A note on partially conformal geodesic transformation on the Kahler manifolds

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Abstract

In this paper, we deal with partially conformal geodesic transformations in Kahler geometry by using Fermi coordinates when the submanifold is a geodesic sphere. We derive the necessary and sufficient condition for the existence of such transformation in terms of the Jacobi operator and its derivative.

0. Historical background and introduction

In 1972, S. Tochibana introduced the notion of a geodesic conformal transformations around submanifolds in a Riemannian manifold. These transformations are extensions of geodesic symmetries and local reflections with respect to submanifolds. The notion of a reflections generalize that of reflections with respect to linear subspaces in Euclidean space. Recently, E. Garcia-Rio, L. Vanhecke and B. Y. Chen begun a systematic study of geodesic conformal transformation. They show that conformality is a strong condition and motivated the study of the notion of a partially conformal geodesic transformation.

We focus on partially conformal geodesic transformations in Kahler manifolds when the submanifold is a geodesic sphere. This note are devoted to characterizations of complex space forms by using non-Euclidean inversions as defining partially conformal geodesic transformations.

* This paper was supported by Wonkwang University in 2003.
1. Kahler manifolds and Fermi coordinates

Let \((M, g)\) be a connected smooth Riemannian manifold and \(\nabla\) its Levi Civita connection. Denote by \(R\) its associated Riemannian curvature tensor defined by
\[
R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]
\]
for all vector fields \(X, Y \in \mathfrak{X}(M)\). We put
\[
R_{XYZW} = g( R_{XY}Z, W )
\]
Let \(M\) be a \(n\)-dimensional Kahler manifold with structure \((M, g, J)\):
\[
J^2 = -1, \quad g(JX, JY) = g(X, Y), \quad \nabla_X(JY) = 0
\]
for all vector fields \(X, Y \in \mathfrak{X}(M)\). Then
\[
R(X, Y)J = JR(X, Y), \quad R(JX, JY) = R(X, Y).
\]
A plane section of the tangent space \(T_pM\) at a point \(p \in M\) is called a holomorphic section if it is spanned by vectors \(X\) and \(JX\) in \(T_pM\). The sectional curvature of a holomorphic section is called a holomorphic sectional curvature. A Kahler manifold of constant holomorphic sectional curvature \(c\) is called a complex space form and its curvature tensor is given by
\[
R_{XY} = \frac{c}{4} \{ g(X, Z)Y - g(Y, Z)X + g(JX, Z)JY - g(JY, Z)JX + 2g(JX, Y)JZ \}.
\]
A Kahler manifold \(M\) of dimension \(\geq 4\) is a complex space form if and only if, for every vector field \(X\) on \(M\), \(R_{XXX}X\) is collinear with \(JX\).

Let \(B\) be a embedded submanifold of \(M\) with dim \(B = q\) and \(\exp\), the exponential map of the normal bundle \(\nu = T^\perp B\) of \(B\) and \(m \in B\) and \(\{E_1, ..., E_n\}\) a local orthonormal frame field of \(M\) defined along \(B\) in a neighborhood of \(m\). We specialize the fields such that \(E_1, ..., E_q\) are tangent to \(B\) and \(E_{q+1}, ..., E_n\) normal vector fields of \(B\). For a system of coordinates \((y^1, ..., y^q)\) of \(B\) in a neighborhood of \(m\) such that \((\partial/\partial y^i)(m) = E_i(m), i = 1, ..., q\), the Fermi coordinates \((x^1, ..., x^n)\) with respect to \(m\), \((y^1, ..., y^q)\) and \((E_{q+1}, ..., E_n)\) are defined by
\[ x'(\exp_y(\sum_{a=1}^q t_a E_a)) = y', \quad i = 1, \ldots, q, \]
\[ x'(\exp_y(\sum_{a=1}^n t_a E_a)) = t^a, \quad a = q+1, \ldots, n \]
in an open neighborhood \( U_m \) of \( m \in M \).

Put \( s(r) = \rho(r)r \), where \( r \) is the normal distance. Then \( \gamma^2 = \sum_{a=q+1}^n (x^a)^2 \).

