2_a([0,T])} \exp \left\{ 2n|2q_0|^{-1/2} \int_0^T |v(s)|d[a](s) \right\} |df(v)| < \infty
$$

for $n = 1, 2, \cdots$, then

$$
(3.18) \quad \int_{C_a[0,T]}^{anf_{q_n}} \int_{C_a[0,T]}^{anf_{q_1}} \cdots F(y_1 + \cdots + y_n)d\mu(y_1) \cdots d\mu(y_n)
$$

where $\alpha_n = \frac{\sum_{j=1}^{n} \frac{q_{k-j}}{q_j}}{\sum_{j=1}^{n} \frac{q_1}{q_j}}$ and $F_{q_1, \cdots, q_n}$ is given by equation (3.21) below.

In addition, both expressions in (3.18) are given by the expression

$$
(3.19) \quad \int_{L^2_a([0,T])} \exp \left\{ \frac{i}{2} \sum_{j=1}^{n} \frac{1}{q_j} (v^2, b') + i \sum_{j=1}^{n} \left( \frac{i}{q_j} \right)^{\frac{1}{2}} (v, a') \right\} |df(v)|.
$$

Proof. Using equation (3.6) repeatedly, we obtain that

$$
(3.20) \quad \int_{C_a[0,T]}^{anf_{q_n}} \int_{C_a[0,T]}^{anf_{q_1}} \cdots F(y_1 + \cdots + y_n)d\mu(y_1) \cdots d\mu(y_n)
= \int_{C_a[0,T]}^{anf_{q_n}} \int_{C_a[0,T]}^{anf_{q_3}} \cdots \int_{C_a[0,T]}^{anf_{q_{1+q_2}}} F_{q_1, q_2}(z_1 + y_3 + \cdots + y_n)d\mu(z_1)d\mu(y_3) \cdots d\mu(y_n)
$$
\[
\begin{align*}
&= \int_{C_{a,b}[0,T]}^{\infty} \cdots \int_{C_{a,b}[0,T]}^{\infty} f_{q_1, \cdots, q_n}(s_2 + y_4 + \cdots + y_n) \\
&= \cdots \\
&= \int_{C_{a,b}[0,T]}^{\infty} F_{q_1, \cdots, q_n}(x) d\mu(x)
\end{align*}
\]

where

\begin{equation}
F_{q_1, \cdots, q_n}(x) = \int_{L^2_{a,b}[0,T]} \exp\{i(v, x)\} df_{q_1, \cdots, q_n}(v)
\end{equation}

and

\begin{equation}
F_{q_1, \cdots, q_n}(E) = \int_E \exp\left\{i \left( \sum_{j=1}^{n} \left( \frac{i}{q_j} \right) \left( v, a' \right) - \left( \sum_{j=1}^{n} \frac{i}{q_j} \right) \left( v, a' \right) \right) \right\} df(v)
\end{equation}

for every \( E \in B(L^2_{a,b}[0,T]) \). Finally, we have that

\begin{equation}
\|f_{q_1, \cdots, q_n}\| = \int_{L^2_{a,b}[0,T]} |df_{q_1, \cdots, q_n}(v)|
\end{equation}

\begin{align*}
&\leq \int_{L^2_{a,b}[0,T]} \exp\left\{ \sum_{j=1}^{n} |2q_j|^{-1/2} \int_0^T |v(s)| |d|a|(s)| \right\} \\
&\quad \cdot \exp\left\{ |2\alpha_n|^{-1/2} \int_0^T |v(s)| |d|a|(s)| \right\} |df(v)| \\
&\leq \int_{L^2_{a,b}[0,T]} \exp\left\{ 2 \sum_{j=1}^{n} |2q_j|^{-1/2} \int_0^T |v(s)| |d|a|(s)| \right\} |df(v)| \\
&\leq \int_{L^2_{a,b}[0,T]} \exp\left\{ 2n |2q_0|^{-1/2} \int_0^T |v(s)| |d|a|(s)| \right\} |df(v)| < \infty.
\end{align*}

Hence \( f_{q_1, \cdots, q_n} \) is an element of \( M(L^2_{a,b}[0,T]) \) and so \( F_{q_1, \cdots, q_n} \) is in \( S(L^2_{a,b}[0,T]) \). Thus we have the desired results.

