Journal of the Korea Society of Mathematical Education Series D:
Research in Mathematical Education
Vol. 7, No. 3, September 2003, 191–209

韓國數學敎育學會誌 시리즈 D:
<數學敎育硏究>
제 7 권 제 3 호 2003 년 9 월, 191–209

The Learning and Teaching of Transcendental Functions through Sound and Music[1]

Choi, Jong Sool
School of Computer Aided Science, Inje University, 607 Eobang-dong, Gimhae, Gyeongnam 621-749, Korea; Email: choijon01@hanmail.net

Kim, Hyang Sook
School of Computer Aided Science, Inje University, 607 Eobang-dong, Gimhae, Gyeongnam 621-749, Korea; Email: mathkim@ijnc.inje.ac.kr

(Received September 2, 2003)

In this paper, we present a new environment of learning and teaching of trigonometric, exponential and logarithmic functions, the most difficult parts for students to learn among functions, through sound and music, students like the most. First, by using sound and music, we try to arouse student's interest. Second, we let students see and hear properties of transcendental functions so that students can understand and remember them easily. Finally we encourage students to compose their favorite song using transcendental functions so that they can experience the practicality of transcendental functions.

Key words: teaching and learning, transcendental functions, computer and calculator.
ZDM classification: C70
MSC2000 classification: 97D40, 97U50

I. INTRODUCTION

The widespread use of technologies, such as computer and calculator, have influenced greatly to all aspects of human activities. Accordingly, students' ways of thinking and expectations about mathematics also have changed. Mathematics teachers have tried endlessly to develop better learning and teaching environments for stimulating students' interest about mathematics and maintaining it, and for understanding mathematical concepts.

On the other hand, no one can deny that music have influenced greatly on human

[1] This paper will be presented as at the Eighth International Seminar on Education of Talented Children in Mathematics at the Daegu National University of Education, Daegu, Korea, October 3, 2003.

emotion and behavior throughout human history. Such effect was positive on one hand, but negative on the other hand. So many sociologists had to worry about the negative impact of music, especially to adolescent. Recently, due to the widespread use of technology, the influence of music to adolescent has become even greater. Thus only if we can use music in our learning of mathematics effectively, the impact would be great.

In this paper, we suggest learning and teaching environment of trigonometric, exponential and logarithmic functions, the most difficult parts for students to learn among functions, through sound and music. We expect that this new learning and teaching environment stimulate students' interest and emphasize the practicality of transcendental functions. We also hope that, by listening to music composed of trigonometric and exponential functions, students become interested in the long common history of mathematics and music and their related area, and notice the importance of technology and the beauty of mathematics.

II. MAIN

We believe that finding various methods of teaching abstract mathematical concepts is the main road of teaching mathematics with technology. In harmony with that main stream, we show how to use sound and music to instruct transcendental functions.

1. Graphs of trigonometric functions

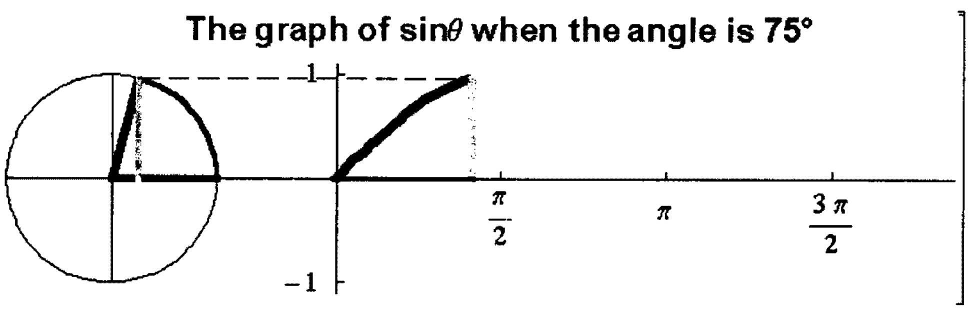
First, let's look at the shape of graphs of trigonometric functions. We try to find the shape of graph, periodicity, frequency and amplitude while considering the connection between sound and sine function.

(1) Sine curve

Let's draw graphs of sine functions as the angle θ changes (The Mathematica program for this graph appears in the Appendix [1].)

The following shows the process of drawing sine curve through animation

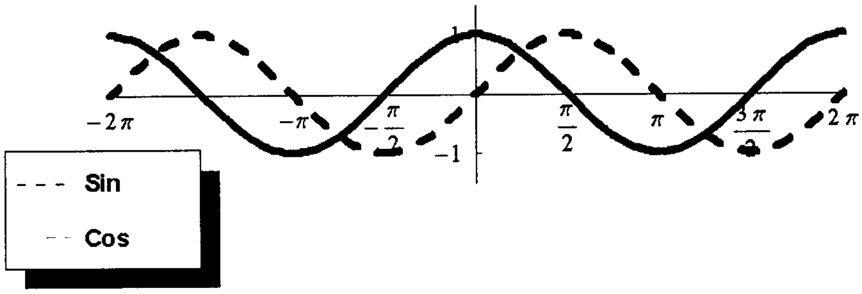
Do[SinPlot[θ],{θ,(π/180)*15, 2π, (π/180)*15}];



It shows that the sine curve is a periodic function with period 2π.

• The following shows graphs of sine curve and cosine curve together.

```
Needs["Graphics`Legend`"];
Plot[{Sin[θ],Cos[θ]},{θ,-2π,2π},
    PlotStyle→{{Dashing[{0.02,0.03}], RGBColor[0,0,1],Thickness[0.01]},
    {Hue[0.9], Thickness[0.01]}},
    DefaultFont→{"Times",14},
    PlotLegend→{FontForm["Sin",{"Arial-Bold",14}], FontForm["Cos",{"Arial-
    Bold",14}]},
    LegendSize→{0.5,0.3},
    Ticks→{{-2π, -π, -π/2, 0, π/2, π, 3π/2, 2π},{-1,0,1}},
    AspectRatio→Automatic, ImageSize→500,
    PlotRange→{{-2π, 2π},{-1.5,1.5}}];
```



The cosine curve is a translation of the sine curve, so it is a periodic function with the same period 2π.

