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SOME REMARKS ON M-IDEALS AND STRONG PROXIMINALITY

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Abstract. We prove that every M-ideal is strongly proximinal and that, for any Banach space X, $K(X, c_0)$ is an M-ideal in $L(X, \ell^\infty)$.

1. Introduction

Many papers ([2], [10]-[14], [16], ...) are concerned with the phenomenon that for certain Banach spaces X and Y, the Banach space $K(X,Y)$ of compact linear operators from X to Y is an M-ideal in $L(X,Y)$, the Banach space of bounded linear operators from X to Y. This special attention is due to the facts that, for example, when this happens :

- every element of $L(X,Y)$ has a best compact approximant (cf. Theorem A and Theorem B),
- one has the uniqueness of Hahn-Banach extensions from $K(X,Y)$ to $L(X,Y)$ ([14]).

This article extends some results of [2]. The authors proved that if X is a Banach space with an unconditional shrinking basis then $K(X, c_0)$ is proximinal in $L(X, \ell^\infty)$. Here, we prove that every M-ideal is strongly proximinal. We also show that $K(X, c_0)$ is an M-ideal in $L(X, \ell^\infty)$. This implies that, without any assumption on X, $K(X, c_0)$ is strongly proximinal in $L(X, \ell^\infty)$.

Let us recall the notions mentioned above.

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Let J be a closed subspace of a Banach space X. For $x \in X$, let

$$P_J(x) := \{j \in J : \|x - j\| = d(x, J)\}$$

where $d(x, J) := \inf\{\|x - j\| : j \in J\}$. The subspace J is said to be *proximinal* in X, if for each x in X, the set $P_J(x)$ is non-empty. An element of $P_J(x)$ is called a *best approximant* of x in J. Every closed subspace of a finite dimensional space or of a uniformly convex space (and then of a reflexive space) is clearly proximinal. It is known that a Banach space X is reflexive if and only if every closed hyperplane is proximinal in X. If J is a proximinal subspace of finite codimension in X then $J^\perp := \{x^\star \in X^\star : x^\star_{\restriction J} \equiv 0\}$ is contained in $NA(X)$, the subset of $X^\star$ consisting of all norm attaining functionals on X ([5]). The converse fails in general (see [15]) but it is true for c_0 and its subspaces ([6]).

When each "nearly best approximant" of x in J is necessarily close to an actual best approximant one says that J is *strongly proximinal* in X. More precisely, for $\delta > 0$ let $P_J(x, \delta) := \{j \in J : \|x - j\| < d(x, J) + \delta\}$, then J is strongly proximinal in X if and only if for each x in X and each $\epsilon > 0$ there is $\delta > 0$ such that $d(j, P_J(x)) < \epsilon$ for all $j \in P_J(x, \delta)$. Every strongly proximinal subspace is proximinal. It is easy to prove that every closed subspace of a finite dimensional space is strongly proximinal. But in general proximinality does not imply strong proximinality. For example, every proximinal hyperplane is strongly proximinal if and only if $NA_1(X) := NA(X) \cap S(X^*)$ coincides with the set $S := \{x^* \in S(X^*) : \|\cdot\|_{X^*}$ is *SSD* at $x^*\}$ ([7]). A norm $\|\cdot\|$ on X is said to be *strongly subdifferentiable* (*SSD*) at x if the one-side limit $\lim_{t\to 0^+} \frac{1}{t}\left(\|x + th\| - \|x\|\right)$ exists uniformly in $h \in S(X)$. In [4], it is shown that $S \subseteq NA_1(X)$. Then, in a reflexive infinite dimensional space with a dual norm which is not everywhere SSD, there exists a hyperplane which is not strongly proximinal. In ℓ_1, there exist non strongly proximinal hyperplanes since the canonical sup-norm on ℓ_∞ is not everywhere SSD (see [3]). In [8], the authors proved that a finite codimensional subspace J of $K(\ell^2)$ is strongly proximinal if and only if $J^\perp$ is contained in $NA(K(\ell^2))$.

A closed subspace J in X is said to be an *M-ideal* in X if and only if there exists a linear projection P from $X^\star$ to $J^\perp$ such that :

$$\forall x^\star \in X^\star : \|x^\star\| = \|Px^\star\| + \|x^\star - Px^\star\|.$$

This notion is due to Alfsen and Effros [1] and studied in detail in [9].

