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NONVANISHING OF A PLURIGENUS OF A THREEFOLD OF GENERAL TYPE

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ABSTRACT. When X is a threefold of general type, it is well known $h^0(X, \mathcal{O}_X(nK_X)) \geq 1$ for a sufficiently large n. When $\chi(\mathcal{O}_X) > 0$, it is not easy to obtain such an integer n. A. R. Fletcher showed that $h^0(X, \mathcal{O}_X(nK_X)) \geq 1$ for $n = 12$ when $\chi(\mathcal{O}_X) = 1$. We introduce a technique different from A. R. Fletcher's. Using this technique, we also prove the same result as he showed and have a new result.

Throughout this paper, we are working over the complex number field $\mathbb{C}$.

When X is a threefold of general type with a canonical divisor K_X, it is well known that $h^0(X, \mathcal{O}_X(nK_X))$ is not zero for a sufficiently large n. When $\chi(\mathcal{O}_X) \leq 0$, we have $h^0(X, \mathcal{O}_X(nK_X)) \neq 0$ for $n \geq 2$. But when $\chi(\mathcal{O}_X) > 0$, it is not easy to obtain such an integer n.

M. Reid and A. R. Fletcher gave the formula for $\chi(\mathcal{O}_X(nK_X))$ in Reid [3] and Fletcher [1]:

$$\chi(\mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12}{K_X}^3 + (1-2n)\chi(\mathcal{O}_X) + \sum_Q l(Q,n),$$

where the summation is over a basket of singularities. In fact, singularities in the basket are not necessarily the singularities which appear on X. However, the singularities of X make the contribution as if they were in the basket. For detailed explanations about a basket of singularities, see Reid [3] or Kawamata [2]. The exact formula for $l(Q,n)$ is given as follows:

$$l(Q,n) = \sum_{i=1}^{n-1} \frac{\overline{ib}(r-\overline{ib})}{2r},$$

where Q is a singularity of type $\frac{1}{r}(1,-1,b)$, $(r,b)=1$ and $\overline{ib}$ is the least residue of ib modulo r. For detailed explanations about $l(Q,n)$ and types

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of singularities, see Reid [3] or Fletcher [1]. For the sake of simplicity, let's denote $\sum_Q l(Q,n)$ by $l(n)$ and $\frac{1}{r}(1,-1,b)$ by b/r, unless there is some confusion. Let's identify the type b/r with the number b/r in the interval $(0,1]$. Then we can describe our situation for a computation of $l(n)$ very easily.

Combining above formula with the Kawamata-Viehweg Vanishing Theorem, we have the following proposition. Since a plurigenus is a birational invariant, we may assume that X is a canonical threefold from now on.

PROPOSITION 1. *For all $n \geq 2$,*

$$p_n \overset{def}{:=} h^0(X, \mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12} K_X{}^3 + (1-2n)\chi(\mathcal{O}_X) + l(n).$$

A. R. Fletcher introduced an interesting method. Using his method, he showed $p_{12} \geq 1$ when $\chi(\mathcal{O}_X) = 1$ (see Fletcher [1]). We are going to introduce a technique different from A. R. Fletcher's. Using this technique, we have also the same result as he showed and a new result. See Theorem 1 and Theorem 2. To get these results, we are going to find necessary conditions to be $p_{12} = 0$ which are different from A. R. Fletcher's. Our necessary conditions are mutually disjoint tables T_1, T_2 and T_3 which are given in the end of this paper. Tables T_1, T_2 and T_3 help us to find the combinatorial ideas and the directions for the proofs.

Let's state our preliminary lemmas.