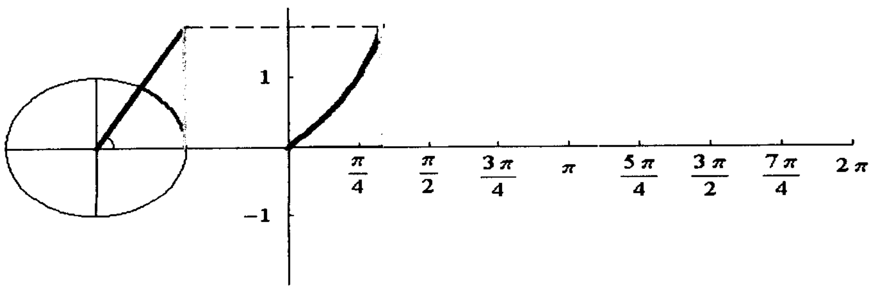
(2) Tangent curve

Let's draw the graph of a tangent function as the angle changes (The Mathematica program for this graph appears in the Appendix [2].)

• The following shows the process of drawing tangent curve through animation.

```
Do [TanPlot[θ],{θ, (π/180)*15, 2π, N[(π/180)*15]}]
```

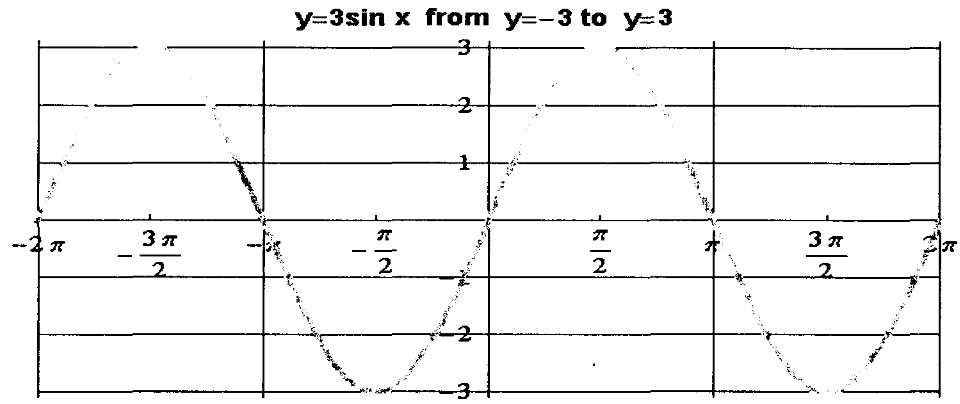
The graph of tanθ when the angle is 60.°



It shows that the tangent curve is a periodic function with period π.

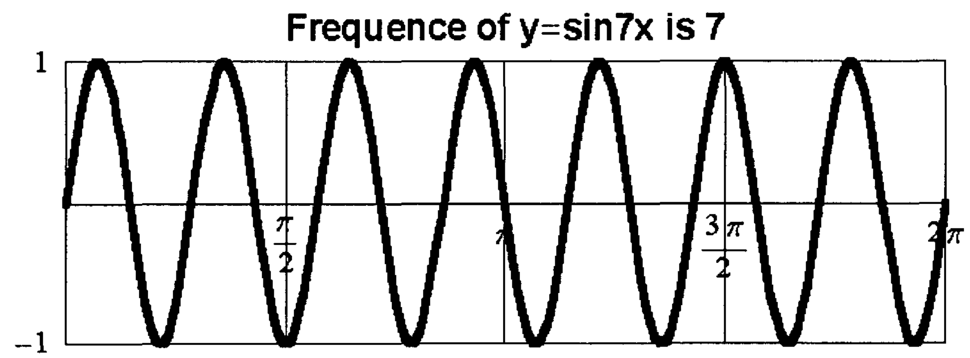
To understand the relation between sine function and sound, let's draw two functions that have to do with amplitude and frequency of sound. We draw $a\sin(x)$ and $\sin(bx)$ for arbitrary real number a, b.

Let's examine the change of the graph of $a\sin(x)$ as the amplitude of $a\sin(x)$, a, changes (The `Mathematica` program for this graph appears in the Appendix [3].)



• Let's examine the change of the graph of $\sin(bx)$ as the frequency of $\sin(bx)$, b, changes.

```
Table[Plot[Sin[ b x],{x,0,2π},
        PlotStyle→{RGBColor[0,0,1],Thickness[0.01]},
        PlotLabel→FontForm[" Frequence of y=sin"<>ToString[b]<>
  "x is"<>ToString[b]<> " " ,{"Helvetica-Bold",16}],
Background→GrayLevel[1], DefaultFont→{"Times",14},
        GridLines→{{0, π/2, π, 3π/2, 2π}, {-1,0,1}},
        Ticks→{{0, π/2, π, 3π/2, 2π}, {-1,1}},
        AspectRatio→Automatic, ImageSize→500,
        PlotRange→{{0,2π}, {-1,1}}];,{b,1,10,1}];
```



2. Sound

Sound is a phenomenon caused by vibration of air. Let's consider whether sine functions can make sound or not.