Let \( u \in T_m^1 \subset T_m M \) and \( \gamma(\tau) = \exp_m(\tau u) \) the normal geodesics with \( \gamma(0) = m \), \( \gamma'(0) = u = E_n(m) \). Denote by \( \{F_1, \ldots, F_n\} \) the frame field along \( \gamma \) obtained by parallel translating \( \{E_1(m), \ldots, E_n(m)\} \) along \( \gamma \). Consider the \( n-1 \) Jacobi vector fields \( Y_a, \quad a = 1, \ldots, n-1 \) along \( \gamma \), determined by the initial conditions

\[ Y_i(0) = E_i(m), \quad Y_i'(0) = (\nabla_u \partial/\partial x^i)(m), \quad i = 1, \ldots, q, \]
\[ Y_a(0) = 0, \quad Y_a'(0) = E_a(m), \quad a = q+1, \ldots, n. \]

Then \( Y_i(\tau) = \frac{\partial}{\partial x^i} (\gamma(\tau)), \quad Y_a(\tau) = r \frac{\partial}{\partial x^a} (\gamma(\tau)). \)

Put \( Y_a(\tau) = D_u(\tau)F_a, \quad a = 1, \ldots, n-1 \). Then \( D_u \) satisfies the Jacobi equation

\[ D_u'' + R \cdot D_u = 0 \]
where \( R(x)X = R_{\gamma'(\tau)x} \gamma'(\tau). \)

Using the initial conditions for \( Y_a \),

\[ D_u(0) = \begin{pmatrix} I \circ \partial & 0 \\ 0 & 0 \end{pmatrix}, \quad D_u'(0) = \begin{pmatrix} -T(u) & 0 \\ -\frac{1}{\nu} \partial \perp(u) & I_{n-q-1} \end{pmatrix} \]
where \( T(u) = g(T(u)E_i, E_j)(m) \), \( \perp(u) = g(\perp \circ E_a, E_a)(m) \)
and \( \perp \circ N(m) = (\perp \circ X N)(m) \).

Then \( g_{ij}(\tau) = (D_u D_u)_{ij}(\tau), \quad g_{ab}(\tau) = \frac{1}{\nu} (D_u D_u)_{ab}(\tau), \)
\( g_{ia}(\tau) = g_{ia} = 0, \quad g_{nn} = 1 \)
for \( i, j = 1, \ldots, q \) and \( a, b = q+1, \ldots, n-1 \).

2. Main results

We consider the local diffeomorphi
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\[ \phi_B : p = \exp_u (ru) \mapsto \phi_B(p) = \exp_u (s(r)u) \]

for \( u \in T^*_m \), \( \|u\| = 1 \). \( \phi_B \) is called the geodesic transformation with respect to \( B \), which is locally given by

\[ \phi_B : (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^q, \rho(r)x^{q+1}, \ldots, \rho(r)x^n) \]

Then we have

\[ \phi_B^* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, q, \]

\[ \phi_B^* \frac{\partial}{\partial x^a} = \rho \frac{\partial}{\partial x^a} + \rho \frac{\partial}{\partial r}, \quad a = q + 1, \ldots, n. \]

Let \( \eta \) be the one form defined by \( \eta(X) = g(X, JN) \).

If \( \phi_B^*g = e^{2\sigma}g + f \eta \otimes \eta \) for some function \( f \) which is depends only on the normal distance function \( r \), \( \phi_B \) is said to be partially conformal.

**Lemma 1.** A geodesic transformation \( \phi_B \) with respect to \( B \) is partially conformal if and only if

\[ g_u(\phi_B(p)) = (e^{2\sigma}g + f \eta \otimes \eta)_u(p), \quad \rho g_u(\phi_B(p)) = (e^{2\sigma}g + f \eta \otimes \eta)_u(p), \]

\[ \rho^2 g_{ii}(\phi_B(p)) = (e^{2\sigma}g + f \eta \otimes \eta)_{ii}(p), \quad e^{2\sigma} = (\rho r + \rho)^2 = (s')^2 \]

where \( i, j = 1, \ldots, q \) and \( a, b = q + 1, \ldots, n - 1 \).

**proof.** Using Fermi coordinates

\[ (\phi_B^*g)_{ii}(p) = g_u(\phi_B(p)), \quad (\phi_B^*g)_{ab}(p) = \rho g_u(\phi_B(p)) \]

\[ (\phi_B^*g)_{ab}(p) = \rho^2 g_{ab}(\phi_B(p)), \quad (\phi_B^*g)_{nn}(p) = (\rho' r + \rho)^2 g_{nn}(\phi_B(p))(p)) \]

By the definition of partially conformal, we have the desired result.

**Lemma 2.** Let \( (M, g, J) \) be a Kahler manifold and \( B \) a real hypersurface. If \( \phi_B \) is a partially conformal geodesic transformation with respect to \( B \), then \( B \) is a Hopf hypersurface with two constant principal curvatures.

**proof.** By Lemma 1

\[ g_u(s(r)) = e^{2\sigma}g_u(r) + f(r)(\eta \otimes \eta)_u(r). \quad (\ast) \]

Taking limits for \( r = 0 \), \( \delta_u = s'(0)^2 \delta_u + f(0) \delta_{ii} \delta_{ij} \). Thus \( s'(0)^2 = 1 \) and \( f(0) = 0 \).