Examples of M-ideals are [9] : c_0 in its bidual ℓ^∞, $K(H)$ in $L(H)$ where H is an Hilbert space, $K(X, c_0)$ in $L(X, c_0)$ for every Banach space X, or $K(\ell^p, \ell^q)$ in $L(\ell^p, \ell^q)$ for $1 < p, q < +\infty$, while it is not true for $K(X)$ in $L(X)$ if $X = \ell^\infty$ or $X = L^p(0,1)$, $p \in [1, +\infty] \setminus \{2\}$.

In [12], it is shown that $K(L^p, \ell^p)$ is an M-ideal in $L(L^p, \ell^p)$ if $1 < p \leq 2$ and it is not true if $p > 2$. In [10], it is proved for $2 \leq p < +\infty$ and subspaces X of quotients of L^p with a 1-unconditional finite dimensional Schauder decomposition that $K(X, \ell^p)$ is an M-ideal in $L(X, \ell^p)$.

In [1], the authors also gave the following equivalent condition for J to be an M-ideal in X which avoids mentioning the dual space $X^\star$.

THEOREM A. *For a closed subspace J of a Banach space X, the following assertions are equivalent :*

1. *J is an M-ideal,*
2. *The n-ball property.*
   For all $n \in \mathbb{N}$ and all families $(B(x_i, r_i))_{i=1}^n$ of n closed balls satisfying $B(x_i, r_i) \cap J \neq \emptyset$ for all $i = 1, \ldots, n$ and $\bigcap_{i=1}^n B(x_i, r_i) \neq \emptyset$ the conclusion $\bigcap_{i=1}^n B(x_i, r_i + \epsilon) \cap J \neq \emptyset$ for all $\epsilon > 0$ holds.
3. *The [restricted] 3-ball property.*
   For all $j_1, j_2, j_3 \in B(J)$, all $x \in B(X)$ and all $\epsilon > 0$ there is $j \in J$ satisfying : $\|x + j_i - j\| \leq 1 + \epsilon$ for $i = 1, 2, 3$.

Following [16], we say that a closed subspace J of a Banach space X has the $1\frac{1}{2}$-*ball property in X* if the conditions

$$x \in X, j \in J, \quad \|x - j\| \leq r_1 + r_2 \quad \text{and} \quad B(x, r_2) \cap J \neq \emptyset$$

implies that

$$B(j, r_1) \cap B(x, r_2) \cap J \neq \emptyset.$$

This is equivalent to requiring the (*strict*) *2-ball property* subject to the restriction that one of the centers lies in J:

$$\text{if} \quad x \in X, j \in J, \quad B(j, r_1) \cap B(x, r_2) \neq \emptyset \quad \text{and} \quad B(x, r_2) \cap J \neq \emptyset$$
$$\text{then} \quad B(j, r_1) \cap B(x, r_2) \cap J \neq \emptyset.$$

Let us note that the $1\frac{1}{2}$-ball property is not a sufficient condition to be an M-ideal : $K(\ell^1)$ has the $1\frac{1}{2}$-ball property in $L(\ell^1)$ but it is not an M-ideal (see [9], [16]).

THEOREM B. [16] *Let J be a closed subspace of a Banach space X. If J has the $1\frac{1}{2}$-ball property in X then J is proximinal in X.*

According to Theorem A and Theorem B, we get

COROLLARY. *Every M-ideal is proximinal.*

2. Results

As mentioned before, the $1\frac{1}{2}$-ball property is a sufficient condition for proximinality. Here we prove that this property implies strong proximinality.

THEOREM 1. *Let J be a closed subspace of a Banach space X. If J has the $1\frac{1}{2}$-ball property in X then J is strongly proximinal in X.*

Proof. The proof follows the ideas of the proof of Theorem B in [16]. Let $x \in X$ be such that $d := d(x, J) > 0$ and $\epsilon > 0$ be fixed. We want to prove that there exists $\delta > 0$ such that :

$$[\forall j \in J : \|x - j\| < d + \delta] \Rightarrow \left[\exists j' \in J : \|j - j'\| < \epsilon \,\text{and}\, \|x - j'\| = d\right].$$