LEMMA 1. *Let Q be a point of type b/r. Let $k = min\{b, r-b\}$. Then we have $\overline{nb}(r-\overline{nb}) = \overline{nk}(r-\overline{nk})$ for a positive integer n.*

PROOF. If $k = r - b$, then we have

$$\overline{nk} \equiv \overline{nr - nb} \equiv -\overline{nb} \mod r.$$

From the graph of $x(r-x)$, we have

$$\overline{nb}(r-\overline{nb}) = \overline{nk}(r-\overline{nk}).$$

□

Note that $k \leq \frac{r}{2}$. By Lemma 1, we may assume that the basket contains only types of the form k/r because b/r and k/r produce the same $l(Q,n)$. In our sense, a singularity of type k/r stands for a singularity of type k/r or $(r-k)/r$. In either case, we have the same value for $l(Q,n)$. Thus, it is enough to consider a point of type k/r for a computation of $l(Q,n)$.

LEMMA 2. *If the basket contains a singularity of type k/r such that*

$$r \geq \frac{(12\chi(\mathcal{O}_X)-1)(24\chi(\mathcal{O}_X)-1)}{2(6\chi(\mathcal{O}_X)-1)},$$

then $p_{12\chi(\mathcal{O}_X)} \geq 1$.

For a proof of Lemma 2, see Fletcher [2, 4.9].

When $\chi(\mathcal{O}_X) = 1$, Lemma 2 implies $p_{12} \neq 0$ if a basket contains a point of type $r \geq 26$. Thus, it is enough to consider finite cases k/r, $(r < 26)$ to find possible candidates for the basket to be $p_{12} = 0$ when $\chi(\mathcal{O}_X) = 1$.

LEMMA 3. (1) *If* $p_2 = p_3 = 0$, *then* $\chi(\mathcal{O}_X) = \frac{1}{10}\sum k$.

(2) *If* $p_2 = p_4 = 0$, *then* $\chi(\mathcal{O}_X) = \frac{1}{35}\sum \frac{11kr - 9k^2 - \overline{3k}(r-\overline{3k})}{2r}$.

PROOF. Eliminate ${K_X}^3$ from the system of equations $p_2 = p_3 = 0$. Express $\chi(\mathcal{O}_X)$ in other terms. Then we have $\chi(\mathcal{O}_X) = \frac{1}{10}\sum k$. We can get also the other case by the same process from the system of equations $p_2 = p_4 = 0$. □

The term $\frac{11kr - 9k^2 - \overline{3k}(r-\overline{3k})}{2r}$ in (2) of Lemma 3 can be expressed as follows:

$$\frac{11kr - 9k^2 - \overline{3k}(r-\overline{3k})}{2r} = \begin{cases} 4k & \text{if } k/r \leq 1/3 \\ r+k & \text{if } 1/3 < k/r \leq 1/2. \end{cases}$$

Note that $r+k$ has the smallest value 3 which comes from one point of type $1/2$. The next one is 6 which comes from two points of type $1/2$.

Let X be a canonical threefold with $\chi(\mathcal{O}_X) = 1$.

Suppose that $p_{12} = 0$. Then we have $p_2 = p_3 = p_4 = p_6 = 0$.

Now, we are going to consider the following equations which are linear combinations of several $p_n = 0$. The reason for these equations is to get equations of ${K_X}^3$ and $l(n)$ only:

$$\begin{aligned} 0 &= p_{12} - p_4 - 2p_3 - 2p_2 \\ 0 &= p_{12} - p_3 - 6p_2 \\ 0 &= p_6 - p_3 - 2p_2. \end{aligned}$$

Then from Proposition 1, we have

$$0 = 240{K_X}^3 + l(12) - l(4) - 2l(3) - 2l(2)$$
$$0 = \frac{495}{2}{K_X}^3 + l(12) - l(3) - 6l(2)$$
$$0 = 24{K_X}^3 + l(6) - l(3) - 2l(2).$$

Each of $l(12)-l(4)-2l(3)-2l(2)$, $l(12)-l(3)-6l(2)$ and $l(6)-l(3)-2l(2)$ must be negative, since ${K_X}^3$ is positive. It means that the basket must contain points Q_1, Q_2 and Q_3 each of which contributes a negative value to the corresponding equation, i.e., Q_1, Q_2 and Q_3 such that

$$g_1(Q_1) := l(Q_1, 12) - l(Q_1, 4) - 2l(Q_1, 3) - 2l(Q_1, 2) < 0$$
$$g_2(Q_2) := l(Q_2, 12) - l(Q_2, 3) - 6l(Q_2, 2) < 0$$
$$g_3(Q_3) := l(Q_3, 6) - l(Q_3, 3) - 2l(Q_3, 2) < 0.$$

For the sake of simplicity, let's denote each of above equations by g_i respectively.