Choosing \( q_j = q \) for \( j = 1, \cdots, n \), we obtain the following corollary to Theorem 3.4.
COROLLARY 3.5. Let $q_0$ be a nonzero real number and let $F$ be an element of $S(L^2_{a,b}[0,T])$ given by (2.9) whose associated measure $f$ satisfies the condition

$$\int_{L^2_{a,b}[0,T]} \exp\left\{2n|2q_0|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\} |df(v)| < \infty$$

for $n = 1, 2, \ldots$. Then for all real $q$ with $|q| \geq |q_0|$,

$$\int_{C_{a,b}[0,T]} \cdots \int_{C_{a,b}[0,T]} F(y_1 + \cdots + y_n) d\mu(y_1) \cdots d\mu(y_n)$$

$$= \int_{C_{a,b}[0,T]} F_{q_1,\ldots,q_n}(x) d\mu(x)$$

where

$$F_{q_1,\ldots,q_n}(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, x \rangle\} df_{q_1,\ldots,q_n}(v)$$

and

$$f_{q_1,\ldots,q_n}(E) = \int_E \exp\left\{\frac{n}{2}\left(\frac{v}{q}\right)^{\frac{1}{2}}(v, a') - i\left(\frac{in}{q}\right)^{\frac{1}{2}}(v, a')\right\} df(v)$$

for every $E \in \mathcal{B}(L^2_{a,b}[0,T])$.

REMARK 3.6. Note that each of the iterated integrals in equation (3.18) above can also be expressed in $(n! - 1)$ other similar ways; for example, all of the expressions in (3.18), also equal the expression

$$\int_{C_{a,b}[0,T]} \cdots \int_{C_{a,b}[0,T]} F_{q_1,\ldots,q_n}(y_1 + x) d\mu(y_1) d\mu(x)$$

where

$$F_{q_1,\ldots,q_n}(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, x \rangle\} df_{q_1,\ldots,q_n}(v)$$

and

$$f_{q_1,\ldots,q_n}(E) = \int_E \exp\left\{\frac{n}{2}\left(\frac{v}{q_j}\right)^{\frac{1}{2}}(v, a') - i\left(\frac{n}{q_j}\right)^{\frac{1}{2}}(v, a')\right\} df(v)$$

for every $E \in \mathcal{B}(L^2_{a,b}[0,T])$. 
**Lemma 3.7.** Let $q_0$ be a nonzero real number and let $F$ be an element of $S(L^2_{a,b}[0,T])$ given by (2.9) whose associated measure $f$ satisfies the condition (2.18) with $q$ replaced with $q_0$. Then for all nonzero real number $q$ and for all $\alpha > 0$ with $|\alpha q| \geq |q_0|$, 

\[
\int_{C_{a,b}[0,T]}^{\infty} F(x) d\mu(x) = \int_{C_{a,b}[0,T]}^{\infty} F\left(\frac{x}{\sqrt{\alpha}}\right) d\mu(x).
\]  

\[
\text{Proof.} \text{ By using (2.15), we see that}
\]

\[
\int_{C_{a,b}[0,T]}^{\infty} F(x) d\mu(x)
\]

\[
= \int_{L^2_{a,b}[0,T]} \exp\left\{ -\frac{i(v^2, b')}{2\alpha q} + i\left(\frac{i}{\alpha q}\right)^{\frac{1}{2}} (v, a') \right\} df(v)
\]

\[
= \int_{L^2_{a,b}[0,T]} \exp\left\{ -\frac{i}{2q} (v', a') + i\left(\frac{i}{q}\right)^{\frac{1}{2}} (v, a') \right\} df(v)
\]

\[
= \int_{C_{a,b}[0,T]}^{\infty} F\left(\frac{x}{\sqrt{\alpha}}\right) d\mu(x).
\]  

The generalized analytic Feynman integral in equation (3.32) exists because 

\[
\int_{L^2_{a,b}[0,T]} \left| \exp\left\{ -\frac{i}{2\alpha q} (v^2, b') + i\left(\frac{i}{\alpha q}\right)^{\frac{1}{2}} (v, a') \right\} \right| df(v)
\]

\[
\leq \int_{L^2_{a,b}[0,T]} \left| 2\alpha q \right|^{-1/2} \left| \int_0^T |v(s) d|a(s) \right| df(v)
\]

\[
\leq \int_{L^2_{a,b}[0,T]} \left| 2q_0 \right|^{-1/2} \left| \int_0^T |v(s) d|a(s) \right| df(v) < \infty.
\]  

Hence we have the desired result. \(\square\)

