SOME REMARKS ON $M$-IDEALS
AND STRONG PROXIMALITY

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Abstract. We prove that every $M$-ideal is strongly proximinal
and that, for any Banach space $X$, $K(X,c_0)$ is an $M$-ideal in
$L(X,\ell^\infty)$.

1. Introduction

Many papers ([2], [10]-[14], [16], ...) are concerned with the phe-
nomenon that for certain Banach spaces $X$ and $Y$, the Banach space
$K(X,Y)$ of compact linear operators from $X$ to $Y$ is an $M$-ideal in
$L(X,Y)$, the Banach space of bounded linear operators from $X$ to $Y$.
This special attention is due to the facts that, for example, when this
happens:

- every element of $L(X,Y)$ has a best compact approximant (cf.
Theorem A and Theorem B),
- one has the uniqueness of Hahn-Banach extensions from $K(X,Y)$
to $L(X,Y)$ ([14]).

This article extends some results of [2]. The authors proved that if $X$
is a Banach space with an unconditional shrinking basis then $K(X,c_0)$
is proximinal in $L(X,\ell^\infty)$. Here, we prove that every $M$-ideal is strongly
proximinal. We also show that $K(X,c_0)$ is an $M$-ideal in $L(X,\ell^\infty)$.
This implies that, without any assumption on $X$, $K(X,c_0)$ is strongly
proximinal in $L(X,\ell^\infty)$.

Let us recall the notions mentioned above.

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got.
Let \( J \) be a closed subspace of a Banach space \( X \). For \( x \in X \), let

\[
P_J(x) := \{ j \in J : \| x - j \| = d(x, J) \}
\]

where \( d(x, J) := \inf \{ \| x - j \| : j \in J \} \). The subspace \( J \) is said to be \textit{proximinal} in \( X \), if for each \( x \) in \( X \), the set \( P_J(x) \) is non-empty. An element of \( P_J(x) \) is called a \textit{best approximant} of \( x \) in \( J \). Every closed subspace of a finite dimensional space or of a uniformly convex space (and then of a reflexive space) is clearly proximinal. It is known that a Banach space \( X \) is reflexive if and only if every closed hyperplane is proximinal in \( X \). If \( J \) is a proximinal subspace of finite codimension in \( X \) then \( J^\perp := \{ x^* \in X^* : x^* |_J \equiv 0 \} \) is contained in \( NA(X) \), the subset of \( X^* \) consisting of all norm attaining functionals on \( X \) ([5]). The converse fails in general (see [15]) but it is true for \( c_0 \) and its subspaces ([6]).

When each “nearly best approximant” of \( x \) in \( J \) is necessarily close to an actual best approximant one says that \( J \) is \textit{strongly proximinal} in \( X \). More precisely, for \( \delta > 0 \) let \( P_J(x, \delta) := \{ j \in J : \| x - j \| < d(x, J) + \delta \} \), then \( J \) is strongly proximinal in \( X \) if and only if for each \( x \) in \( X \) and each \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( d(j, P_J(x)) < \epsilon \) for all \( j \in P_J(x, \delta) \). Every strongly proximinal subspace is proximinal. It is easy to prove that every closed subspace of a finite dimensional space is strongly proximinal. But in general proximinality does not imply strong proximinality. For example, every proximinal hyperplane is strongly proximinal if and only if \( NA_1(X) := NA(X) \cap S(X^*) \) coincides with the set \( S := \{ x^* \in S(X^*) : \| \cdot \|_{X^*} \text{ is SSD at } x^* \} \) ([7]). A norm \( \| \cdot \| \) on \( X \) is said to be \textit{strongly subdifferentiable} (SSD) at \( x \) if the one-side limit \( \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( \| x + \epsilon h \| - \| x \| \right) \) exists uniformly in \( h \in S(X) \). In [4], it is shown that \( S \subseteq NA_1(X) \). Then, in a reflexive infinite dimensional space with a dual norm which is not everywhere SSD, there exists a hyperplane which is not strongly proximinal. In \( \ell_1 \), there exist non strongly proximinal hyperplanes since the canonical sup-norm on \( \ell_\infty \) is not everywhere SSD (see [3]). In [8], the authors proved that a finite codimensional subspace \( J \) of \( K(\ell^2) \) is strongly proximinal if and only if \( J^\perp \) is contained in \( NA(K(\ell^2)) \).

