Bull. Korean Math. Soc. **40** (2003), No. 3, pp. 531–536

ROUGHNESS OF GAMMA-SUBSEMIGROUPS/IDEALS IN GAMMA-SEMIGROUPS

YOUNG BAE JUN

ABSTRACT. We discuss the roughness of Γ-subsemigroups and ideals/bi-ideals in Γ-semigroups.

1. Introduction

The notion of rough sets was introduced by Z. Pawlak in his paper [3]. The theory of rough sets has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy, or incomplete information. It is turning out to be methodologically significant to the domains of artificial intelligence and cognitive sciences, especially in the representation of and reasoning with vague and/or imprecise knowledge, data classification, data analysis, machine learning, and knowledge discovery [4, 8]. In connection with algebraic structures, R. Biswas and S. Nanda [1] gave the notion of rough subgroups, and N. Kuroki and P. P. Wang [2] introduced the notion of a rough subgroup with respect to a normal subgroup of a group, and gave some properties of the lower and the upper approximations in a group. In this paper, we will discuss the lower and upper approximations with respect to congruences in Γ-semigroups which are introduced and studied by N. K. Saha and M. K. Sen [5, 6, 7].

2. Preliminaries

Let $S = \{x, y, z, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ be two nonempty sets. Then S is called a Γ-*semigroup* if it satisfies $x\alpha y \in S$ and $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

Received January 28, 2002.
2000 Mathematics Subject Classification: 20M12, 18B40.
Key words and phrases: upper (lower) rough Γ-subsemigroup, upper (lower) rough ideal/bi-ideal.

A nonempty subset A of a Γ-semigroup S is called a Γ-*subsemigroup* of S if $A\Gamma A \subseteq A$, a *right* (*left*) *ideal* of S if $A\Gamma S \subseteq A$ ($S\Gamma A \subseteq A$), and an *ideal* of S if it is both a right and a left ideal of S.

Let the domain U of discourse (also called *universe*) be a nonempty set, and Θ an equivalence relation on U. The pair $\langle U, \Theta\rangle$ is called an *approximation space* [3].

3. Roughness of Γ-subsemigroups/ideals

In what follows, let S denote a Γ-semigroup unless otherwise specified. Let Θ be a congruence relation on S, that is, Θ is an equivalence relation on S such that

$$(a, b) \in \Theta \text{ implies } (a\gamma x, b\gamma x) \in \Theta \text{ and } (x\gamma a, x\gamma b) \in \Theta$$

for all $x \in S$ and $\gamma \in \Gamma$. We denote by $[a]_\Theta$ the congruence class containing the element $a \in S$. A congruence relation Θ on S is said to be *complete* if $[a]_\Theta\gamma[b]_\Theta = [a\gamma b]_\Theta$ for all $a, b \in S$ and $\gamma \in \Gamma$. Let A be a nonempty subset of S and Θ a congruence relation on S. Then the Θ-*lower* and Θ-*upper approximations* of A are defined to be the sets

$$\underline{\Theta}(A) := \{x \in S \mid [x]_\Theta \subseteq A\}$$

and

$$\overline{\Theta}(A) := \{x \in S \mid [x]_\Theta \cap A \neq \emptyset\},$$

respectively.

PROPOSITION 3.1. *Let Θ and Ψ be congruence relations on S and let A and B be nonempty subsets of S. Then*