Now, we are going to search points Q_i of type k/r such that $g_i(Q_i) < 0$. By Lemma 2 and Lemma 3, it is enough to test the type k/r such that $2 \leq r \leq 25$ and $1 \leq k \leq 10$.

Finally, we obtained tables T_i from each of $g_i < 0\,(1 \leq i \leq 3)$ using the computer program. Note that T_1, T_2 and T_3 are mutually disjoint. Thus, the basket must contain the point Q_i from each of the tables. We attach tables T_1, T_2 and T_3 at the end. We also attach some explanation about the tables.

Now, we are ready to describe the possible basket to be $p_{12} = 0$.

(1) The basket must contain at least one point from each table of T_1, T_2 and T_3, since the tables are disjoint.

(2) By Lemma 3, the points in the basket must satisfy the following :

$$10\chi(\mathcal{O}_X) = \sum k$$
$$35\chi(\mathcal{O}_X) = 4t + s,$$

where $4t$ is the nonnegative term due to the points of type$\leq 1/3$, and s is the term due to the points of type$> 1/3$.

Note that the point of the same type can appear several times in the basket. Note that the several points in the same table can appear in the basket.

The following result is already known in Fletcher [1]. We are going to prove again using the technique different from he used.

THEOREM 1. *For a smooth threefold X of general type with $\chi(\mathcal{O}_X) = 1$, then we have $p_{12} \geq 1$.*

PROOF. We may assume that X is a canonical model.

Assume that $p_{12} = 0$. As we explained above, the basket must contain at least one point from each of tables T_1, T_2 and T_3 which are mutually disjoint.

Comparing the first and the third columns of two tables T_1 and T_3, we can notice that the positive value is bigger or equal to the negative value except the type 2/5. Thus, consider the add-up of the following two equations:

$$p_{12} - p_4 - 2p_3 - 2p_2 = 0 \text{ and } p_6 - p_3 - 2p_2 = 0.$$

Then we have

$$0 = 264{K_X}^3 + (l(12) - l(4) - 2l(3) - 2l(2)) + (l(6) - l(3) - 2l(2)).$$

Since ${K_X}^3 > 0$, the basket must contain a point which contributes a negative value to

$$(l(12) - l(4) - 2l(3) - 2l(2)) + (l(6) - l(3) - 2l(2)),$$

i.e., a point Q such that $g_1(Q) + g_3(Q) < 0$. Clearly, such a point must come from table T_1 or T_3. As we mentioned above, the point of type 2/5 is the only point such that $g_1(Q) + g_3(Q) < 0$. Hence the basket of singularities should be of the following form:

$$\{ \text{ some point from the table } T_1,\ 1/2,\ 2/5,\ \cdots\}.$$

Comparing the first and the second columns of two tables T_1 and T_2, we can notice that the positive value is bigger or equal to the negative value except the type 1/2 and 4/11. Since the type 1/2 must appear in the basket, we are going to divide into the following two cases:

Case (1) the basket does not contain a point of type 4/11 in T_1,
Case (2) the basket contains a point of type 4/11 in T_1.

Case (1). The basket does not contain a point of type 4/11 in T_1.

One single point of type 1/2 in table T_2 contributes only -0.25 to $l(12) - l(3) - 6l(2)$. One single point in table T_1 contributes to $l(12) - l(3) - 6l(2)$ at least $0.57\cdots$ since the type 4/11 is not in the basket by our assumption. Since the basket must contain a point from table T_1, the basket must contain at least 3 points of type 1/2 to get a negative value of $l(12) - l(3) - 6l(2)$.

