GEOMETRIC PROBABILITY
IN THE MINKOWSKI PLANE

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ABSTRACT. In this paper, we get the measure of strips and lines in the Minkowski plane $M^2$ that meet a fixed compact convex body in $M^2$. From this we also investigate the probability that strips and lines meet a fixed compact convex body in $M^2$.

1. Introduction

In [2], G. D. Chakerian computed the measure of lines that meet a fixed curve, compact convex body in the Minkowski plane. The study on integral geometry was initiated by W. Blaschke. Many problems treated in integral geometry had their roots in geometric probability ([4]). Now many problems in integral geometry for the Minkowski spaces are open. In this paper, we have some integral formulas for the strip in the Minkowski plane. We also get the measure of strips and lines in the Minkowski plane $M^2$ that meet a fixed compact convex body in $M^2$. From this we get the probability that strips and lines meet a fixed compact convex body in $M^2$.

2. Basic concepts for the Minkowski plane

The Euclidean plane $E^2$ can be viewed as a normed space with an arbitrary norm $\| \cdot \|$. Then the norm $\| \cdot \|$ makes $E^2$ a new metric space $(M^2, \| \cdot \|)$ called the Minkowski plane. Let $K = \{ x \in M^2 : \| x \| \leq 1 \}$ be a unit ball in $M^2$. And we denote the boundary of $K$ by $bdK$. We shall assume throughout that $bdK$ is sufficiently differentiable and has positive finite curvature everywhere. Then $bdK$ is a centrally symmetric
closed convex curve. Assume that $bdK$ has enclosing Euclidean area $\pi$. Now we parametrize $bdK$ as follows:

We write the equation of $bdK$ as $t = t(\phi), 0 \leq \phi \leq 2\pi$. Then $\|\overrightarrow{t}\| = 1$, where $\overrightarrow{t}$ is the vector from the origin $O$ to the point corresponding to $t(\phi)$ on $bdK$. And the sectorial area from the positive $x$-axis to $\overrightarrow{t}$ is $\frac{\phi}{2}([1])$.

From now on we do not distinguish $\overrightarrow{t}$ from $t$. Now we define $n(\phi)$ by $n(\phi) = \frac{dt(\phi)}{d\phi}, 0 \leq \phi \leq 2\pi$. We denote by $bdK^{*}$ the curve with equation $n = n(\phi), 0 \leq \phi \leq 2\pi$. Then $bdK^{*}$ is the polar dual of $bdK$ with respect to the Euclidean unit circle, rotated through $\frac{\pi}{2}$. The function $\lambda = \lambda(\phi), 0 \leq \phi \leq 2\pi$, defined by $d\lambda(\phi) = -\frac{t(\phi)}{\lambda(\phi)}$ is called the Minkowskian curvature of $bdK^{*}$ at a point where the tangent has direction $t(\phi)([2])$. For $(x, y) = (x_{1}, x_{2}), (y_{1}, y_{2})$ in $M^{2}$ we denote $[x, y] = x_{1}y_{2} - x_{2}y_{1}$. Put

$$A(p, t(\phi)) = (0, \infty) \times [0, 2\pi) \cup 0 \times [0, \pi)$$

and let $L(M^{2})$ be the set of all straight lines on $M^{2}$. Then there is a one-to-one correspondence $T$ from $L(M^{2})$ onto $A(p, t(\phi))$ as follows:

If $G \in L(M^{2})$ is a line parallel to the direction $t(\phi)$ and the Euclidean area of the parallelogram with the sides $t(\phi)$ and $x \in G$ is $p$, then $T(G) = (p, t(\phi))$.

Here the equation of $G$ is:

$$[t(\phi), x] = p.$$ 

Then the density of $G$ (see [2]) is

$$dG = \lambda^{-1}(\phi)d\phi.$$ 

3. Main results

By a strip $B \subset M^{2}$ we mean the closed region between two parallel straight lines. The position of $B$ can be determined by the coordinates $p$ and $t(\phi)$ of its midparallel. So the density for sets of strips is

$$dB = \lambda^{-1}(\phi)d\phi.$$ 

For a compact convex body $U$ with the origin in its interior, the Minkowski length $L(bdU)$ of $bdU$ is defined by

$$L(bdU) = \int_{-\pi}^{\pi} \left( h_{U}(\theta) + h_{U}''(\theta) \right) \rho(\theta + \frac{\pi}{2} \epsilon)d\theta,$$
where \( \epsilon = \pm 1 \), \( h_U(\cdot) \) is a support function of \( U \), and \( \rho(\cdot) \) is a support function of \( K^*([3]) \). From now on we denote the Minkowski length by \( L(\cdot) \). We have the following theorem.