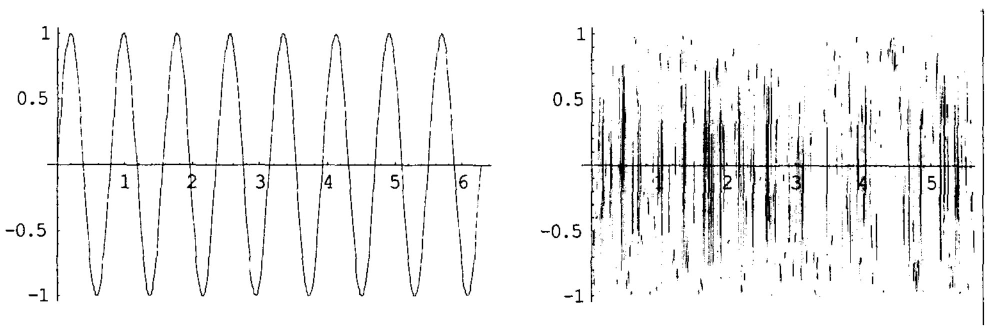
Do trigonometric functions make sound? If they do, what factors contribute to sound? Do sine, cosine, and tangent function make same sound? Have you ever heard about Pythagorean law of music? From now on, we examine sound.

Let's examine how amplitude and frequency contribute to sound.

(1) Frequency

Which one among loudness, pitch and quality of sound has to do with the frequency of sound?

- The following shows that we cannot hear the sound when the frequency is low
  Play[Sin[8*x*],{*x*,0,1}];
- Audible frequency of sound differs among animals. It is known that human can hear the sound of frequency between 220Hz~40000Hz.
  Play [Sin[400*x*],{*x*,0,1}];
  Play [Sin[800*x*],{*x*,0,1}];
- Let's confirm the difference in freqency in graphs of above two functions.
  s2=Plot[Sin[8*x*],{*x*,0,2π}, PlotStyle→{RGBColor[0,0,1]},
  ImageSize→400, DisplayFunction→Identity];
  s4=Plot[Sin[800*x*],{*x*,0,2π}, PlotStyle→{RGBColor[1,0,0]},
  ImageSize→400, DisplayFunction→Identity];
  Show[GraphicsArray[{s2,s4}], ImageSize→550,
  DisplayFunction→$DisplayFunction];



- Let's compare sound of two sine functions in different frequency. Listen carefully.
  Play[Sin[700*x*],{*x*,0,1}];
  Play[Sin[1400*x*] ,{*x*,0,1}];
- When we listen to them consecutively, we can distinguish their different pitch.

Play[Which[0<*x*<1, Sin[700*x*],
1<*x*<2, Sin[1400(*x*–1)]], {*x*,0,2}];

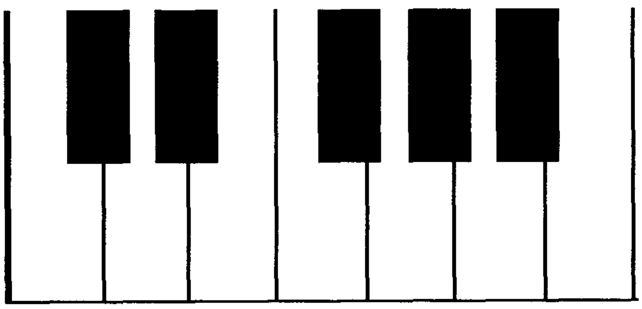
When the frequency is high, the sound is high. Pythagoras discovered that when the string of harp is short, the frequency is high; and when the frequency is high, the sound is high. This is the Pythagorean law in music.

(2) Quality of Sound

The quality of sound has to do with the shape of wave. When the wave is smooth, the quality of sound is good, but when the wave is not smooth, then it makes noise. Let's make sure that a beautiful wave makes beautiful sound by comparing the sound of trigonometric functions (sine, cosine, tangent).

- The shapes of sine and cosine functions are the same since one is a translation of another, but the shape of tangent is different from others. Let's confirm it through the quality of sound.
  Play [Sin[1500*x*], {*x*,0,1}];
- Cos[1500x]and Sin[1500x]have the same sound.
  Play [Cos[1500*x*], {*x*,0,1}];
- We can confirm it when we listen to them consecutively.
  Play[Which[0<*x*<1, Sin[1500*x*] ,
  1<*x*<2, Cos[1500(*x*–1)]], {*x*,0,2}];
- Let's listen another sound made by sine functions.
  Play [Sin[20 *t*] Sin[23 *t*] Sin[2000 *t*], {*t*, 0, 2.05}];
- Tan[1500*x*]makes noise.
  Play [Tan[1500*x*], {*x*,0,1}];
- Sec[1500*x*]makes noise too.
  Play [Sec[1500*x*], {*x*,0,1}];

Keyboard of Piano



Do Re Mi Fa Sol La Si Do2: Scale

$2^{\frac{0}{12}}$ $2^{\frac{2}{12}}$ $2^{\frac{4}{12}}$ $2^{\frac{5}{12}}$ $2^{\frac{7}{12}}$ $2^{\frac{9}{12}}$ $2^{\frac{11}{12}}$ 2: Ratio of frequency

Let's confirm through scale that sine and cosine functions make beautiful sound.
Listen 'Do', 'Mi', 'Sol' in which sine function is used.

$$\text{Play[Which[}0<x<1,\ \frac{\sin[1000x2^{\frac{0}{12}}]^2}{e^{x2^{\frac{0}{12}}}},\ 1<x<2,\ \frac{\sin[1000(x-1)2^{\frac{4}{12}}]^2}{e^{(x-1)2^{\frac{4}{12}}}},$$

$$2<x<3,\ \frac{\sin[1000(x-2)2^{\frac{7}{12}}]^2}{e^{(x-2)2^{\frac{7}{12}}}}],\ \{x, 0, 3\}];$$

Pythagorean 12 scales consist of above 7 scales and black keyboards, the ratio of frequency of two consecutive scales is $\sqrt[12]{2}$. That is, they make a geometric sequence with the first term 1 and the ratio $\sqrt[12]{2}$.