One single point of type 2/5 in table T_3 contributes -0.2 to $l(6) - l(3) - 2l(2)$. One single point in table T_1 contributes to $l(6) - l(3) - 2l(2)$ at least $0.28\cdots$ since the type $4/11$ is not in the basket by our assumption. To get a negative value of $l(6) - l(3) - 2l(2)$, the basket must contain an additional point from table T_3. We already have 3 points of type $1/2$, a point from T_1 and a point of type $2/5$ which must appear in the basket. Thus, the basket can contain no other additional point from T_3 than a point of type $2/5$, since $\sum_i k_i = 10$ by (1) of Lemma 3. Hence the basket of singularities should be of the following form:

$$\{\text{some point from the table } T_1,\ 3 \times 1/2,\ 2 \times 2/5,\ \cdots\}.$$

Moreover, by (1) of Lemma 3 again, a point of type $2/7$ or $3/11$ is only a candidate for a point from T_1 which must appear in the basket. A single point of type $3/11$ contributes at least 1.0 to $l(12) - l(3) - 6l(2)$. If a point of type $3/11$ is contained in the basket, the basket must have at least 5 points of type $1/2$ to get a negative value of $l(12) - l(3) - 6l(2)$. It contradicts (1) of Lemma 3. Hence, by (1) of Lemma 3, the basket for case (1) should be of the following form:

$$\{2/7,\ 3 \times 1/2,\ 2 \times 2/5,\ \frac{1}{r}\}.$$

By (2) of Lemma 3, we have the following equations:

$$4 \cdot 2 + 3(1+2) + 2(2+5) + \beta = 35,$$

where β are the term due to the last point of type $1/r$ in the basket. We have $\beta = 4$.

Let's determine r. From $p_2 = 0$, we have

$$\begin{aligned} 0 = p_2 &= \frac{1}{2} K_X{}^3 - 3\chi(\mathcal{O}_X) + l(2) \\ &= \frac{1}{2} K_X{}^3 - 3 + \frac{5}{7} + \frac{3}{4} + \frac{6}{5} + \frac{r-1}{2r}. \end{aligned}$$

Since $K_X{}^3$ is positive, $0 < \frac{1}{2}K_X{}^3 = \frac{1}{2r} - \frac{23}{140}$. The integer ≥ 2 which satisfies above inequality are 2 or 3. When $r = 2$, we have $\beta = 3$ by the explanation just below Lemma 3. Hence we have a following possible candidate for a basket to be $p_{12} = 0$:

$$\{2/7,\ 3 \times 1/2,\ 2 \times 2/5,\ 1/3\}.$$

Case (2). The basket contains a point of type $4/11$ in table T_1.

In this case, the possible candidate is the following:

$$\{4/11,\ 1/2,\ 2/5,\ \cdots\}.$$

By Lemma 3, we have the following:

$$4+1+2+\alpha=10$$
$$15+3+7+\beta=35,$$

where $\alpha,\ \beta$ are the nonnegative terms due to the other points not listed explicitly in the above basket. We have $\alpha=3,\ \beta=10$. Since β is of the form $4t+s$ ($s=0$, or $s\geq 3$), by the explanation just below Lemma 3, there are two possible combinations for $\beta=10$:

$$t=0,\ s=10 \text{ or } t=1,\ s=6.\ \cdots\ (*)$$

The basket $\{4/11,\ 1/2,\ 2/5,\ \cdots\}$ must have more additional points but contain at most one more additional point of type $\leq 1/3$ since $0\leq t\leq 1$. The fact $\alpha=3$ implies that the basket can have at most three more additional points since $\sum_i k_i=10$. Hence let's divide our case into the three subcases depending on the number of the additional points we can add to the basket.

Subcase 2-(1). The basket contains 3 more extra points $k_i/r_i(1\leq i\leq 3)$.