Since \( \phi_B \) is non-trivial, \( s'(0) = -1 \). Using power series expansion for both side of
(\star), we get
\[ \delta_y - 2rT_y + O(r^2) = \delta_y + (2T_y - 2s'(0)\delta_y + f'(0)\delta_1,\delta_1)r + O(r^2). \]
Hence \[ T_y = \frac{1}{2} (s''(0)\delta_y - \frac{1}{2} f'(0)\delta_1,\delta_1). \]
Therefore \[ k_1 = \frac{1}{2} s''(0) - \frac{1}{2} f'(0) \] and \[ k_2 = \cdots = k_{n-1} = \frac{1}{2} s''(0). \]

**Theorem 3.** \((M, g, J)\) is an \(n\)-dimensional Kahler manifold of complex space form \(M_n(c), c \neq 0\) if and only if the non-Euclidean inversion
\[
\tan(s + a) = \frac{\sqrt{c}}{4} \tan(r + a) = \tan^2 a \sqrt{c} \quad (**) \]
defines a partially conformal geodesic transformation with respect to each geodesic sphere \(G(a)\) of small radius \(a\).

**proof.** Let \((**\)) be a partially conformal geodesic transformation with respect to \(G(a)\). Then \(G(a)\) is a hypersurface of \(M\). Put
\[
s + a = \frac{4}{\sqrt{c}} \tan^{-1} t, \quad r + a = \frac{4}{\sqrt{c}} \tan^{-1} t \quad \text{and} \quad D = \tan^2 a \sqrt{c}.
\]
Then \((**\)) takes the form \(t = D\) and \[ s = \frac{4}{\sqrt{c}} \tan^{-1} (D/t) - a. \]
By the power series expansion and lemma 2,
\[
s = -r = \frac{1}{2} r^2 \sqrt{c} \cot a \sqrt{c} + O(r^3),
\]
\[
f = -\left( \frac{\sin(s + a)\sqrt{c}}{\sin(r + a)\sqrt{c}} \right)^2 + \left( \frac{\sin(s + a)\sqrt{c}}{\sin(r + a)\sqrt{c}} \right)^2.
\]
Hence \[ k_1 = \sqrt{c} \cot a \sqrt{c} \] and \[ k_2 = \cdots = k_{n-1} = \frac{\sqrt{c}}{2} \cot a \frac{\sqrt{c}}{2}. \]
Thus \((M, g, J)\) is a complex space \(M_n(c)\).
Conversely, suppose \(M = M_n(c)\) and \(c\) to be positive. Then
\[
R = \begin{pmatrix} c & 0 \\ 0 & \frac{c}{4} I_{n-2} \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{\sqrt{c}} \sin a \sqrt{c} & 0 \\ \frac{2}{\sqrt{c}} \sin a \frac{\sqrt{c}}{2} & I_{n-2} \end{pmatrix}.
\]
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Hence \[ T(\exp_\nu(ru)) = \begin{pmatrix} \sqrt{c} \cos a \sqrt{c} & 0 \\ 0 & \frac{\sqrt{c}}{2} \cos a \frac{\sqrt{c}}{2} \cdot I_{n-2} \end{pmatrix}. \]

Since \[ D_\nu(r) = (\cos r\sqrt{c})D_\nu(0) + \left( \frac{-\sin r\sqrt{c}}{\sqrt{c}} \right)D_\nu(0), \]

\[ g_{ij}(\exp_\nu(ru)) = (D_\nu D_\nu)_{ij}(r) = \left( \cos r\frac{\sqrt{c}}{2} + \cot a \frac{\sqrt{c}}{2} \sin r\frac{\sqrt{c}}{2} \right) \delta_{ij}. \]

From \[ e^{2a}g_{ij}(\rho) = g_{ij}(\phi_\rho(\rho)), \] we get

\[ e^{2a} \left( \cos r\frac{\sqrt{c}}{2} + \cot a \frac{\sqrt{c}}{2} \sin r\frac{\sqrt{c}}{2} \right) \delta_{ij} = \left( \cos s(r)\frac{\sqrt{c}}{2} + \cot a \frac{\sqrt{c}}{2} \sin s(r)\frac{\sqrt{c}}{2} \right) \delta_{ij}. \]

Thus \[ \frac{ds}{dr} \sin (r + a) \frac{\sqrt{c}}{2} = \pm \sin (s + a) \frac{\sqrt{c}}{2}. \] Therefore

\[ \frac{ds}{\sin (s + a) \frac{\sqrt{c}}{2}} = \pm \frac{dr}{\sin (r + a) \frac{\sqrt{c}}{2}} \quad (***) \]

So (***) is the only solution of (***), leaving \( G(a) \) invariant.

References