- Listen 'Do', 'Mi', 'Sol' in which cosine function is used.

$$\text{Play[Which[}0<x<1,\ \frac{\cos[1000x2^{\frac{0}{12}}]^2}{e^{x2^{\frac{0}{12}}}},\ 1<x<2,\ \frac{\cos[1000(x-1)2^{\frac{4}{12}}]^2}{e^{(x-1)2^{\frac{4}{12}}}},$$

$$2<x<3,\ \frac{\cos[1000(x-2)2^{\frac{7}{12}}]^2}{e^{(x-2)2^{\frac{7}{12}}}}],\ \{x, 0, 3\}];$$

- Listen 'Do', 'Mi', 'Sol' in which tangent function is used.

$$\text{Play[Which[}0<x<1,\ \frac{\tan[1000x2^{\frac{0}{12}}]^2}{e^{x2^{\frac{0}{12}}}},\ 1<x<2,\ \frac{\tan[1000(x-1)2^{\frac{4}{12}}]^2}{e^{(x-1)2^{\frac{4}{12}}}},$$

$$2<x<3,\ \frac{Tan[1000(x-2)2^{\frac{7}{12}}]^2}{e^{(x-2)2^{\frac{7}{12}}}}],\ \{x,0,3\}];$$

How about the sound of composite functions of trigonometric functions?

- First, input a function and composite itself n-times, and then check the graph of the result.

```
f[x_]=Input["f(x)="]
g[x_, n_]:= Plot[Nest[f, x, n], {x, -12, 12}, PlotRange -> {{-11, 11}, {-1.5, 1.5}},
   PlotStyle -> {Hue[0.55], Thickness[0.009]},
   Epilog -> {Text[StyleForm["The number of composition ="<> ToString[n],
      FontSize -> 16, FontColor -> Hue[0.6]], {1, 1.3}, {-1, 0}]},
      AspectRatio -> 1/2,
   ImageSize -> 600];
```

$g[x,10]$;
Table[$g[x,n]$, $\{n,2,20,2\}$];

Let' compare the sound of Sin(400x) with the sound of the composite function in which we composite Sine(400x) itself 1000 times.

Play[Which[$0 < x < 1$, Nest[Sin, 400x $2^{\frac{4}{12}}$, 1000], $1 < x < 2$,

Sin[400 $(x-1)$ $2^{\frac{4}{12}}$]], $\{x, 0, 2\}$];

From above two sounds, let's think about the relation between sound and amplitude, and check our guess in the following subsection.

(3) Loudness of Sound

Let's listen sound of two sine functions with different amplitude.

- It is not easy to distinguish the difference when we listen to them separately.

Play[Sin[1000× $2^{\frac{3}{12}}$] Sin[1000× $2^{\frac{3}{12}}$], $\{x, 0, 1\}$];

Play[8Sin[1000× $2^{\frac{3}{12}}$] Sin[1000× $2^{\frac{3}{12}}$], $\{x, 0, 1\}$];

- It is easier to distinguish the loudness of sound when we listen consecutively.

Play[Which[$0<x<1$, Sin[1000× $2^{\frac{3}{12}}$] Sin[1000× $2^{\frac{3}{12}}$],

1<x<2, 8Sin[1000× $2^{\frac{3}{12}}$] Sin[1000× $2^{\frac{3}{12}}$]], $\{x, 0, 2\}$];

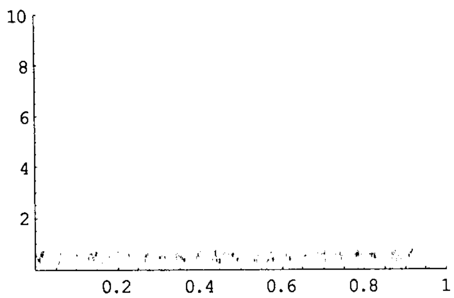
The loudness of sound is in proportion to amplitude. That is, when amplitude is higher, the sound is louder.

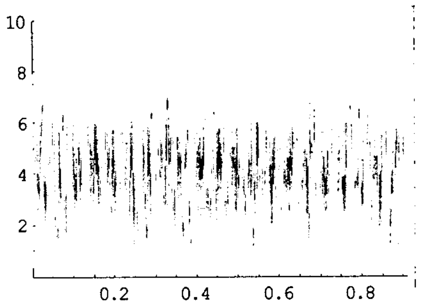
- Let's confirm the above fact with graphs of two functions with different amplitude.

f2 = Plot[Sin[1000× $2^{\frac{3}{12}}$] Sin[1000× $2^{\frac{3}{12}}$], $\{x, 0, 1\}$,
PlotStyle → {RGBColor[0.3, 0.7, 0]},
ImageSize → 400, PlotRange → {{0, 1}, {0, 10}},
DisplayFunction → Identity];

f4 = Plot[8Sin[1000× $2^{\frac{3}{12}}$] Sin[1000× $2^{\frac{3}{12}}$], $\{x, 0, 1\}$,
PlotStyle → {RGBColor[0.3, 0.7, 0]},
ImageSize → 400, PlotRange → {{0, 1}, {0, 10}},
DisplayFunction → Identity];

Show[GraphicsArray[{f2, f4}],
ImageSize → 550, DisplayFunction → $DisplayFunction];





We noticed that the frequency, the amplitude and the shape of the graph of a trigonometric function determines the pitch, the loudness and the quality of sound respectively. With this fact in mind, from now on, we examine the shape of graphs of exponential and logarithmic function through sound. Furthermore we examine the inverse relation of two inverse functions each other through animation with the change of base.

(4) Study of Exponential and Logarithmic Functions through Sound

Music and Mathematics have long common history and many mathematical theories contributed to the development of music, but most mathematicians are unaware that music is mathematics. That is, so far, mathematician themselves have thought that the practicality of mathematics is not part of their research area. But recently many people are aware that mathematics with practicality is more appealing to students than mathematics alone. Thus to teach the practicality of mathematics, to motivate and to show the beauty of mathematics, we teach transcendental functions (trigonometric, exponential and logarithmic functions), the hardest part of mathematics for a lot of students, through sound and music.

First of all, let students guess shape of graphs of exponential and logarithmic functions through sound of these functions.