Then each k_i must be 1 since $\alpha=3$. Moreover, $\beta\neq 10$ from three points of type $>1/3$ since each k_i is 1. Thus, by $(*)$, the basket must contain one additional point of type $\leq 1/3$ and two additional points of type $>1/3$. Hence, the additional points which are not listed explicitly in the basket are two points of type $1/2$ and one point of type $1/r$ ($r\geq 3$) since all the k_i are 1. Thus, we have the following basket:

$$\{4/11,\ 3\times 1/2,\ 2/5,\ 1/r(r\geq 3)\}.$$

Now, let's determine r. From $p_2=0$, we have

$$0=p_2=\frac{1}{2}K_X{}^3-3+\frac{14}{11}+\frac{3}{4}+\frac{3}{5}+\frac{r-1}{2r}.$$

Since $K_X{}^3$ is positive, we must have

$$0<\frac{1}{2}K_X{}^3=\frac{1}{2r}-\frac{27}{220}.$$

The integer which satisfies above inequality is 3 or 4. Hence we have the following possible candidates for a basket to be $p_{12}=0$:

$$\{4/11,\ 3\times 1/2,\ 2/5,\ 1/3\} \text{ or } \{4/11,\ 3\times 1/2,\ 2/5,\ 1/4\}.$$

Subcase 2-(2). The basket contains 2 more extra points k_i/r_i ($i=1,\ 2$).

In this case, we may assume $k_1=1,\ k_2=2$ since $\sum_i k_i=3$.

If $1/r_1 \leq 1/3$, then the type of the other point is greater than $1/3$ because the basket can contain at most one additional point of type $\leq 1/3$ more by $(*)$. Since $\beta = 10$, we must have

$$10 = 4 + (2 + r_2).$$

Hence $r_2 = 4$. It is impossible since k_2 is relatively prime to r_2. Hence $1/r_1 > 1/3$, i.e., $r_1 = 2$.

If $2/r_2 \leq 1/3$, then we must have $\beta = 11$ which contradicts. Thus $2/r_2 > 1/3$. The only possible integer for r_2 is 5 by $(*)$ since $\beta = 10$. Hence we have a following possible candidate for a basket to be $p_{12} = 0$:

$$\{4/11,\ 2 \times 1/2,\ 2 \times 2/5\}.$$

Subcase 2-(3). The basket contains only one more extra point k/r. Then $k = 3$. If $3/r \leq 1/3$, then $\beta = 12$ which contradicts. Thus $3/r > 1/3$. Then $r = 7$ by $(*)$. Hence we have a following possible candidate for a basket to be $p_{12} = 0$:

$$\{4/11,\ 1/2,\ \ 2/5,\ 3/7\}.$$

Now, we have all possible baskets to be $p_{12} = 0$:

$$\{2/7, 3 \times 1/2, 2 \times 2/5, 1/3\},\ \{4/11, 3 \times 1/2, 2/5, 1/3\},$$

$$\{4/11, 3 \times 1/2, 2/5, 1/4\},\ \{4/11, 2 \times 1/2, 2 \times 2/5\},\ \{4/11, 1/2,\ 2/5, 3/7\}.$$

The last step is as follows: for $(n = 2, 3, 4, 6, 12)$,

(1) plugging in all these baskets into equations $p_n = 0$,

(2) check and compare $K_X{}^3$ from each equation $p_n = 0$ which must be positive and equal to each other.

Using a computer program, we check all the baskets given above. Then we have that $K_X{}^3$ from $p_2 = 0$ is not equal to $K_X{}^3$ from $p_{12} = 0$ in all the cases. For example, for the basket $\{2/7,\ 3 \times 1/2,\ 2 \times 2/5,\ 1/3\}$ $K_X{}^3$ from $p_2 = 0$ is $0.0047\cdots$, but $K_X{}^3$ from $p_{12} = 0$ is $0.000809\cdots$.

Therefore, we have $p_{12} \neq 0$. $\square$

Using this technique, we have the following results - Theorem 2 and Corollary 1.