**Theorem 1.** For a compact convex body \( U \subset M^2 \) and a strip \( B \) of breadth \( Br \), we have

\[
\int_{B \cap U \neq \emptyset} dB = L(bdU) + \frac{1}{2} BrP(K^*),
\]

where \( P(\cdot) \) is the Euclidean perimeter.

**Proof.** \( B \) meets \( U \) if and only if the midparallel of \( B \) meets the parallel body \( U_{1/2} Br \) of \( U \) at distance \( 1/2 Br \). Let \( G \) be a midparallel of \( B \). For the equality \( \int_{G \cap U_{1/2} Br \neq \emptyset} dG = L(bdU_{1/2} Br) \), let \( x(s) \) be the equation of \( bdU_{1/2} Br \) with Minkowski arc length \( s \). Now we can assume that \( x \) is on the intersection of \( bdU_{1/2} Br \) with \( G \) of the equation \([t(\phi), x] = p \). Let \( \overline{u} = \frac{du}{ds} \) be the Minkowski unit tangent vector at \( x(s) \). Then

\[
(1) \quad \int_{G \cap U_{1/2} Br \neq \emptyset} dG = \frac{1}{2} \int_{bdU_{1/2} Br} \int_0^\pi \|[t(\phi), \overline{u}]||\lambda^{-1}(\phi) d\phi ds
\]

\[
= \int_{bdU_{1/2} Br} ds = L(bdU_{1/2} Br).
\]

In the equation (1) we can assume that \( \overline{u} \) moves from the direction \( t(0) \) to the direction \( t(\pi) \) without loss of generality. Thus we get

\[
\int_{B \cap U \neq \emptyset} dB = \int_{G \cap U_{1/2} Br \neq \emptyset} dG
\]

\[
= L(bdU_{1/2} Br)
\]

\[
= \int_{-\pi}^\pi \left( h_{U_{1/2} Br}(\theta) + h_{U_{1/2} Br}(\theta) \right) \rho(\theta + \frac{\pi}{2} \epsilon) d\theta
\]

\[
= \int_{-\pi}^\pi \left( \frac{1}{2} Br + h_U(\theta) + h_U(\theta) \right) \rho(\theta + \frac{\pi}{2} \epsilon) d\theta
\]

\[
= L(bdU) + \frac{1}{2} BrP(K^*).
\]

This completes the proof. \( \square \)

From the Theorem 1 we have immediately the following corollary.
\textbf{Corollary 1.} For a fixed point \( x \in M^2 \) and a strip \( B \) of breadth \( Br \), we have
\[
\int_{x \in B} dB = \frac{1}{2} BrP(K^*).
\]

Now we compute the integral of some geometric quantity on the set of strips that meet a fixed compact convex body in \( M^2 \). First, we have the following theorem.

\textbf{Theorem 2.} Let \( U \) be a compact convex body in \( M^2 \) and let \( B \) be a strip of breadth \( Br \). Then
\[
\int_{B \cap U \neq \emptyset} A(B \cap U)dB = \frac{1}{2} BrP(K^*)A(U),
\]
where \( A(\cdot) \) is the area.

\textit{Proof.} Let \( G_1 \) and \( G_2 \) be any lines in \( M^2 \). Now we consider the integral
\[
\int_{G_1 \cap G_2 \cap B \cap U \neq \emptyset} dG_1dG_2dB.
\]
Fixing the lines \( G_1 \) and \( G_2 \) and then integrating over the strips \( B \), gives
\[
\int_{G_1 \cap G_2 \cap B \cap U \neq \emptyset} dG_1dG_2dB = \int_{G_1 \cap U \neq \emptyset} \int_{G_2 \cap (G_1 \cap U) \neq \emptyset} \int_{B \cap (G_1 \cap G_2 \cap U) \neq \emptyset} dBdG_2dG_1 = \frac{1}{2} BrP(K^*) \int_{G_1 \cap U \neq \emptyset} \int_{G_2 \cap (G_1 \cap U) \neq \emptyset} dG_2dG_1
\]
\[
= BrP(K^*) \int_{G_1 \cap U \neq \emptyset} L(G_1 \cap U) dG_1 = BrP(K^*) A(K^*) A(U).
\]
The second equality follows from Corollary 1 immediately. On the other hand, first fixing the strip \( B \) and then integrating over the lines \( G_1 \) and
$G_2$ gives

\[
\int_{G_1 \cap G_2 \cap B \cap U \neq \emptyset} dG_1 dG_2 dB
\]

\[
= \int_{B \cap U \neq \emptyset} \int_{G_1 \cap (B \cap U) \neq \emptyset} \int_{G_2 \cap (G_1 \cap B \cap U) \neq \emptyset} dG_2 dG_1 dB
\]

\[
= 2 \int_{B \cap U \neq \emptyset} \int_{G_1 \cap (B \cap U)} L(G_1 \cap B \cap U) dG_1 dB
\]

\[
= 2A(K^*) \int_{B \cap U \neq \emptyset} A(B \cap U) dB.
\]

From this we have the result. \( \Box \)

Now we investigate the measure of the set of lines and strips that meet a fixed compact convex body \( U \).