1) Comparison in Sound using Exponential and Logarithmic Functions

Let's compare the "Do" sound using exponential and logarithmic functions with the "Do" sound without using them.

First, let's compare "Do" using exponential function with "Do" without using exponential function.

- Listen to the "Do" sound.

$$\text{Play}[\sin[1000x2^{\frac{0}{12}}]^2, \{x, 0, 1\}];$$

- Let's examine the "Do" sound with exp(x) ($0 \le x \le 1$) in denominator and its wave shape.

$$\text{Play}[\frac{\sin[1000x2^{\frac{0}{12}}]^2}{e^x}, \{x, 0, 1\}];$$

- Let's examine the "Do" sound with |exp(x)| ($0 \le x \le 1$) in denominator and its wave shape.

$$\text{Play}[\frac{\sin[1000x2^{\frac{0}{12}}]^2}{|e^x|}, \{x, 0, 1\}];$$

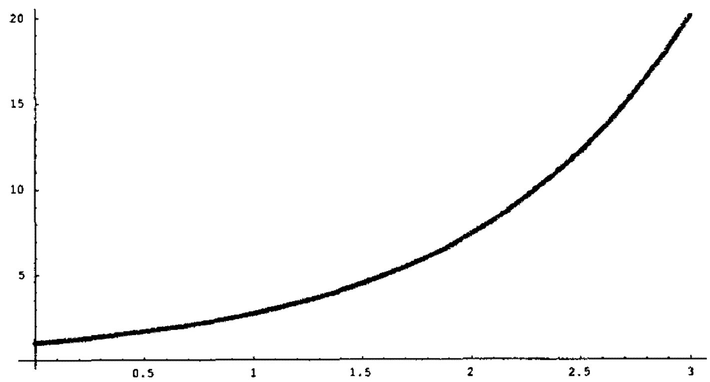
When we compare "Do" using exponential function with "Do" without using it, the sound is lower when we have exponential function in denominator. So when x changes, exp(x) ($0 < x \le 1$) is an increasing function with positive value. Thus when we divide sine function by exp(x), we can make decreasing sound effect. Therefore the sound of sine function divided by exp(x) makes sound like the trailing notes of a piano keyboard.

- With extended time, let's examine the sound of "Do" with exp(x) ($0 \le x \le 3$) in denominator and its wave shape.

$$\text{Play}[\frac{\sin[1000x2^{\frac{0}{12}}]^2}{e^x}, \{x, 0, 3\}];$$

When the time is extended, the sound of sine function divided by exp(x) makes better effect that sounds like the trailing note of a piano keyboard.

- Let's check our guessed results from above facts in the graph of exp(x) ($0 \le x \le 3$).
  Plot[e^x, {x, 0, 3}, PlotStyle →{Hue[0.55], Thickness[0.009]},
  ImageSize →500];



Next, let's compare "Do" using logarithmic function, inverse function of exponential function, with "Do" without using logarithmic function.

- Let's examine the "Do" sound divided by Log(x) with domain [0,2] and its wave shape.

$$\text{Play}[\frac{\sin[1000x2^{\frac{0}{12}}]^2}{\log[x]}, \{x, 0, 2\}];$$

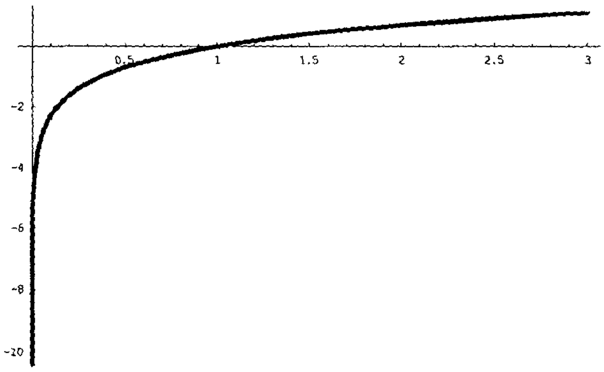
- Let's examine the "Do" sound divided by |Log(*x*)| with domain [0,2] and its wave shape.

$$\text{Play}[\frac{\sin[1000x2^{\frac{0}{12}}]^2}{|\log[x]|}, \{x, 0, 2\}];$$

When we divide the given sine function by a logarithmic function with Domain [0, 2], the sound becomes louder and then lower. Thus we can guess that when x ($0<x\leq 2$) changes, log(*x*) is a function that the absolute value of function value becomes smaller and then larger. Also we can guess the change of sign of function value as we divide it by |log(*x*)|. Thus when we divide the given sine function by log(*x*) ($0<x\leq 2$), we can make sound effect that becomes louder and then lower.

Let's examine the graph of Log(*x*) ($0\leq x\leq 3$).

Plot[Log[*x*],{*x*,0,3},PlotStyle→Hue[0.9],Thickness[0.009]], ImageSize→500];



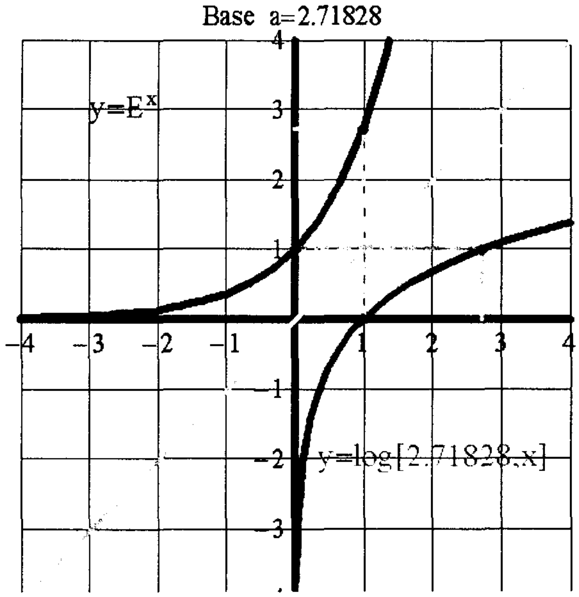
From the above observation on graph of exponential and logarithmic function, we can encourage students to guess the relation between them. We notify that above two exponential and logarithmic functions are with base *e*. We can also encourage students to guess the relation between a^x and $\log_a x$ as *a* changes. From now on, let's examine the relation between exponential function a^x and logarithmic function $\log_a x$.