Let us take $\delta = \epsilon$ and let $j_1 \in J$ be such that $\|x - j_1\| < d + \delta = (d + \frac{\delta}{2}) + \frac{\delta}{2}$. By definition of d, $B(x, d + \frac{\delta}{2}) \cap J \neq \emptyset$ and then by the $1\frac{1}{2}$-ball property, we have

$$\exists j_2 \in J : j_2 \in B\left(j_1, \frac{\delta}{2}\right) \cap B\left(x, d + \frac{\delta}{2}\right).$$

We have now $\|x - j_2\| \leq (d + \frac{\delta}{2}) = (d + \frac{\delta}{4}) + \frac{\delta}{4}$ and by the $1\frac{1}{2}$-ball property, we have :

$$\exists j_3 \in J : j_3 \in B\left(j_2, \frac{\delta}{4}\right) \cap B\left(x, d + \frac{\delta}{4}\right).$$

So, inductively, we construct a sequence $(j_n)_{n \geq 1} \subset J$ such that : $\forall n \geq 1$,

$$\|x - j_n\| \leq d + \frac{\delta}{2^{n-1}} \tag{1}$$

$$\|j_n - j_{n+1}\| \leq \frac{\delta}{2^n}. \tag{2}$$

By (2), $(j_n)_{n \geq 1}$ is a Cauchy sequence in J. Since J is closed, there exists $j' \in J$ such that $j' = \lim_{n \to +\infty} j_n$. By (1), we have $\|x - j'\| = d$. By (2) again,

$$\begin{aligned} \forall n \geq 1 \quad : \quad \|j_1 - j_n\| &\leq \|j_1 - j_2\| + \|j_2 - j_3\| + \cdots + \|j_{n-1} - j_n\| \\ &\leq \delta \sum_{n=1}^{+\infty} \frac{1}{2^n} = \delta. \end{aligned}$$

Then, $\|j_1 - j'\| \leq \delta = \epsilon$. □

By Theorem A and Theorem 1, we have the following

COROLLARY 2. *Every M-ideal is strongly proximinal.*

Following ideas described in [9], we prove

PROPOSITION 3. *For every Banach space X, $K(X,c_0)$ is an M-ideal in $L(X,\ell^\infty)$.*

Proof. By Theorem A, it suffices to prove that $K(X,c_0)$ satisfies the 3-ball property in $L(X,\ell^\infty)$. Let $S\in B(L(X,\ell^\infty))$, $T_i\in B(K(X,c_0))$ $(i=1,2,3)$ and $\epsilon>0$. We want to find $T\in K(X,c_0)$ such that :

$$\|S+T_i-T\|\le 1+\epsilon \quad \text{for } i=1,2,3.$$

For $a=(a_j)_{j\ge1}\in\ell^\infty$, let us put : $P_n(a)=(a_1,a_2,\dots,a_n,0\dots)$. Since $T_i\in K(X,c_0)$ $(i=1,2,3)$, we have :

$$\|P_nT_i-T_i\|<\epsilon \quad \text{for } i=1,2,3 \text{ and } n \text{ big enough.}$$

Moreover,

$$\|P_nT_i+(Id_{\ell^\infty}-P_n)S\|\le 1 \quad (i=1,2,3),$$

since $\|P_nT_i+(Id_{\ell^\infty}-P_n)S\|\le\max\{\|P_nT_i\|,\|(Id_{\ell^\infty}-P_n)S\|\}$.

If we take $T=P_nS$ (for n big enough), then for $i=1,2,3$:

$$\begin{aligned}\|T_i+S-P_nS\| &\le \|T_i-P_nT_i\|+\|P_nT_i+(Id_{\ell^\infty}-P_n)S\|\\ &\le \epsilon+1.\end{aligned}$$

□

By Corollary 2, we then have

COROLLARY 4. *For every Banach space X, $K(X,c_0)$ is strongly proximinal in $L(X,\ell^\infty)$.*

REMARK 5. It is clear from the proof of Proposition 3 that if (K_α) is a net of compact operators on Y satisfying :

$$K_\alpha\longrightarrow Id_Y \quad \text{in norm and}$$
$$\limsup_\alpha\|K_\alpha U+(Id_Y-K_\alpha)W\|\le\max\{\|U\|,\|W\|\}$$
$$\text{for all } U,W \text{ in } L(X,Y)$$

then $K(X,Y)$ is an M-ideal in $L(X,Y)$.

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