(i) $\underline{\Theta}(A) \subseteq A \subseteq \overline{\Theta}(A)$.
(ii) $\overline{\Theta}(A \cup B) = \overline{\Theta}(A) \cup \overline{\Theta}(B)$.
(iii) $\underline{\Theta}(A \cap B) = \underline{\Theta}(A) \cap \underline{\Theta}(B)$.
(iv) $A \subseteq B$ *implies* $\underline{\Theta}(A) \subseteq \underline{\Theta}(B)$ *and* $\overline{\Theta}(A) \subseteq \overline{\Theta}(B)$.
(v) $\underline{\Theta}(A) \cup \underline{\Theta}(B) \subseteq \underline{\Theta}(A \cup B)$.
(vi) $\overline{\Theta}(A \cap B) \subseteq \overline{\Theta}(A) \cap \overline{\Theta}(B)$.
(vii) $\Theta \subseteq \Psi$ *implies* $\underline{\Psi}(A) \subseteq \underline{\Theta}(A)$ *and* $\overline{\Theta}(A) \subseteq \overline{\Psi}(A)$.
(viii) $\overline{\Theta}(A)\Gamma\overline{\Theta}(B) \subseteq \overline{\Theta}(A\Gamma B)$.
(ix) *If Θ is complete, then* $\underline{\Theta}(A)\Gamma\underline{\Theta}(B) \subseteq \underline{\Theta}(A\Gamma B)$.
(x) $\overline{\Theta \cap \Psi}(A) \subseteq \overline{\Theta}(A) \cap \overline{\Psi}(A)$.
(xi) $\underline{\Theta \cap \Psi}(A) = \underline{\Theta}(A) \cap \underline{\Psi}(A)$.

Proof. (i) If $x \in \underline{\Theta}(A)$, then $x \in [x]_\Theta \subseteq A$. Hence $\underline{\Theta}(A) \subseteq A$. Now let $x \in A$. Since $x \in [x]_\Theta$, we have $[x]_\Theta \cap A \neq \emptyset$, and so $x \in \overline{\Theta}(A)$. Thus $A \subseteq \overline{\Theta}(A)$.

(ii) and (iii) Note that

$$\begin{aligned} x \in \overline{\Theta}(A \cup B) &\Leftrightarrow [x]_\Theta \cap (A \cup B) \neq \emptyset \\ &\Leftrightarrow \big([x]_\Theta \cap A\big) \cup \big([x]_\Theta \cap B\big) \neq \emptyset \\ &\Leftrightarrow [x]_\Theta \cap A \neq \emptyset \text{ or } [x]_\Theta \cap B \neq \emptyset \\ &\Leftrightarrow x \in \overline{\Theta}(A) \text{ or } x \in \overline{\Theta}(B) \\ &\Leftrightarrow x \in \overline{\Theta}(A) \cup \overline{\Theta}(B), \end{aligned}$$

and

$$\begin{aligned} x \in \underline{\Theta}(A \cap B) &\Leftrightarrow [x]_\Theta \subseteq A \cap B \\ &\Leftrightarrow [x]_\Theta \subseteq A \text{ and } [x]_\Theta \subseteq B \\ &\Leftrightarrow x \in \underline{\Theta}(A) \text{ and } x \in \underline{\Theta}(B) \\ &\Leftrightarrow x \in \underline{\Theta}(A) \cap \underline{\Theta}(B). \end{aligned}$$

Hence (ii) and (iii) are valid.

(iv) Since $A \subseteq B$ if and only if $A \cap B = A$, it follows from (iii) that

$$\underline{\Theta}(A) = \underline{\Theta}(A \cap B) = \underline{\Theta}(A) \cap \underline{\Theta}(B)$$

so that $\underline{\Theta}(A) \subseteq \underline{\Theta}(B)$. Note also that $A \subseteq B$ if and only if $A \cup B = B$. Thus, by (ii), we have

$$\overline{\Theta}(B) = \overline{\Theta}(A \cup B) = \overline{\Theta}(A) \cup \overline{\Theta}(B),$$

which implies that $\overline{\Theta}(A) \subseteq \overline{\Theta}(B)$.

(v) and (vi) Using (iv), we get $\underline{\Theta}(A) \subseteq \underline{\Theta}(A \cup B)$ and $\underline{\Theta}(B) \subseteq \underline{\Theta}(A \cup B)$. Hence $\underline{\Theta}(A) \cup \underline{\Theta}(B) \subseteq \underline{\Theta}(A \cup B)$. Also $\overline{\Theta}(A \cap B) \subseteq \overline{\Theta}(A)$ and $\overline{\Theta}(A \cap B) \subseteq \overline{\Theta}(B)$, which yields $\overline{\Theta}(A \cap B) \subseteq \overline{\Theta}(A) \cap \overline{\Theta}(B)$.