2) The Relation between exponential and logarithmic function.

Let us use a program that show the relation between log(*x*) and exp(*x*) (The Mathematica program for this can be found in the Appendix [4].)

Let's examine the inverse relation between exponential function e^x and logarithmic function $\log_e x$, of which we examined through sound.

- We use the program [4] to get k[e].



We may need to examine the relation of two functions in case domain is $0 < x \le 1$ and in case domain is $x \ge 1$ separately.

- The following shows the relation, between y= a^x and $\log_a x$, in which the base is bigger than 1(when base a = 2).
  k[2]
- The following shows an animation of the relation, between y= a^x and $\log_a x$, in which the base is bigger than 1(base a changes from 1.1 to 4).
- Table[k[i],{i,1.1,4,0.2}];
- To see the relation, between y= a^x and $\log_a x$ when the base is lesser than 1(when base a = 0.5), we use the following.
  k[0.5]
- The following shows an animation of the relation, between y= a^x and $\log_a x$ when the base is lesser than 1(base a changes from 0.1 to 0.9).
  Table[k[i],{i,0.1,0.9,0.1}];
- Combination of above : An animation of the relation, between y= a^x and $\log_a x$ when the base is between 0.1 and 3.1 can be seen in the following program;
  Table[k[i],{i,0.1,3.1,0.2}];

3. Music

By encouraging students to write music to a known song with transcendental functions, let students feel the practicality and curiosity of mathematics. Let's compose "Song-A-Gi"(meaning a calf).

We compose it with sine function in Mathematica (The Mathematica program for this can be found in the Appendix [5]).

A lot of students listen to music even when they are sleeping and studying. By letting them examine that their beloved music is a mathematical data consisting of lots of coordinates, we can teach them that music is mathematics. By encouraging students to ask themselves what difference of mathematical, statistic data of classic music and bit music makes them different music, we can help them realize that our actual life and mathematics are strongly connected.

III. CONCLUSION

We tried to set up a new environment of learning and teaching trigonometric, exponential and logarithmic functions, the most difficult parts for students to learn among functions, through sound and music, students like the most. Although this method doesn't cover from genesis to application of transcendental functions, we believe that it helps students who lost interest in mathematics and does not have motivation to study mathematics because of its abstractness, difficulty and their lack of understanding on practicality of mathematics can enjoy studying mathematics. First, since most students do like sound and music, it arouses student's interest. Second, we let students to see and hear properties of transcendental functions so that they can understand and remember them easily. We also encourage students to compose their favorite song using transcendental functions so that they can experience the practicality of them. We believe that it will help them to keep their interest in transcendental functions. We hope it help to enhance student's interest and attitude about all parts of mathematics.

We believe that to keep the enhanced attitude and interest in mathematics permanently, their experience of practicality and arousing interest should not be just once. So we should continue putting our efforts to keep enhanced interest. Since it is evident that music is one of the greatest influential things to adolescent, we think that further research is necessary in the field of using sound and music to learn and teach mathematics

REFERENCES

Kim, H. S.; Kim, T. H.; Kim, Y. M. & Choi, J. S. (2003): Teaching of Transcendental functions through sound and music. In: *Proceeding of Summer Conference* (pp. 611–637). Seoul: Korea

Society of Educational Studies in Mathematics
Wolfram, S. (2000): *Mathematica (4th Ed.)*. Cambridge, UK: Cambridge University Press.

http://mttc.inje.ac.kr
http://socas.inje.ac.kr/hskim

Appendix (The Mathematica program used in MAIN.)

[1] Program [1]

```
SinPlot[θ_]:=
    Module[{lns},
      lns={{Thickness[0.001], Line[{{-π+1,1},{-π+1,-1}}]},
           {Thickness[0.008], Line[{{-π+1,0},{-π+2, 0}}]},
           {Thickness[0.008], Line[{{-π+1,0},{-π+1+Cos[θ],Sin[θ]}}]},
           {Thickness[0.008],   Hue[0.4],    Line[{{-π+1+Cos[θ],0},{-π+1+Cos[θ],
               Sin[θ]}}]},
           {Dashing[{0.02,0.01}], { Thickness[0.001], Line[{{-π+1+Cos[θ],
               Sin[θ]},{θ, Sin[θ]}}]},
            {Thickness[0.01], Hue[0.4], Line[{{θ,Sin[θ]},{θ,0}}]}},
             {Thickness[0.005], Hue[0.7], Line[{{0,0},{θ,0}}]}};
      cls={Thickness[0.001], Circle[{-π+1,0},1],
          {Thickness[0.005], Hue[0.7], Circle[{-π+1,0},1,{0,θ}]},
              Circle[{-π+1,0},1.0,{0, Mod[θ,2π]}]}};
          Plot[Sin[t],{t,0,θ},
          PlotStyle->{RGBColor[1,0,0],Thickness[0.01]},
          PlotLabel-> FontForm["The graph of sinθ when the angle is"<>
ToString[θ 180/π]<>"°",{"Helvetica-Bold", 16}],
Background-> GrayLevel[1],
         Epilog->{cls,lns}, DefaultFont->{"Times",14},
         Ticks->{{0, π/2, π, 3π/2, 2π}, {-1,0,1}},
         AspectRatio-> Automatic, ImageSize->600,
         PlotRange->{{-π,2π},{-1.1,1.1}}];];
```

