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SEMI-PRIMENESS OF THE ENDOMORPHISM RING OF A PROJECTIVE MODULE

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ABSTRACT. Semi-meet-prime projective modules and fully invariant meet-prime submodules of a projective module are studied. Actually a generalization of the Schur's lemma and semi-prime endomorphism rings of projective modules are considered.

1. Introduction

Assume that ring R is any associative ring with identity. The ring of all R-endomorphisms on a left R-module ${}_RM$, denoted by $End_R(M)$, will be written on the right side of M as right operators on ${}_RM$, that is, ${}_RM_{End_R(M)}$ will be considered in this paper.

A module ${}_RM$ is said to be simple if 0 and M are the only submodules of ${}_RM$.

For any subset J of $End_R(M) = S$, let $\text{Im}J = MJ = \sum_{f\in J} \text{Im}f = \sum_{f\in J} Mf$ be the sum of images of endomorphisms in J.

Also we call N an *open* submodule if $N{=}N^o, N^o{=}\sum_{f\in S, \text{Im}f\leq N} \text{Im}f$, is the sum of all images of endomorphisms contained in N.

A left R-module ${}_RM$ is said to be *openly simple* if every *open* submodule is improper, that is, every *open* submodule is either 0 or M.

THEOREM 1.1 ([5]). **(Generalized Schur's Lemma I)** *Each projective and openly simple module ${}_RT$ has a division endomorphism ring $End_R(T)$.*

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Definition 1.2 ([4]). For a submodule $P \leq M$ of a left R-module ${}_RM$, P is said to be a *meet-prime* submodule of ${}_RM$ if it satisfies the following conditions: for any *open* submodules $A, B \leq M$ with $P^o + A \neq M$ or $P^o + B \neq M$,

(1) if $A \cap B \leq P$, then $A \leq P$ or $B \leq P$,
(2) if $(P \cap A \cap B)^o \neq 0$, then $A \leq P$ or $B \leq P$,
(3) if $P \cap A = 0$, then $A = 0$ or $P + A = M$.

One of the important results related to meet-prime submodules is as follows: If $P \leq M$ is any fully invariant meet-prime submodule of ${}_RM$, then

$$I^P = \{ f \in S \mid \mathrm{Im} f \leq P \} \trianglelefteq S \quad \text{is a prime ideal of } S.$$

Any ring R is said to be *semi-prime* if it has the zero intersection of all prime ideals, i.e., $\cap P_\alpha = 0$, for which P_α is a prime ideal of R.

Any left R-module ${}_RM$ is said to be *semi-meet-prime* if it has the zero intersection of all *fully invariant meet-prime* open submodules of ${}_RM$, i.e., $\cap P_\alpha = 0$, for which P_α is a *fully invariant meet-prime* open submodule of ${}_RM$.

Since the next proposition is the same as ${}_RM$ with epimorphisms $\phi_\gamma : {}_RM \to {}_RM_\gamma$ is a subdirect product of the modules ${}_RM_\gamma$ if and only if $\cap_\gamma \ker \phi_\gamma = 0$, the definition of P-reject of a module studied in [1] will not be introduced here.

Proposition 1.3 [2]. *An R-module M is a subdirect product of a class $\mathcal{U}$ of left R-modules if and only if the P-reject of M in $\mathcal{U}$ is zero.*

2. Results

Theorem 2.1. *For any module ${}_RM$, the following are equivalent:*

(1) *${}_RM$ is semi-meet prime;*
(2) *${}_RM$ is a subdirect product of openly simple modules.*

Proof. Since the meet-prime radical $rad(M) = \cap_\alpha P_\alpha = \cap_\alpha {P_\alpha}^o$, for which P_α is a meet-prime submodule of ${}_RM$ for α the proof is completed easily by applying the Corollary 2.10 in [4]. □

Since distinct fully invariant meet-prime open submodules $P \leq M$ and $Q \leq M$ of any left R-module ${}_RM$ have the sum $P + Q = M$, we have that ${}_RM/P = {}_R(P+Q)/P \simeq {}_RQ/(P\cap Q) \to {}_RP/(P\cap Q) \simeq {}_RM/Q$ is the only trivial homomorphism. Thus we have a next remark.

REMARK 2.2. If P and Q are distinct fully invariant meet-prime open submodules of any self-generated left R-module ${}_RM$, then the additive group

$$Hom_R(M/P, M/Q) = 0.$$

LEMMA 2.3. *For a fully invariant meet-prime submodule P of a left R-module ${}_RM$, if ${}_RM$ is self-generated, then we have the openly simple quotient R-module ${}_RM/P$.*

PROOF. It is elementary. □

THEOREM 2.4. *For any self-generated R-module ${}_RM$, if ${}_RM$ is semi-meet prime, then the endomorphism ring $End_R(M)$ is a sudirect sum of prime rings. Furthermore the endomorphism ring $End_R(M)$ is a semi-prime ring.*

PROOF. The proof is completed by Theorem 2.1 and Lemma 2.3. □

Assume that a left R-module ${}_RM$ is projective. Then for any fully invariant meet-prime submodule $P \leq M$ in ${}_RM$, we let $f : {}_RM/P \to {}_RM/P$ be any endomorphism on the quotient module ${}_RM/P$ over ring R. Then we have that $P \leq K = \pi^{-1}(\mathrm{Im}f) \leq M$ and $K/P = \mathrm{Im}f$, where $\pi : {}_RM \to {}_RM/P$ is the projection. It suffices to show that K is an open submodule of ${}_RM$, because it follows that $K = M$ or $K = P$ from the meet-primeness of P. More precisely, to show that K is open in ${}_RM$ consider the following diagram:

$$\begin{array}{ccccc} {}_RM & \xrightarrow{\exists\, k} & {}_RK & & \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_K} & & \\ {}_RM/P & \xrightarrow{f} & {}_RK/P & \longrightarrow & 0. \end{array}$$