A closed subspace \( J \) in \( X \) is said to be an \textit{\( M \)-ideal} in \( X \) if and only if there exists a linear projection \( P \) from \( X^* \) to \( J^\perp \) such that:

\[
\forall x^* \in X^* : \| x^* \| = \| Px^* \| + \| x^* - Px^* \|.
\]

This notion is due to Alfsen and Effros [1] and studied in detail in [9].

Examples of \( M \)-ideals are [9] : \( c_0 \) in its bidual \( \ell^\infty \), \( K(H) \) in \( L(H) \) where \( H \) is an Hilbert space, \( K(X, c_0) \) in \( L(X, c_0) \) for every Banach
space $X$, or $K(\ell^p, \ell^q)$ in $L(\ell^p, \ell^q)$ for $1 < p, q < +\infty$, while it is not true for $K(X)$ in $L(X)$ if $X = \ell^\infty$ or $X = L^p(0, 1)$, $p \in [1, +\infty) \setminus \{2\}$.

In [12], it is shown that $K(L^p, \ell^p)$ is an $M$-ideal in $L(L^p, \ell^p)$ if $1 < p \leq 2$ and it is not true if $p > 2$. In [10], it is proved for $2 \leq p < +\infty$ and subspaces $X$ of quotients of $L^p$ with a $1$-unconditional finite dimensional Schauder decomposition that $K(X, \ell^p)$ is an $M$-ideal in $L(X, \ell^p)$.

In [1], the authors also gave the following equivalent condition for $J$ to be an $M$-ideal in $X$ which avoids mentioning the dual space $X^*$.

**Theorem A.** For a closed subspace $J$ of a Banach space $X$, the following assertions are equivalent:

1. $J$ is an $M$-ideal,
2. The $n$-ball property.
   For all $n \in \mathbb{N}$ and all families $(B(x_i, r_i))_{i=1}^n$ of $n$ closed balls satisfying $B(x_i, r_i) \cap J \neq \emptyset$ for all $i = 1, \ldots, n$ and $\bigcap_{i=1}^n B(x_i, r_i) \neq \emptyset$ the conclusion $\bigcap_{i=1}^n B(x_i, r_i + \epsilon) \cap J \neq \emptyset$ for all $\epsilon > 0$ holds.
3. The [restricted] $3$-ball property.
   For all $j_1, j_2, j_3 \in B(J)$, all $x \in B(X)$ and all $\epsilon > 0$ there is $j \in J$ satisfying $\|x + j_i - j\| \leq 1 + \epsilon$ for $i = 1, 2, 3$.

Following [16], we say that a closed subspace $J$ of a Banach space $X$ has the $1\frac{1}{2}$-ball property in $X$ if the conditions

$x \in X, j \in J, \quad \|x - j\| \leq r_1 + r_2 \quad \text{and} \quad B(x, r_2) \cap J \neq \emptyset$

implies that

$B(j, r_1) \cap B(x, r_2) \cap J \neq \emptyset.$

This is equivalent to requiring the (strict) $2$-ball property subject to the restriction that one of the centers lies in $J$:

if $x \in X, j \in J, \quad B(j, r_1) \cap B(x, r_2) \neq \emptyset \quad \text{and} \quad B(x, r_2) \cap J \neq \emptyset$

then $B(j, r_1) \cap B(x, r_2) \cap J \neq \emptyset.$

Let us note that the $1\frac{1}{2}$-ball property is not a sufficient condition to be an $M$-ideal: $K(\ell^1)$ has the $1\frac{1}{2}$-ball property in $L(\ell^1)$ but it is not an $M$-ideal (see [9], [16]).

**Theorem B.** [16] Let $J$ be a closed subspace of a Banach space $X$. If $J$ has the $1\frac{1}{2}$-ball property in $X$ then $J$ is proximinal in $X$.

According to Theorem A and Theorem B, we get

**Corollary.** Every $M$-ideal is proximinal.
2. Results

As mentioned before, the $1^\frac{1}{2}$-ball property is a sufficient condition for proximinality. Here we prove that this property implies strong proximinality.

**Theorem 1.** Let $J$ be a closed subspace of a Banach space $X$. If $J$ has the $1^\frac{1}{2}$-ball property in $X$ then $J$ is strongly proximinal in $X$.