[2] Program[2]

```
TanPlot[θ_]:=
    Module[{lns},
      lns={{Thickness[0.001],Line[{{-π+1,1},{-π+1,-1}}]},
          {Thickness[0.008],Line[{{-π+1,0}, {-π+1+1,Tan[θ]}}],
            Line[{{-π+1,0},{-π+1+Cos[θ], Sin[θ]}}]},
          {Thickness[0.008],Hue[0.4], Line[{{-π+1+1,0},{-π+1+1,Tan[θ]}}]},
             {Dashing[{0.02, 0.01}],
          {Thickness[0.001], Line[{{-π+1+1,Tan[θ]},{θ,Tan[θ]}}]},
```

```
            {Thickness[0.008],Hue[0.4], Line[{{θ,Tan[θ]},{θ,0}}]}},
            {Thickness[0.005],Hue[0.7],Line[{{0,0},{θ,0}}]}};
      cls={Thickness[0.001],Circle[{-π+1,0},1],
                        Circle[{-π+1,0},0.2,{0,Mod[θ,2π]}],
            {Thickness[0.005],Hue[0.7],Circle[{-π+1,0},1,{0,θ}]}};
      Plot[Tan[t],{t,0,θ},
        PlotStyle→{RGBColor[1,0,0],Thickness[0.01]},
        PlotLabel→ FontForm["The graph of tanθ when the angle is"<>
          ToString[θ 180/π]<>"°",{"Helvetica-Bold",16}],
        Background→GrayLevel[1],
        Epilog→{cls,lns}, DefaultFont→{"Times",14},
        Ticks→{{0,π/4, π/2, 3π/4, π, 5π/4, 3π/2, 7π/4, 2π},{-1,0,1}},
        AspectRatio→Automatic, ImageSize→450,
        PlotRange→{{-π, 2π}, {-3.8,3.8}}];];
```

[3] Program[3]

```
Table[Plot[a Sin[x],{x,-2π,2π},
        PlotStyle→{RGBColor[0,1,0],Thickness[0.01]},
        PlotLabel→FontForm[
          "y=" <>ToString[a]<> "sin x   from   y="<>ToString[-a]<> " to   y="<>
              ToString[a]<> " ", {"Helvetica-Bold",16}],
        Background→GrayLevel[1], DefaultFont→{"Times",14},
        Ticks→{{-2π, -3π/2, -π, -π/2, 0, π/2, π, 3π/2, 2π},
              {-5,-4,-3,-2,-1,0,1,2,3,4,5}},
        GridLines→{{-2π, -π, 0, π, 2π},{-5,-4,-3,-2,-1,0,1,2,3,4,5}},
        AspectRatio→Automatic, ImageSize→430,
        PlotRange→{{-2π, 2π},{-5,5}}];,{a,1,5,1}];
```

[4] Program[4]

```
Clear[k, a1];
  k[a1_] :=
    Module[{},
lns = {{Thickness[0.007], Hue[0.1], Line[{{-4, -4}, {4, 4}}]},
  {Thickness[0.01], Hue[0.3], Dashing[{0.02, 0.02}],
If[a1 > 1, Line[{{1, a1^1}, {1, 0}}], Line[{{-1, a1^1}, {-1, 0}}]]},
  {Thickness[0.01], Hue[0.3], Dashing[{0.02, 0.02}],
If[a1 > 1, Line[{{1, a1^1}, {0, a1^1 }}], Line[{{-1, a1^-1}, {0, a1^-1 }}]]},
```

```
{Thickness[0.01], Hue[0.33], Dashing[{0.02, 0.02}],
If[a1 > 1, Line[{{a1, 1}, {a1, 0}}], Line[{{a1^-1, -1}, {0, -1}}]]},
{Thickness[0.01], Hue[0.33], Dashing[{0.02, 0.02}],
If[a1 > 1, Line[{{a1, 1}, {0, 1}}], Line[{{a1^-1, -1}, {a1^-1, 0}}]]}};
pts = {{PointSize[0.02], Hue[0], Point[{0, 1}]},
{PointSize[0.02], Hue[0], Point[{1, 0}]},
{PointSize[0.02], Hue[0], If[a1 > 1, Point[{1, a1^1}], Point[{-1, a1^-1}]]},
{PointSize[0.02], Hue[0], If[a1 > 1, Point[{a1, 1}], Point[{a1^-1, -1}]]}};
Plot[{a1^x, Log[a1, x]}, {x, -4, 4},
AspectRatio -> Automatic, GridLines -> Automatic,
PlotStyle -> {{Thickness[0.015], Hue[0.7]},
{Thickness[0.015], Hue[0.9]}},
AxesStyle -> {Thickness[0.015]},
Epilog -> {lns, pts}, PlotRange -> {{-4, 4}, {-4, 4}}, ImageSize -> 450,
Prolog -> {{Text[StyleForm["y=" <> ToString[a1] <> ""^x,
FontSize -> 24, FontColor -> Hue[ .7]], {-2.5, 3}]},
{Text[StyleForm["y=log[" <> ToString[N[a1]] <> ",x]",
FontSize ->24, FontColor -> Hue[ .9]], {2, -2}]},
{Text[StyleForm["y=x", FontSize ->24, FontColor -> Hue[ .1]], {3.2, 2.4}]}},
DefaultFont ->{"Times", 20},
PlotLabel ->"Base   a=" <> ToString[N[a1]]];]
```

[5] Program[5]

Play[Which[0<x<0.5, $\dfrac{\sin[1000x2^{\frac{3}{12}}]}{e^{x2^{\frac{3}{12}}}}\sin[1000x2^{\frac{3}{12}}]$,

0.5<x<0.75, $\dfrac{\sin[1000(x-0.5)2^{\frac{3}{12}}]}{e^{(x-0.5)2^{\frac{3}{12}}}}\sin[1000(x-0.5)2^{\frac{3}{12}}]$,