(vii) Assume that $\Theta \subseteq \Psi$. If $x \in \underline{\Psi}(A)$, then $[x]_\Theta \subseteq [x]_\Psi \subseteq A$, that is, $x \in \underline{\Theta}(A)$. Now if $y \in \overline{\Theta}(A)$, then $[y]_\Theta \cap A \neq \emptyset$, and so there exists $a \in S$ such that $a \in [y]_\Theta$ and $a \in A$. Since $\Theta \subseteq \Psi$, we get $a \in [y]_\Theta \subseteq [y]_\Psi$. It follows that $a \in [y]_\Psi \cap A$ so that $y \in \overline{\Psi}(A)$.

(viii) Let $w \in \overline{\Theta}(A)\Gamma\overline{\Theta}(B)$. Then $w = x\gamma y$ with $x \in \overline{\Theta}(A)$, $y \in \overline{\Theta}(B)$ and $\gamma \in \Gamma$, and therefore there exist $a, b \in S$ such that $a \in [x]_\Theta \cap A$ and $b \in [y]_\Theta \cap B$. Since Θ is congruence, it follows that

$$a\gamma b \in [x]_\Theta \gamma [y]_\Theta \subseteq [x\gamma y]_\Theta.$$

On the other hand, since $a\gamma b \in A\Gamma B$, we have $a\gamma b \in [x\gamma y]_\Theta \cap A\Gamma B$, and so $w = x\gamma y \in \overline{\Theta}(A\Gamma B)$. This shows that $\overline{\Theta}(A)\Gamma\overline{\Theta}(B) \subseteq \overline{\Theta}(A\Gamma B)$.

(ix) Assume that Θ is complete and let $u \in \underline{\Theta}(A)\Gamma\underline{\Theta}(B)$. Then $u = x\gamma y$ with $x \in \underline{\Theta}(A)$, $y \in \underline{\Theta}(B)$ and $\gamma \in \Gamma$. It follows that $[x]_\Theta \subseteq A$ and $[y]_\Theta \subseteq B$. Since Θ is complete, we have

$$[x\gamma y]_\Theta = [x]_\Theta\gamma[y]_\Theta \subseteq A\Gamma B,$$

and so $u = x\gamma y \in \underline{\Theta}(A\Gamma B)$. Hence $\underline{\Theta}(A)\Gamma\underline{\Theta}(B) \subseteq \underline{\Theta}(A\Gamma B)$.

(x) and (xi) Note that $\Theta \cap \Psi$ is also a congruence relation on S. Let $x \in \overline{\Theta \cap \Psi}(A)$. Then $[x]_{\Theta\cap\Psi} \cap A \neq \emptyset$. Taking $a \in [x]_{\Theta\cap\Psi} \cap A$, then $(a,x) \in \Theta \cap \Psi$, that is, $(a,x) \in \Theta$ and $(a,x) \in \Psi$. It follows that $a \in [x]_\Theta \cap A$ and $a \in [x]_\Psi \cap A$ so that $x \in \overline{\Theta}(A) \cap \overline{\Psi}(A)$. Hence $\overline{\Theta \cap \Psi}(A) \subseteq \overline{\Theta}(A) \cap \overline{\Psi}(A)$. Note that

$$\begin{aligned} x \in \underline{\Theta \cap \Psi}(A) &\Leftrightarrow [x]_{\Theta\cap\Psi} \subseteq A \\ &\Leftrightarrow [x]_\Theta \subseteq A \text{ and } [x]_\Psi \subseteq A \\ &\Leftrightarrow x \in \underline{\Theta}(A) \text{ and } x \in \underline{\Psi}(A) \\ &\Leftrightarrow x \in \underline{\Theta}(A) \cap \underline{\Psi}(A). \end{aligned}$$