**Theorem 3.** If \( B_1 \) and \( B_2 \) are independent strips in \( M^2 \) of breadth \( Br_1, Br_2 \), respectively, that meet a fixed compact convex body \( U \) and the line \( G \in \mathcal{L}(M^2) \) meets \( U \), then the probability that \( B_1 \cap B_2 \cap G \cap U \neq \emptyset \) is

\[
Prob = \frac{P(K^*)A(K^*)A(U)[Br_1 + Br_2] + \frac{1}{4} P^2(K^*)L(bdU)Br_1 Br_2}{L(bdU)[L(bdU) + \frac{1}{2} Br_1 P(K^*)][L(bdU) + \frac{1}{2} Br_2 P(K^*)]}.
\]

**Proof.** We compute the measure of \( B_1, B_2, \) and \( G \) such that \( B_1 \cap B_2 \cap G \cap U \neq \emptyset \).

\[
\int_{B_1 \cap B_2 \cap G \cap U \neq \emptyset} dGdB_1 dB_2
\]

\[
= \int_{B_1 \cap U \neq \emptyset} \int_{G \cap (B_1 \cap U) \neq \emptyset} \int_{B_2 \cap (B_1 \cap G \cap U) \neq \emptyset} dB_2 dG dB_1
\]

\[
= \int_{B_1 \cap U \neq \emptyset} \int_{G \cap (B_1 \cap U) \neq \emptyset} [2L(G \cap B_1 \cap U) + \frac{1}{2} Br_2 P(K^*)] dG dB_1
\]

\[
= \int_{B_1 \cap U \neq \emptyset} [2A(K^*)A(B_1 \cap U) + \frac{1}{2} Br_2 P(K^*)L(bd(B_1 \cap U))] dB_1.
\]

From Theorem 2 we have

\[
\int_{B_1 \cap U \neq \emptyset} 2A(K^*)A(B_1 \cap U) dB_1 = P(K^*)A(K^*)A(U)Br_1.
\]
Now for a line \( G \in \mathcal{L}(M^2) \) and a strip \( B \), consider the measure of \( G \) and \( B \) such that \( G \cap B \cap U \neq \emptyset \).

\[
\int_{G \cap B \cap U \neq \emptyset} dBdG = \int_{B \cap U \neq \emptyset} \int_{G \cap (B \cap U) \neq \emptyset} dGdB = \int_{B \cap U \neq \emptyset} L(bd(B \cap U))dB.
\]

On the other hand,

\[
\int_{G \cap B \cap U \neq \emptyset} dBdG = \int_{G \cap U \neq \emptyset} \int_{B \cap (G \cap U) \neq \emptyset} dBdG = \int_{G \cap U \neq \emptyset} [2L(G \cap U) + \frac{1}{2} BrP(K^*)]dG = 2A(K^*)A(U) + \frac{1}{2} BrP(K^*)L(bdU).
\]

Thus we have

\[
\int_{B \cap U \neq \emptyset} L(bd(B \cap U))dB = 2A(K^*)A(U) + \frac{1}{2} BrP(K^*)L(bdU).
\]

So we have

\[
\int_{B_1 \cap B_2 \cap G \cap U \neq \emptyset} dGdB_1dB_2 = P(K^*)A(K^*)A(U)[Br_1 + Br_2] + \frac{1}{4} P^2(K^*)L(bdU)Br_1Br_2.
\]

From this we get the result easily.

If \( B_1 \) and \( B_2 \) are strips of breadth \( Br \), then we have the following corollary.