**Theorem 3.8.** Let $q_0$ be a nonzero real number and let $F$ be an element of $S(L^2_{a,b}[0,T])$ given by (2.9) whose associated measure $f$ satisfies the condition (3.5). Let $\alpha, \beta > 0$ and let $q_1$ and $q_2$ be nonzero real
numbers with $|q_1|/\alpha^2 \geq |q_0|$, $|q_2|/\beta^2 \geq |q_0|$ and $\beta q_1 + \alpha^2 q_2 \neq 0$. Then

\begin{equation}
\int_{C_{a,b}[0,T]}^{anf_{q_2}} \left[ \int_{C_{a,b}[0,T]}^{anf_{q_1}} F(\alpha y + \beta z) d\mu(y) \right] d\mu(z)
= \int_{C_{a,b}[0,T]}^{anf_{q_1, q_2}} \frac{F_{q_1, \alpha^2, q_2/\beta^2}(x)}{\beta q_1 + \alpha^2 q_2} d\mu(x)
\end{equation}

where $F_{q_1, \alpha^2, q_2/\beta^2}$ is given by (3.37) below.

Also, both expressions in (3.34) are given by the expression

\begin{equation}
\int_{L_{a,b}^2[0,T]} \exp \left\{ -\frac{i}{2} \left( \frac{\alpha^2}{q_1} + \frac{\beta^2}{q_2} \right) \langle v^2, b' \rangle + i \left( \alpha \left( i \frac{1}{q_1} \right) + \beta \left( i \frac{1}{q_2} \right) \right) \langle v, a' \rangle \right\} df(v).
\end{equation}

**Proof.** By using (3.31) and (3.6), we see that

\begin{align}
\int_{C_{a,b}[0,T]}^{anf_{q_2}} \left[ \int_{C_{a,b}[0,T]}^{anf_{q_1}} F(\alpha y + \beta z) d\mu(y) \right] d\mu(z)
&= \int_{C_{a,b}[0,T]}^{anf_{q_2}} \left[ \int_{C_{a,b}[0,T]}^{anf_{q_1, \alpha^2}} F(y + \beta z) d\mu(y) \right] d\mu(z)
&= \int_{C_{a,b}[0,T]}^{anf_{q_1, \alpha^2}} \left[ \int_{C_{a,b}[0,T]}^{anf_{q_2}} F(y + \beta z) d\mu(z) \right] d\mu(y)
&= \int_{C_{a,b}[0,T]}^{anf_{q_1, \alpha^2}} \left[ \int_{C_{a,b}[0,T]}^{anf_{q_2, \beta^2}} F(y + z) d\mu(z) \right] d\mu(y)
&= \int_{C_{a,b}[0,T]}^{anf_{q_1, q_2}} \frac{F_{q_1, \alpha^2, q_2/\beta^2}(x)}{\beta q_1 + \alpha^2 q_2} d\mu(x)
\end{align}

where

\begin{equation}
F_{q_1, \alpha^2, q_2/\beta^2}(x) = \int_{L_{a,b}^2[0,T]} \exp \{ i \langle v, x \rangle \} df_{q_1, \alpha^2, q_2/\beta^2}(v)
\end{equation}

and

\begin{equation}
F_{q_1, \alpha^2, q_2/\beta^2}(E) = \int_{E} \exp \left\{ i \left( \alpha \left( i \frac{1}{q_1} \right) + \beta \left( i \frac{1}{q_2} \right) \right) \langle v, a' \rangle - i \left( i \left( \frac{\beta q_1 + \alpha^2 q_2}{q_1 q_2} \right) \right) \langle v, a' \rangle \right\} df(v)
\end{equation}
for every \( E \in \mathcal{B}(L_{a,b}^2[0,T]) \).

Moreover, we have that

\[
\| f_{q_1/\alpha^2,q_2/\beta^2} \| = \int_{L_{a,b}^2[0,T]} |df_{q_1/\alpha^2,q_2/\beta^2}(v)| \\
\leq \int_{L_{a,b}^2[0,T]} \exp \left\{ \left| \frac{2q_1q_2}{\beta^2q_1 + \alpha^2q_2} \right|^{-\frac{1}{2}} \int_0^T |v(s)|d|a|(s) \right\} \\
\cdot \exp \left\{ |2q_1/\alpha^2|^{-\frac{1}{2}} \int_0^T |v(s)|d|a|(s) \right\} \\
\cdot \exp \left\{ |2q_2/\beta^2|^{-\frac{1}{2}} \int_0^T |v(s)|d|a|(s) \right\} |df(v)| \\
\leq \int_{L_{a,b}^2[0,T]} \exp \left\{ 4|2q_0|^{-\frac{1}{2}} \int_0^T |v(s)|d|a|(s) \right\} |df(v)| < \infty.
\]

(3.39)

Hence \( f_{q_1/\alpha^2,q_2/\beta^2} \) is an element of \( M(L_{a,b}^2[0,T]) \) and so \( F_{q_1/\alpha^2,q_2/\beta^2} \) is in \( S(L_{a,b}^2[0,T]) \). Thus we have the desired results.