This completes the proof. □

Let Θ be a congruence relation on S. Then a nonempty subset A of S is called a Θ-*upper* (resp. Θ-*lower*) *rough* Γ-*subsemigroup* of S if the Θ-upper (resp. Θ-lower) approximation of A is a Γ-subsemigroup of S, a Θ-*upper* (resp. Θ-*lower*) *rough right* (*left*) *ideal* of S if the Θ-upper (resp. Θ-lower) approximation of A is a right (left) ideal of S.

THEOREM 3.2. *Let Θ be a congruence relation on S. Then*

(i) *Every Γ-subsemigroup of S is a Θ-upper rough Γ-subsemigroup of S.*
(ii) *Every right (left) ideal of S is a Θ-upper rough right (left) ideal of S.*

Proof. (i) Let A be a Γ-subsemigroup of S. Then $\emptyset \neq A \subseteq \overline{\Theta}(A)$ by Proposition 3.1(i). It follows from (iv) and (viii) of Proposition 3.1 that

$$\overline{\Theta}(A)\Gamma\overline{\Theta}(A) \subseteq \overline{\Theta}(A\Gamma A) \subseteq \overline{\Theta}(A)$$

so that $\overline{\Theta}(A)$ is a Γ-subsemigroup of S, that is, A is a Θ-upper rough Γ-subsemigroup of S.

(ii) Let A be a right ideal of S. Then, by (iv) and (viii) of Proposition 3.1, we have

$$\overline{\Theta}(A)\Gamma S = \overline{\Theta}(A)\Gamma\overline{\Theta}(S) \subseteq \overline{\Theta}(A\Gamma S) \subseteq \overline{\Theta}(A).$$

Thus $\overline{\Theta}(A)$ is a right ideal of S, that is, A is a Θ-upper rough right ideal of S. The other case can be also proved in a similar way. This completes the proof. □

THEOREM 3.3. *Let A be a nonempty subset of S and let Θ be a complete congruence relation on S such that the Θ-lower approximation of A is nonempty. Then*

(i) *If A is a Γ-subsemigroup of S, then A is a Θ-lower rough Γ-subsemigroup of S.*
(ii) *If A is a right (left) ideal of S, then A is a Θ-lower rough right (left) ideal of S.*

Proof. (i) Let A be a Γ-subsemigroup of S. Applying (iv) and (ix) of Proposition 3.1, we have

$$\underline{\Theta}(A)\Gamma\underline{\Theta}(A) \subseteq \underline{\Theta}(A\Gamma A) \subseteq \underline{\Theta}(A).$$

Thus the Θ-lower approximation of A is a Γ-subsemigroup of S, and so A is a Θ-lower Γ-subsemigroup of S.

(ii) Let A be a right ideal of S. Then, by (iv) and (ix) of Proposition 3.1, we get

$$\underline{\Theta}(A)\Gamma S = \underline{\Theta}(A)\Gamma\underline{\Theta}(S) \subseteq \underline{\Theta}(A\Gamma S) \subseteq \underline{\Theta}(A)$$

which means that the Θ-lower approximation of A is a right ideal of S. Hence A is a Θ-lower rough right ideal of S. The left case is by a similar way. □

A Γ-subsemigroup A of S is called a *bi-ideal* of S if $A\Gamma S\Gamma A \subseteq A$. A nonempty subset A of S is called a Θ-*upper* (resp. Θ-*lower*) *rough bi-ideal* of S if the Θ-upper (resp. Θ-lower) approximation of A is a bi-ideal of S, where Θ is a congruence relation on S.