**Corollary 2.** Let \( U \) be a compact convex body in \( M^2 \) and let \( B_1 \) and \( B_2 \) be strips of breadth \( Br \). Then for a line \( G \in \mathcal{L}(M^2) \) we have

\[
\int_{B_1 \cap B_2 \cap G \cap U \neq \emptyset} dGdB_1dB_2 = 2P(K^*)A(K^*)A(U)Br + \frac{1}{4} P^2(K^*)L(bdU)(Br)^2.
\]

In Theorem 3 we have

\[
\int_{B \cap U \neq \emptyset} L(bd(B \cap U))dB = 2A(K^*)A(U) + \frac{1}{2} BrP(K^*)L(bdU).
\]

Now we consider the measure \( \int_{B \cap U \neq \emptyset} L^2(bd(B \cap U))dB \).
Theorem 4. Let $U$ be a compact convex body in $M^2$ and let $B$ be a strip of breadth $Br$. Then we have
\[
\int_{B \cap U \neq \emptyset} L^2(bd(B \cap U)) dB
= 2A(K^*)A(U) + \frac{1}{2} Br P(K^*) L(bdU) L(bd(B \cap U)).
\]

Proof. Let $G_1, G_2$ be any lines in $\mathcal{L}(M^2)$. Then we have
\[
\int_{G_1 \cap B \cap U \neq \emptyset, G_2 \cap B \cap U \neq \emptyset} dG_1 dG_2 dB
= \int_{B \cap U \neq \emptyset} \int_{G_1 \cap (B \cap U) \neq \emptyset, G_2 \cap (B \cap U) \neq \emptyset} dG_1 dG_2 dB
= \int_{B \cap U \neq \emptyset} \int_{G_1 \cap (B \cap U) \neq \emptyset} dG_1 \int_{G_2 \cap (B \cap U) \neq \emptyset} dG_2 dB
= \int_{B \cap U \neq \emptyset} L^2(bd(B \cap U)) dB.
\]
On the other hand,
\[
\int_{G_1 \cap B \cap U \neq \emptyset, G_2 \cap B \cap U \neq \emptyset} dG_1 dG_2 dB
= \int_{G_1 \cap B \cap U \neq \emptyset} dG_1 \int_{G_2 \cap B \cap U \neq \emptyset} dB dG_2.
\]
Here we get
\[
\int_{G_2 \cap B \cap U \neq \emptyset} dB dG_2 = \int_{G_2 \cap U \neq \emptyset} \int_{B \cap (G_2 \cap U) \neq \emptyset} dB dG_2
= \int_{G_2 \cap U \neq \emptyset} \left[ 2L(G_2 \cap U) + \frac{1}{2} Br P(K^*) \right] dG_2
= 2A(K^*) A(U) + \frac{1}{2} Br P(K^*) L(bdU).
\]
Thus we have
\[
\int_{G_1 \cap B \cap U \neq \emptyset, G_2 \cap B \cap U \neq \emptyset} dG_1 dG_2 dB
= L(bd(B \cap U)) \left[ 2A(K^*) A(U) + \frac{1}{2} Br P(K^*) L(bdU) \right].
\]
This completes the proof. \qed
THEOREM 5. Let $U$ be a compact convex body in $M^2$ and let $B$ be a strip of breadth $Br$. Then for $n \geq 1$ we have

$$\frac{\int_{B \cap U \neq \emptyset} L^{n+1}(bd(B \cap U)) dB}{\int_{B \cap U \neq \emptyset} L^n(bd(B \cap U)) dB} = L(bd(B \cap U)).$$

Proof. Let $G_1, G_2, \ldots, G_n$ be any lines in $\mathcal{L}(M^2)$. Then for $1 \leq i \leq n$, we have

$$\int_{G_i \cap (B \cap U) \neq \emptyset} dG_1 dG_2 \cdots dG_n dB$$

$$= \int_{B \cap U \neq \emptyset} \int_{G_i \cap (B \cap U) \neq \emptyset} dG_1 dG_2 \cdots dG_n dB$$

$$= \int_{B \cap U \neq \emptyset} \int_{G_1 \cap (B \cap U) \neq \emptyset} dG_1 \cdots \int_{G_i \cap (B \cap U) \neq \emptyset} dG_i dB$$

$$= \int_{B \cap U \neq \emptyset} L^n(bd(B \cap U)) dB.$$

On the other hand,

$$\int_{G_i \cap (B \cap U) \neq \emptyset} dG_1 dG_2 \cdots dG_n dB$$

$$= \int_{G_1 \cap (B \cap U) \neq \emptyset} dG_1 \cdots \int_{G_{n-1} \cap (B \cap U) \neq \emptyset} dG_{n-1} \int_{G_n \cap (B \cap U) \neq \emptyset} dB dG_n$$

$$= L^{n-1}(bd(B \cap U)) \left[ 2A(K^*) A(U) + \frac{1}{2} Br P(K^*) L(bd U) \right].$$

This completes the proof. \(\square\)

References


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Geometric probability in the Minkowski plane

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