**Proof.** The proof follows the ideas of the proof of Theorem B in [16]. Let $x \in X$ be such that $d := d(x, J) > 0$ and $\epsilon > 0$ be fixed. We want to prove that there exists $\delta > 0$ such that:

$$\forall j \in J : \|x - j\| < d + \delta \Rightarrow [\exists j' \in J : \|j - j'\| < \epsilon \text{ and } \|x - j'\| = d].$$

Let us take $\delta = \epsilon$ and let $j_1 \in J$ be such that $\|x - j_1\| < d + \delta = (d + \frac{\delta}{2}) + \frac{\delta}{2}$. By definition of $d$, $B(x, d + \frac{\delta}{2}) \cap J \neq \emptyset$ and then by the $1^\frac{1}{2}$-ball property, we have

$$\exists j_2 \in J : j_2 \in B\left(j_1, \frac{\delta}{2}\right) \cap B\left(x, d + \frac{\delta}{2}\right).$$

We have now $\|x - j_2\| \leq (d + \frac{\delta}{2}) = (d + \frac{\delta}{4}) + \frac{\delta}{4}$ and by the $1^\frac{1}{2}$-ball property, we have:

$$\exists j_3 \in J : j_3 \in B\left(j_2, \frac{\delta}{4}\right) \cap B\left(x, d + \frac{\delta}{4}\right).$$

So, inductively, we construct a sequence $(j_n)_{n \geq 1} \subset J$ such that: $\forall n \geq 1$,

(1) $\|x - j_n\| \leq d + \frac{\delta}{2^{n-1}}$

(2) $\|j_n - j_{n+1}\| \leq \frac{\delta}{2^n}$.

By (2), $(j_n)_{n \geq 1}$ is a Cauchy sequence in $J$. Since $J$ is closed, there exists $j' \in J$ such that $j' = \lim_{n \to +\infty} j_n$. By (1), we have $\|x - j'\| = d$. By (2) again,

$$\forall n \geq 1 : \|j_1 - j_n\| \leq \|j_1 - j_2\| + \|j_2 - j_3\| + \cdots + \|j_{n-1} - j_n\| \leq \delta \sum_{n=1}^{+\infty} \frac{1}{2^n} = \delta.$$

Then, $\|j_1 - j'\| \leq \delta = \epsilon$. \qed

By Theorem A and Theorem 1, we have the following

**Corollary 2.** Every $M$-ideal is strongly proximinal.
Following ideas described in [9], we prove

**Proposition 3.** For every Banach space $X$, $K(X,c_0)$ is an $M$-ideal in $L(X,\ell^\infty)$.

**Proof.** By Theorem A, it suffices to prove that $K(X,c_0)$ satisfies the 3-ball property in $L(X,\ell^\infty)$. Let $S \in B(L(X,\ell^\infty))$, $T_i \in B(K(X,c_0))$ $(i = 1, 2, 3)$ and $\varepsilon > 0$. We want to find $T \in K(X,c_0)$ such that:

$$\|S + T_i - T\| \leq 1 + \varepsilon \quad \text{for } i = 1, 2, 3.$$

For $a = (a_j)_{j \geq 1} \in \ell^\infty$, let us put: $P_n(a) = (a_1, a_2, \ldots, a_n, 0, \ldots)$. Since $T_i \in K(X,c_0)$ $(i = 1, 2, 3)$, we have:

$$\|P_n T_i - T_i\| \leq \varepsilon \quad \text{for } i = 1, 2, 3 \text{ and } n \text{ big enough.}$$

Moreover,

$$\|P_n T_i + (Id_{\ell^\infty} - P_n)S\| \leq 1 \quad (i = 1, 2, 3),$$

since $\|P_n T_i + (Id_{\ell^\infty} - P_n)S\| \leq \max \{\|P_n T_i\|, \|(Id_{\ell^\infty} - P_n)S\|\}$.

If we take $T = P_n S$ (for $n$ big enough), then for $i = 1, 2, 3$:

$$\|T_i + S - P_n S\| \leq \|T_i - P_n T_i\| + \|P_n T_i + (Id_{\ell^\infty} - P_n)S\| \leq \varepsilon + 1.$$

□

By Corollary 2, we then have

**Corollary 4.** For every Banach space $X$, $K(X,c_0)$ is strongly proximinal in $L(X,\ell^\infty)$.

**Remark 5.** It is clear from the proof of Proposition 3 that if $(K_\alpha)$ is a net of compact operators on $Y$ satisfying:

$$K_\alpha \longrightarrow Id_Y \quad \text{in norm and}$$

$$\limsup_\alpha \|K_\alpha U + (Id_Y - K_\alpha)W\| \leq \max \{\|U\|, \|W\|\}$$

for all $U, W$ in $L(X,Y)$

then $K(X,Y)$ is an $M$-ideal in $L(X,Y)$.

**References**


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