Since ${}_RM$ is projective for an epimorphic homomorphism $\pi f : {}_RM \xrightarrow{\pi} {}_RM/P \xrightarrow{f} {}_RK/P$, there is an endomorphism $k : {}_RM \to {}_RK \subseteq {}_RM$ such that $\pi f = k\pi_K$, where $\pi_K : {}_RK \to {}_RM/P$ is the restriction of π to K. Therefore $\mathrm{Im}k = K$ follows immediately. Thus $K = M$ or $K = P$ follows from the meet-primeness of $P \leq M$. Therefore the quotient module ${}_RM/P$ is openly simple, for each fully invariant meet-prime submodule $P \leq M$, if ${}_RM$ is projective.

As a result of this, we can generalize the Schur's lemma.

THEOREM 2.5. **(Generalized Schur's Lemma III)** *If ${}_RM$ is projective and if $P \leq M$ is any fully invariant meet-prime submodule of ${}_RM$, then $End_R(M/P)$ is a division ring.*

PROOF. For any fully invariant meet-prime submodule P, the quotient module ${}_RM/P$ is an openly simple module over ring R. Considering the following diagram:

$$\begin{array}{ccccc}
{}_RM & \xleftarrow{\exists g} & {}_RM & & \\
\| \downarrow & & \| \downarrow & & \\
{}_RM & \xrightarrow{\exists f_0} & {}_RM & & \\
\downarrow \pi & & \pi \downarrow & & \\
{}_RM/P & \xrightarrow{f} & {}_RM/P & \longrightarrow & 0 \\
& & \downarrow & & \\
& & 0 & &
\end{array}$$

for any non-zero endomorphism $f : {}_RM/P \to {}_RM/P$ we have an epimorphism f. We claim that f is an automorphism. Since ${}_RM$ is projective and $\pi f : {}_RM \xrightarrow{\pi} {}_RM/P \xrightarrow{f} {}_RM/P$ and $\pi : {}_RM \to {}_RM/P$ are epimorphisms, there are endomorphisms $f_0, g : {}_RM \to {}_RM$ such that $g\pi f = \pi$ and $f_0\pi = \pi f$. Now that f_0 has its induced homomorphism $f : {}_RM/P \to {}_RM/P$, in other words, $f = f_0^*$ is the induced homomorphism by f_0, then we conclude that $g^* f = 1_{{}_RM/P}$ follows from $gf_0\pi = \pi$. Thus f is an automorphism. Therefore the endomorphism ring $End_R(M/P)$ is a division ring. □

REMARK 2.6. Since every simple module ${}_RM$ over any ring R is projective and the trivial $0 \leq M$ is fully invariant meet-prime in ${}_RM$, the above Theorem 2.5 is a generalization of the Schur's lemma.

PROPOSITION 2.7. *For any projective module ${}_RM$, the following are equivalent:*

(1) *${}_RM$ is semi-meet prime;*
(2) *${}_RM$ is a subdirect product of openly simple projective modules;*
(3) *$End_R(M)$ is a subdirect sum of division rings.*

PROOF. It is sufficient to show that (3) implies (2) because the rest parts of proof directly follow from the previous results. Assume that $End_R(M)$ is a subdirect sum of division rings $\{E_\alpha\}$. Then there is a monomorphism $\iota : End_R(M) \to \prod_\alpha E_\alpha$ such that $End_R(M)\iota\pi_\alpha = E_\alpha$ for every α. From a construction of the direct product $\prod_\beta F_\beta \le \prod_\alpha E_\alpha$, with $F_\alpha = 0$ and $F_\beta = E_\beta$ whenever $\beta \ne \alpha$. Then $\iota^{-1}(\prod_\beta F_\beta) \le End_R(M)$ is a subring of the endomorphism ring $End_R(M)$ whose image $\mathrm{Im}(\iota^{-1}(\prod_\beta F_\beta)) = \sum_{f\in\iota^{-1}(\prod_\beta F_\beta)} \mathrm{Im} f \le M$ is an open meet-prime submodule of ${}_RM$, in fact it is the kernel $\ker \iota^{-1}(\prod_\gamma G_\gamma)$ of $\iota^{-1}(\prod_\gamma G_\gamma)$, where $G_\alpha = E_\alpha$ and $G_\gamma = 0$ if $\gamma \ne \alpha$ for every γ. Furthermore ${}_RM$ is a subdirect product of the openly simple modules, images $\{\mathrm{Im}(\iota^{-1}(\prod_\gamma G_\gamma))\}_\alpha$. Since E_α is a division ring for each α, we have an openly simple module $\mathrm{Im}(\iota^{-1}(\prod_\gamma G_\gamma))$ for γ. □

THEOREM 2.8. *For any projective module ${}_RM$, if at least one of the following equivalent conditions is satisfied:*

(1) *${}_RM$ is semi-meet prime;*
(2) *${}_RM$ is a subdirect product of openly simple projective modules;*
(3) *$End_R(M)$ is a subdirect sum of division rings,*

then we have a semi-prime endomorphism ring $End_R(M)$.

PROOF. It follows from Proposition 2.7. □

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