THEOREM 3.4. *Let Θ be a congruence relation on S and let A be a bi-ideal of S. Then*

(i) *A is a Θ-upper rough bi-ideal of S.*
(ii) *If Θ is complete such that the Θ-lower approximation of A is nonempty, then A is a Θ-lower rough bi-ideal of S.*

Proof. (i) Using (iv) and (viii) of Proposition 3.1, we obtain

$$\overline{\Theta}(A)\Gamma S\Gamma\overline{\Theta}(A) = \overline{\Theta}(A)\Gamma\overline{\Theta}(S)\Gamma\overline{\Theta}(A) \subseteq \overline{\Theta}(A\Gamma S\Gamma A) \subseteq \overline{\Theta}(A).$$

Combining this and Theorem 3.2(i), we conclude that the Θ-upper approximation of A is a bi-ideal. Hence A is a Θ-upper rough bi-ideal of S.

(ii) Assume that Θ is complete and the Θ-lower approximation of A is nonempty. Then A is a Θ-lower rough Γ-subsemigroup of S (see Theorem 3.3(i)). Moreover we have

$$\underline{\Theta}(A)\Gamma S\Gamma\underline{\Theta}(A) = \underline{\Theta}(A)\Gamma\underline{\Theta}(S)\Gamma\underline{\Theta}(A) \subseteq \underline{\Theta}(A\Gamma S\Gamma A) \subseteq \underline{\Theta}(A).$$

Hence the Θ-lower approximation of A is a bi-ideal, and consequently A is a Θ-lower rough bi-ideal of S. □

THEOREM 3.5. *Let Θ be a congruence relation on S. If A and B are a right ideal and a left ideal of S, respectively, then*

$$\overline{\Theta}(A\Gamma B) \subseteq \overline{\Theta}(A) \cap \overline{\Theta}(B) \text{ and } \underline{\Theta}(A\Gamma B) \subseteq \underline{\Theta}(A) \cap \underline{\Theta}(B).$$

Proof. If A and B are a right ideal and a left ideal of S, respectively, then $A\Gamma B \subseteq A\Gamma S \subseteq A$ and $A\Gamma B \subseteq S\Gamma B \subseteq B$. Hence $A\Gamma B \subseteq A \cap B$, and by (iii), (iv) and (vi) of Proposition 3.1 we have

$$\overline{\Theta}(A\Gamma B) \subseteq \overline{\Theta}(A \cap B) \subseteq \overline{\Theta}(A) \cap \overline{\Theta}(B)$$

and

$$\underline{\Theta}(A\Gamma B) \subseteq \underline{\Theta}(A \cap B) = \underline{\Theta}(A) \cap \underline{\Theta}(B).$$

This completes the proof. □

References

[1] R. Biswas and S. Nanda, *Rough groups and rough subgroups,* Bull. Polish Acad. Sci. Math. **42** (1994), 251–254.

[2] N. Kuroki and P. P. Wang, *The lower and upper approximations in a fuzzy group,* Inform. Sci. **90** (1996), 203–220.

[3] Z. Pawlak, *Rough sets,* Int. J. Comput. Inform. Sci. **11** (1982), no. 5, 341–356.

[4] ———, *Rough sets: theoretical aspects of reasoning about data,* Kluwer Academic Publishers, Dordrecht, 1991.

[5] N. K. Saha, *On Γ-semigroup-II,* Bull. Calcutta. Math. Soc. **79** (1987), 331–335.

[6] ———, *On Γ-semigroup-III,* Bull. Calcutta. Math. Soc. **80** (1988), 1–12.

[7] M. K. Sen and N. K. Saha, *On Γ-semigroup-I,* Bull. Calcutta. Math. Soc. **78** (1986), 180–186.

[8] R. Slowinski, ed., *Intelligent decision support. Handbook of applications and advances of the rough sets theory,* Kluwer Academic Publishers, Dordrecht, 1992.

DEPARTMENT OF MATHEMATICS EDUCATION AND RINS, GYEONGSANG NATIONAL UNIVERSITY, CHINJU (JINJU) 660-701, KOREA

E-mail: ybjun@nongae.gsnu.ac.kr