Let X be a canonical threefold. Suppose that $p_{24} = 0$. Then we have $p_n = 0$ for $n = 2,\ 3,\ 4$. Consider the following equation:

$$p_{24} - 2p_4 - 3p_3 - 6p_2 = 0.$$

We have

$$\frac{4275}{2} K_X{}^3 + l(24) - 2l(4) - 3l(3) - 6l(2) = 0. \ \cdots\ (**)$$

Since ${K_X}^3$ is positive, the basket must contain a point Q which contributes a negative value to

$$l(24) - 2l(4) - 3l(3) - 6l(2),$$

i.e., a point Q such that

$$g_4 :\stackrel{def}{=} l(Q,24) - 2l(Q,4) - 3l(Q,3) - 6l(Q,2) < 0$$

is negative. The list for such points Q is given in table T_4. By table T_4, we have the following results.

THEOREM 2. *Let X be a canonical threefold. If the basket of X does not contain a point of type k/r such that $r = 2$ or $r > 226$, then we have $p_{24} \neq 0$.*

PROOF. It is clear by table T_4. □

COROLLARY 1. *Let X be a canonical threefold. Suppose that the basket of X does not contain a point of type k/r with $r > 226$. If $p_{24} = 0$, then we have ${K_X}^3 < \frac{1}{855}\chi(\mathcal{O}_X)$.*

PROOF. Since $p_{24} = 0$, the basket must contain a point of type $1/2$ from table T_4. Let t be the number of points of type $1/2$. Then from $(**)$, we have

$$\frac{4275}{2}{K_X}^3 - \frac{1}{4}t < 0.$$

Note that if the basket consists only of points of type $1/2$, then $p_2 \neq 0$. Thus

$$\frac{4275}{2}{K_X}^3 < \frac{1}{4}t < \frac{1}{4}\sum_i k_i = \frac{10}{4}\chi(\mathcal{O}_X).$$

Hence we have ${K_X}^3 < \frac{1}{855}\chi(\mathcal{O}_X)$. □

REMARK 1. Here, we attach the tables which explained above Theorem 1. Each table T_i shows the list of points such that $g_i(Q) < 0$. Three tables have mutually disjoint lists of points. It implies that the basket must contain at least one point from each list. Note that $r \leq 25$ and $k \leq 10$.

Each table T_i shows the negative values of g_i and the values of other functions g_j at Q together. The numbers in the tables are not the exact values but approximations. Approximation value is correct up to 7 digit below a point. The last digit is determined by a round up. Approximation value is enough because we are using the tables only to compare the values.

The different combinations of p_n give the different tables. There are many different combinations of p_n. Maybe, we may expect a better result from different tables from ours.

T_1:Table for $g_1(Q) < 0$

Type	$g_1(Q)$	$g_2(Q)$	$g_3(Q)$
2/7	-0.14285714	0.57142857	0.28571429
3/11	-0.36363636	1.0	0.36363636
4/11	-0.09090909	0	0.09090909
5/18	-0.33333333	1.75	0.66666667
7/25	-0.4	2.4	0.96

T_2: Table for $g_2(Q) < 0$

Type	$g_1(Q)$	$g_2(Q)$	$g_3(Q)$
1/2	0	-0.25	0

T_3:Table for $g_3(Q) < 0$

Type	$g_1(Q)$	$g_2(Q)$	$g_3(Q)$
2/5	0	0	-0.2
5/13	0.46153846	0.53846154	-0.15384615
7/17	1.23529412	1.11764706	-0.17647059
7/18	0.66666667	0.75	-0.33333333
8/21	0.57142857	0.71428571	-0.14285714
9/22	1.36363636	1.25	-0.36363636
9/23	0.78260870	0.86956522	-0.52173913

REMARK 2. The table for $g_4(Q) < 0$ is very simple. There is no point between 1/2 and 113/227.

T_4: Table for $g_4(Q) < 0$

Type	$g_4(Q)$
1/2	-0.25
113/ 227	-0.22907489
$\vdots$	$\vdots$

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