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ON FUZZY IMPLICATIVE FILTERS OF LATTICE IMPLICATION ALGEBRAS

YIQUAN ZHU, QUN ZHANG AND EUN HWAN ROH*

ABSTRACT. We investigate some related properties of fuzzy filters and fuzzy implicative filters in lattice implication algebras. We find a characterization of fuzzy filters and fuzzy implicative filters, and we discuss a relation between fuzzy filters and fuzzy implicative filters in lattice implication algebras. Also we give an extension theorem of fuzzy implicative filters.

1. Introduction

In the field of many-valued logic, lattice-valued logic plays an important role for two aspects: First, it extends the chain-type truth-value field of some well-known presented logics to some relatively general lattices. Second, the incompletely comparable property of truth value characterized by general lattice can more efficiently reflect the uncertainty of people's thinking, judging and decision. Hence, lattice-valued logic is becoming a research field which strongly influences the development of Algebraic Logic, Computer Science and Artificial Intelligence Technology.

In 1993, Xu [10] proposed the concept of lattice implication algebras, which combines lattice with implication algebra. He discussed their some properties in [9] and [10]. Xu and Qin [11] introduced the notion of filter in a lattice implication algebra, and investigated their properties. In [13], Xu and Qin defined the fuzzy filter in a lattice implication algebra, and they discussed their some properties. Roh et al.([6]) introduced the concept of fuzzy implicative filter in lattice implication algebras, and they obtained some their properties. This paper is continuation

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of the paper [6]. We find a characterization of fuzzy filters and fuzzy implicative filters, and we discuss a relation between fuzzy filters and fuzzy implicative filters in lattice implication algebras. Also we give an extension theorem of fuzzy implicative filters.

2. Preliminaries

We recall a few definitions and properties.

DEFINITION 2.1 ([10]). By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\rightarrow$ " satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

for all $x, y, z \in L$. A lattice implication algebra L is called a *lattice H implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in L$.

We can define a partial ordering $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $x \rightarrow y = 1$.

In a lattice implication algebra L, the following hold([10]): for all $x, y, z \in L$,

(1) $0 \rightarrow x = 1, 1 \rightarrow x = x$ and $x \rightarrow 1 = 1$,
(2) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,
(3) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
(4) $x \rightarrow ((x \rightarrow y) \rightarrow y) = 1$.

LEMMA 2.2 ([12]). *Let L be a lattice implication algebra. Then the following are equivalent: for all $x, y, z \in L$,*

(H1) *L is a lattice H implication algebra,*
(H2) $x \rightarrow (x \rightarrow y) = x \rightarrow y$,
(H3) $(x \rightarrow y) \rightarrow x = x$,
(H4) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Definition 2.3 ([12]). Let L be a lattice implication algebra. A subset F of L is called a *filter* of L if it satisfies the following axioms for all $x, y \in L$:

(F1) $1 \in F$,

(F2) $x \in F$ and $x \to y \in F$ imply $y \in F$.

A subset F of L is called an *implicative filter* of L, if it satisfies (F1) and

(F3) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$

for all $x, y, z \in L$.

We have the useful proposition.

Proposition 2.4. *Every filter F of a lattice implication algebra L has the following property:*

$$x \leq y \text{ and } x \in F \text{ imply } y \in F.$$

We now review some fuzzy logic concepts. Let X be a set. A function $\mu : X \to [0, 1]$ is called a *fuzzy subset* on X. Let μ be a fuzzy set in a set X. For $t \in [0, 1]$, the set $\mu_t := \{x \in X | \mu(x) \geq t\}$ is called a *level subset* of μ.

3. Main results

Definition 3.1 ([13]). A fuzzy subset μ of a lattice implication algebra L is called a *fuzzy filter* of L if it satisfies the following axioms:

(FF1) $\mu(1) \geq \mu(x)$ for all $x \in L$,

(FF2) $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$ for all $x, y \in L$.

Proposition 3.2 ([13]). *Let μ be a fuzzy filter of a lattice implication algebra L. Then for all $x, y \in L$, $x \leq y$ implies $\mu(x) \leq \mu(y)$.*

In [14], Xu et al. proved:

Proposition 3.3 ([14]). *Let μ be a fuzzy filter of a lattice implication algebra L. Then $x \leq y \to z$ implies $\mu(z) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y, z \in L$.*

Now we consider that the converse of Proposition 3.3, and obtain the following theorem.