4. Generalized Fourier-Feynman transforms

In this section, we will establish a Fubini theorem for analytic GFFT for functional \( F \in S(L_{a,b}^2[0,T]) \). Then, as corollaries we will obtain several Feynman integration formulas involving analytic GFFT. For simplicity, we restrict our discussion to the case \( p = 1 \); however most of our results hold for all \( p \in [1,2] \).

We state the definition of the analytic GFFT [7, 9].

**Definition 4.1.** For \( \lambda \in \mathbb{C}_+ \) and \( y \in C_{a,b}[0,T] \), let

\[
(4.1) \quad T_\lambda(F)(y) = \int_{C_{a,b}[0,T]}^{a\lambda} F(y + x)d\mu(x).
\]

Then for \( q \in \mathbb{R} - \{0\} \), the \( L_1 \) analytic GFFT, \( T_{q}^{(1)}(F) \) of \( F \), is defined by the formula \( (\lambda \in \mathbb{C}_+) \)

\[
(4.2) \quad T_{q}^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y)
\]
for s.a.e. $y \in C_{a,b}[0,T]$ whenever the limit exists. That is to say,

$$
(4.3) \quad T_t^{(1)}(F)(y) = \int_{C_{a,b}[0,T]} F(y + x) d\mu(x)
$$

for s.a.e. $y \in C_{a,b}[0,T]$.

We note that if $T_t^{(1)}(F)$ exists and if $F \approx G$, then $T_t^{(1)}(G)$ exists and $T_t^{(1)}(F) \approx T_t^{(1)}(G)$.

**Theorem 4.2.** Let $q_0$ be a nonzero real number and let $F$ be an element of $S(L^2_{a,b}(0,T))$ given by (2.9) whose associated measure $f$ satisfies the condition (3.5). Let $r > 0$ and let $q_1$ and $q_2$ be nonzero real numbers with $|q_1| > |q_0|$, $|q_2| > |q_0|$ and $q_1 + q_2 \neq 0$. Then

$$
(4.4) \quad \int_{C_{a,b}[0,T]} T_{q_1}^{(1)}(F)(\sqrt{r}z) d\mu(z) = \int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]} F_{q_1,q_2}(x) d\mu(x)
$$

$$
= \int_{C_{a,b}[0,T]} T_{q_2}^{(1)}(F)(\sqrt{r}y) d\mu(y)
$$

where $F_{q_1,q_2}$ is given by (3.11).

**Proof.** Using equations (4.3) and (3.34) with $\alpha = 1$, $\beta = \sqrt{r}$, we obtain that

$$
(4.5) \quad \int_{C_{a,b}[0,T]} T_{q_1}^{(1)}(F)(\sqrt{r}z) d\mu(z)
$$

$$
= \int_{C_{a,b}[0,T]} \left[ \int_{C_{a,b}[0,T]} F(\sqrt{r}z + y) d\mu(y) \right] d\mu(z)
$$

$$
= \int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]} F_{q_1,q_2}(x) d\mu(x)
$$

$$
= \int_{C_{a,b}[0,T]} F_{q_1,q_2}(x) d\mu(x).
$$

By the same argument in equation (4.5) with $\alpha = \sqrt{r}$, $\beta = 1$, we have that

$$
(4.6) \quad \int_{C_{a,b}[0,T]} T_{q_2}^{(1)}(F)(\sqrt{r}y) d\mu(y) = \int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]} F_{q_1,q_2}(x) d\mu(x).
$$

Now equation (4.4) follows from equations (4.5) and (4.6). \qed
COROLLARY 4.3. Let \( q_0 \) and \( F \) be as in Theorem 4.2. Then for all nonzero real numbers \( q_1 \) and \( q_2 \) with \( |q_1| > |q_0|, |q_2| > |q_0| \) and \( q_1 + q_2 \neq 0 \),

\[
\int_{C_{a,b}[0,T]}^{anf_{q_2}} T_{q_1}(F)(z)d\mu(z) = \int_{C_{a,b}[0,T]}^{anf_{q_1}} T_{q_2}(F)(y)d\mu(y).
\]

COROLLARY 4.4. Let \( q_0 \) and let \( F \) be as in Theorem 4.2. Then for all nonzero real number \( q \) with \( |q| > |q_0| \),

\[
\int_{C_{a,b}[0,T]}^{anf_q} T_{q}^{(1)}(F)(y)d\mu(y) = \int_{C_{a,b}[0,T]}^{anf_{q/2}} F_{q,q}(x)d\mu(x)
\]

\[
= \int_{C_{a,b}[0,T]}^{anf_q} F_{q,q}(\sqrt{2}x)d\mu(x)
\]

where \( F_{q,q} \) is given by (3.15).