0.75<x<1, $\dfrac{\sin[1000(x-0.75)2^{\frac{5}{12}}]}{e^{(x-0.75)2^{\frac{5}{12}}}}\sin[1000(x-0.75)2^{\frac{5}{12}}]$,

1<x<2, $\dfrac{\sin[1000(x-1)2^{\frac{3}{12}}]}{e^{(x-1)2^{\frac{3}{12}}}}\sin[1000(x-1)2^{\frac{3}{12}}]$,

$$2< x <2.5,\quad \frac{\sin[1000(x-2)2^{\frac{0}{12}}]}{e^{(x-2)2^{\frac{0}{12}}}}\sin[1000(x-2)2^{\frac{0}{12}}],$$

$$2.5< x <2.75,\quad \frac{Sin[1000(x-2.5)2^{\frac{0}{12}}]}{e^{(x-2.5)2^{\frac{0}{12}}}}\sin[1000(x-2.5)2^{\frac{0}{12}}],$$

$$2.75< x <3,\quad \frac{\sin[1000(x-2.75)2^{\frac{1}{12}}]}{e^{(x-2.75)2^{\frac{1}{12}}}}\sin[1000(x-2.75)2^{\frac{1}{12}}],$$

$$3< x <4,\quad \frac{\sin[1000(x-3)2^{\frac{0}{12}}]}{e^{(x-3)2^{\frac{0}{12}}}}\sin[1000(x-3)2^{\frac{0}{12}}],$$

$$4< x <4.75,\quad \frac{\sin[1000(x-4)2^{\frac{-2}{12}}]}{e^{(x-4)2^{\frac{-2}{12}}}}\sin[1000(x-4)2^{\frac{-2}{12}}],$$

$$4.75< x <5,\quad \frac{\sin[1000(x-4.75)2^{\frac{-4}{12}}]}{e^{(x-4.75)2^{\frac{-4}{12}}}}\sin[1000(x-4.75)2^{\frac{-4}{12}}],$$

$$5< x <5.5,\quad \frac{\sin[1000(x-5)2^{\frac{-7}{12}}]}{e^{(x-5)2^{\frac{-7}{12}}}}\sin[1000(x-5)2^{\frac{-7}{12}}],$$

$$5.5< x <6,\quad \frac{\sin[1000(x-5.5)2^{\frac{-4}{12}}]}{e^{(x-5.5)2^{\frac{-4}{12}}}}\sin[1000(x-5.5)2^{\frac{-4}{12}}],$$

$$6< x <8,\quad \frac{\sin[1000(x-6)2^{\frac{-9}{12}}]}{e^{(x-6)2^{\frac{-9}{12}}}}\sin[1000(x-6)2^{\frac{-9}{12}}],$$

$$8< x <8.5,\quad \frac{\sin[1000(x-8)2^{\frac{-7}{12}}]}{e^{(x-8)2^{\frac{-7}{12}}}}\sin[1000(x-8)2^{\frac{-7}{12}}],$$

$$8.5< x <9,\quad \frac{\sin[1000(x-8.5)2^{\frac{-4}{12}}]}{e^{(x-8.5)2^{\frac{-4}{12}}}}\sin[1000(x-8.5)2^{\frac{-4}{12}}],$$

$$9< x <9.5,\quad \frac{\sin[1000(x-9)2^{\frac{-7}{12}}]}{e^{(x-9)2^{\frac{-7}{12}}}}\sin[1000(x-9)2^{\frac{-7}{12}}],$$

$$9.5< x <10,\ \frac{\sin[1000(x-9.5)2^{\frac{-9}{12}}]}{e^{(x-9.5)2^{\frac{-9}{12}}}}\sin[1000(x-9.5)2^{\frac{-9}{12}}],$$

$$10< x <10.5,\ \frac{\sin[1000(x-10)2^{\frac{-4}{12}}]}{e^{(x-10)2^{\frac{-4}{12}}}}\sin[1000(x-10)2^{\frac{-4}{12}}],$$

$$10.5\ x <11,\ \frac{\sin[1000(x-10.5)2^{\frac{0}{12}}]}{e^{(x-10.5)2^{\frac{0}{12}}}}\sin[1000(x-10.5)2^{\frac{0}{12}}],$$

$$11< x <12,\ \frac{\sin[1000(x-11)2^{\frac{3}{12}}]}{e^{(x-11)2^{\frac{3}{12}}}}\sin[1000(x-11)2^{\frac{3}{12}}],$$

$$12< x <12.5,\ \frac{\sin[1000(x-12)2^{\frac{3}{12}}]}{e^{(x-12)2^{\frac{3}{12}}}}\sin[1000(x-12)2^{\frac{3}{12}}],$$

$$12.5< x <12.75,\ \frac{\sin[1000(x-12.5)2^{\frac{5}{12}}]}{e^{(x-12.5)2^{\frac{5}{12}}}}\sin[1000(x-12.5)2^{\frac{5}{12}}],$$

$$12.75< x <13,\ \frac{\sin[1000(x-12.75)2^{\frac{3}{12}}]}{e^{(x-12.75)2^{\frac{3}{12}}}}\sin[1000(x-12.75)2^{\frac{3}{12}}],$$

$$13< x <13.5\ \frac{\sin[1000(x-13)2^{\frac{0}{12}}]}{e^{(x-13)2^{\frac{0}{12}}}}\sin[1000(x-13)2^{\frac{0}{12}}],$$

$$13.5<x<14,\ \frac{\sin[1000(x-13.5)2^{\frac{-2}{12}}]}{e^{(x-13.5)2^{\frac{-2}{12}}}}\sin[1000(x-13.5)2^{\frac{-2}{12}}],$$

$$14<x<16,\ \frac{\sin[1000(x-14)2^{\frac{-4}{12}}]}{e^{(x-14)2^{\frac{-4}{12}}}}\sin[1000(x-14)2^{\frac{-4}{12}}]],\{x,0,16\}];$$