Theorem 3.4. *A fuzzy subset μ of L is a fuzzy filter of a lattice implication algebra L if and only if for any $x, y, z \in L$, $x \leq y \to z$ implies that $\mu(z) \geq \min\{\mu(x), \mu(y)\}$.*

PROOF. Suppose that μ is a fuzzy subset of L and satisfying that $x \le y \to z$ implies $\mu(z) \ge \min\{\mu(x), \mu(y)\}$ for all $x, y, z \in L$. Since $x \le x \to 1$ for all $x \in L$, we have $\mu(1) \ge \{\mu(x), \mu(x)\} = \mu(x)$, and hence μ satisfies (FF1). Since $y \le (y \to x) \to x$, it follows that $\mu(x) \ge \min\{\mu(y \to x), \mu(y)\}$. Thus μ satisfies (FF2). Therefore μ is a fuzzy filter of L. □

Motivated by Theorem 4.7 in [6], Roh et al. stated the fuzzification of implicative filters.

DEFINITION 3.5 ([6]). A fuzzy subset μ of a lattice implication algebra L is called a *fuzzy implicative filter* if it satisfies (FF1) and

(FF3) $\mu(x) \ge \min\{\mu(z \to ((x \to y) \to x)), \mu(z)\}$

for all $x, y, z \in L$.

In [6], Roh et al. gave a relation between a fuzzy filter and a fuzzy implicative filter.

LEMMA 3.6 ([6]). *Any fuzzy implicative filter of a lattice implication algebra L is a fuzzy filter of L.*

The converse of Lemma 3.6 is not true as shown in the following example.

EXAMPLE 3.7. Let $L := \{0, a, b, c, d, 1\}$ be a set with Figure 1 as a partial ordering. Define a unary operation "′"and a binary operation "$\to$"on L as follows (Tables 1 and 2 respectively):

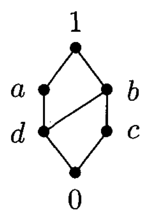

Figure 1

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

Table 1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Table 2

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (x \to y) \to y,$$
$$x \wedge y := ((x' \to y') \to y')',$$

for all $x, y \in L$. Then L is a lattice implication algebra. A fuzzy subset μ on L defined by

$$\mu(x) := \begin{cases} 0.9 & \text{if } x = 1, \\ 0.2 & \text{otherwise,} \end{cases}$$

for all $x \in L$ is a fuzzy filter of L, but μ is not a fuzzy implicative filter of L since $\mu(b) \ngeq \mu(1) = \min\{\mu(1 \to (b \to 0) \to b)), \mu(1)\}$.

In the following theorem, we can see that the converse of Lemma 3.6 also holds in a lattice H implication algebra.

THEOREM 3.8. *If L is a lattice H implication algebra, then every fuzzy filter of L is a fuzzy implicative filter of L.*

PROOF. Suppose that L is a lattice H implication algebra and μ is a fuzzy filter of L. Then by Lemma 2.2 (H3), we have $x = (x \to y) \to x$ for all $x, y \in L$. Since μ is a fuzzy filter of L, by (FF2) we get $\mu(x) \geq \min\{\mu(z \to x), \mu(z)\} \geq \min\{\mu(z \to ((x \to y) \to x)), \mu(z)\}$, hence μ is a fuzzy implicative filter of L. □

THEOREM 3.9. *A fuzzy subset μ of a lattice implication algebra L is a fuzzy implicative filter of L if and only if , for each $t \in [0,1]$, μ_t is either empty or an implicative filter of L.*

PROOF. Suppose that μ is a fuzzy implicative filter of L and $\mu_t \neq \emptyset$ for all $t \in [0,1]$. It is clear that $1 \in \mu_t$ since $\mu(1) \geq t$. Let $x, y, z \in L$ be such that $z \to ((x \to y) \to x) \in \mu_t$ and $z \in \mu_t$. Then we have $\mu(z \to ((x \to y) \to x)) \geq t$ and $\mu(z) \geq t$. It follows from (FF3) that

$$\mu(x) \geq \min\{\mu(z \to ((x \to y) \to x)), \mu(z)\} \geq t,$$

i.e., $x \in \mu_t$. This shows that μ_t is an implicative filter of L.