**Proof.** The first equality in (4.8) follows by letting \( r = 1 \) and \( q_1 = q_2 = q \) in equation (4.4). The second equality follows from Lemma 3.7. \( \Box \)

THEOREM 4.5. Let \( q_0, q_1, \cdots, q_n \), and let \( F \) be as in Theorem 3.4. Then for s-a.e. \( z \in C_{a,b}[0,T] \),

\[
T_{q_n}(T_{q_{n-1}}^{(1)}(\cdots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F))))\cdots))(z)
\]

\[
= \int_{C_{a,b}[0,T]}^{anf_{\alpha_n}} F_{q_1,\cdots,q_n}(z + x)d\mu(x)
\]

\[
= T_{\alpha_n}^{(1)}(F_{q_1,\cdots,q_n})(z)
\]

where \( F_{q_1,\cdots,q_n} \) is given by equation (3.21) and \( \alpha_n \) is as in Theorem 3.4.

**Proof.** Using equations (4.3) and (3.18), we obtain that

\[
T_{q_n}^{(1)}(T_{q_{n-1}}^{(1)}(\cdots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F))))\cdots))(z)
\]

\[
= \int_{C_{a,b}[0,T]}^{anf_{\alpha_n}} \cdots \int_{C_{a,b}[0,T]}^{anf_{q_1}} F(z + y_1 + \cdots + y_n)d\mu(y_1)\cdots d\mu(y_n)
\]

\[
= \int_{C_{a,b}[0,T]}^{anf_{\alpha_n}} F_{q_1,\cdots,q_n}(z + x)d\mu(x)
\]

\[
= T_{\alpha_n}^{(1)}(F_{q_1,\cdots,q_n})(z)
\]

for s-a.e. \( z \in C_{a,b}[0,T] \). \( \Box \)

Choosing \( q_j = q \) for \( j = 1, \cdots, n \), we obtain the following corollary to Theorem 4.5.
COROLLARY 4.6. Let \( q_0 \) and \( F \) be as in Theorem 4.5 and let \( q \) be a nonzero real number with \( |q| \geq |q_0| \). Then for s.a.e. \( z \in C_{a,b}[0,T] \),

\[
T_{q}^{(1)}(T_{q}^{(1)}(F))(z) = T_{q/2}^{(1)}(F_{q,q})(z) = \int_{C_{a,b}[0,T]}^{\text{anf}_q} F_{q,q}(z + \sqrt{2}x) d\mu(x),
\]

(4.11)

\[
T_{q}^{(1)}(T_{q}^{(1)}(T_{q}^{(1)}(F)))(z) = T_{q/3}^{(1)}(F_{q,q,q})(z) = \int_{C_{a,b}[0,T]}^{\text{anf}_q} F_{q,q,q}(z + \sqrt{3}x) d\mu(x),
\]

(4.12)

and in general,

\[
T_{q}^{(1)}(T_{q}^{(1)}(\cdots (T_{q}^{(1)}(F)) \cdots ))(z) = T_{q/n}^{(1)}(F_{q,...,q})(z) = \int_{C_{a,b}[0,T]}^{\text{anf}_q} F_{q,...,q}(z + \sqrt{n}x) d\mu(x).
\]

(4.13)

COROLLARY 4.7. Let \( q_0 \) and \( F \) be as in Theorem 4.5 and let \( q_1 \) and \( q_2 \) be nonzero real numbers with \( |q_1| \geq |q_0|, |q_2| \geq |q_0| \), and \( q_1 + q_2 \neq 0 \). Then for s.a.e. \( z \in C_{a,b}[0,T] \),

\[
T_{q_2}^{(1)}(T_{q_1}^{(1)}(F))(z) = T_{q_1 + q_2}^{(1)}(F_{q_1,q_2})(z) = T_{q_1}^{(1)}(T_{q_2}^{(1)}(F))(z)
\]

(4.14)

where \( F_{q_1,q_2} \) is given by (3.11).

References


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