Conversely, suppose that for each $t \in [0,1]$, μ_t is either empty or an implicative filter of L. For any $x \in L$, let $\mu(x) = t$. Then we get $x \in \mu_t$. Since $\mu_t (\neq \emptyset)$ is an implicative filter of L, therefore $1 \in \mu_t$ and hence $\mu(1) \geq t = \mu(x)$. Thus $\mu(1) \geq \mu(x)$ for all $x \in L$. Now we only need to show that μ satisfies (FF3). If not, then there exist $x', y', z' \in L$ such that $\mu(x') < \min\{\mu(z' \to ((x' \to y') \to x')), \mu(z')\}$. Taking

$$t_0 := \frac{1}{2}\{\mu(x') + \min\{\mu(z' \to ((x' \to y') \to x')), \mu(z')\}\},$$

then we have that $\mu(x') < t_0 < \min\{\mu(z' \to ((x' \to y') \to x')), \mu(z')\}$. Hence, $x' \notin \mu_{t_0}, z' \to ((x' \to y') \to x') \in \mu_{t_0}$ and $z' \in \mu_{t_0}$, i.e., μ_{t_0} is not an implicative filter of L, which is a contradiction. Therefore, μ is a fuzzy implicative filter of L. □

NOTATION. For any fuzzy subsets μ and ν in a lattice implication algebra L, we write $\mu \leq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in L$.

LEMMA 3.10 ([2]). *Let F and G be filters of a lattice implication algebra L with $F \subseteq G$. If F is an implicative filter, then so is G.*

Next we give an extension theorem of fuzzy implicative filters.

THEOREM 3.11. *Let μ and ν be fuzzy filters of a lattice implication algebra L with $\mu \leq \nu$ and $\mu(1) = \nu(1)$. If μ is a fuzzy implicative filter of L, then so is ν.*

PROOF. To prove that ν is a fuzzy implicative filter of L it suffices to show that for any $t \in [0,1], \nu_t$ is either empty or an implicative filter of L. If the level subset ν_t is nonempty, then $\mu_t \neq \emptyset$ and $\mu_t \subseteq \nu_t$. In fact, if $x \in \mu_t$, then $t \leq \mu(x)$ and so $t \leq \nu(x)$ i.e., $x \in \nu_t$. Hence $\mu_t \subseteq \nu_t$. Since μ is a fuzzy implicative filter of L, by the hypothesis it follows from Theorem 3.9 that μ_t is an implicative filter of L. By Lemma 3.10, ν_t is also an implicative filter of L. Hence ν is a fuzzy implicative filter of L. □

THEOREM 3.12. *If F is an implicative filter of a lattice implication algebra L, then there is a fuzzy implicative filter μ of L such that $\mu_t = F$ for some $t \in (0,1)$.*

PROOF. Let μ be a fuzzy set in L defined by

$$\mu(x) := \begin{cases} t & \text{if } x \in F, \\ 0 & \text{if } x \notin F, \end{cases}$$

where t is a fixed number $(0 < t < 1)$. Now we verify that μ is a fuzzy implicative filter of L. Let $x, y, z \in L$ be such that $z \to ((x \to y) \to x) \in F$ and $z \in F$, then $x \in F$ by (F4). Thus we have

$$\mu(z \to ((x \to y) \to x)) = \mu(z) = \mu(x) = t,$$

and so $\mu(x) = \min\{\mu(z \to ((x \to y) \to x)), \mu(z)\}$. If at least one of $z \to ((x \to y) \to x)$ and z is not in F, then at least one of $\mu(z \to ((x \to y) \to x))$ and $\mu(z)$ is 0. Hence we get $\mu(x) \geq \min\{\mu(z \to ((x \to y) \to x)), \mu(z)\}$. Summarizing the above results, we have $\mu(x) \geq \min\{\mu(z \to ((x \to y) \to x)), \mu(z)\}$ for all $x, y, z \in L$. Since $1 \in F$, $\mu(1) = t \geq \mu(x)$ for all $x \in L$. Therefore μ is a fuzzy implicative filter of L. Obviously, $\mu_t = F$. □

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Yiquan Zhu
Department of Mathematics
Zhaoqing University
Zhaoqing 526061, Guangdong, China
E-mail: zhuyiquan@21cn.com

Qun Zhang
College of computers science
South-Central University for Nationalities
Wuhan, Hubei 430074, China

Eun Hwan Roh
Department of Mathematics Education
Chinju National University of Education
Jinju(Chinju) 660-756, Korea
E-mail: ehroh@cue.ac.kr