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ON RINGS CONTAINING A P-INJECTIVE MAXIMAL LEFT IDEAL

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ABSTRACT. We investigate in this paper rings containing a finitely generated p-injective maximal left ideal. We show that if R is a semiprime ring containing a finitely generated p-injective maximal left ideal, then R is a left p-injective ring. Using this result we are able to give a new characterization of von Neumann regular rings with nonzero socle.

0. Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. It is well-known that every maximal left ideal of a ring R is injective if and only if R is semisimple Artinian. Osofsky [3] proved that if R is a left self-injective left hereditary ring, then R is semisimple Artinian. Based on these results, Yuechiming [7] proposed the following question: If R is a left hereditary ring containing an injective maximal left ideal, is R semisimple Artinian? However, Zhang and Du [8] constructed a counterexample to settle in the negative, and then they proved that a ring R is semiprime left hereditary containing an injective maximal left ideal if and only if R is semisimple Artinian.

As the same direction to Zhang and Du, we investigate in this paper rings containing a finitely generated p-injective maximal left ideal. We show that if R is a semiprime ring containing a finitely generated p-injective maximal left ideal, then R is a left p-injective ring. Using this result we are able to give a new characterization of von Neumann regular rings with nonzero socle. Actually we prove that a ring R is von Neumann regular with nonzero socle if and only if R is a semiprime left p.p.-ring containing a finitely generated p-injective maximal left ideal.

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Recall that a ring R is called a *left p.p.-ring* if every principal left ideal of R is projective. A left R-module M is called to be *left p-injective* if every left R-homomorphism from a principal left ideal Ra to M extends to one from ${}_RR$ to M. It is well-known that R is a von Neumann regular ring if and only if every cyclic left R-module is p-injective if and only if R is a left p-injective left p.p.-ring. For any nonempty subset X of a ring R, the left annihilator of X will be denoted by $\ell(X)$.

We first recall the following three results:

(1) Let R be a ring, and $0 \to K \to F \to A \to 0$ be an exact sequence of right R-modules, where F is free. Then A is flat if and only if for any $u \in K$, there exists a homomorphism $f : F \to K$ such that $f(u) = u$ [1, Proposition 2.2].

(2) If M is a maximal left ideal of R which is two-sided, then R/M is flat as a right R-module if and only if R/M is p-injective as a left R-module [4, Proposition 1.4].

(3) If I is a finitely generated p-injective left ideal of R, then I is a direct summand of ${}_RR$ [5, Lemma 1.2].

1. Rings containing a p-injective maximal left ideal

We start with the following lemma.

LEMMA 1. *Let M_1 and M_2 be left R-modules. If M_1 and M_2 are p-injective, then $M_1 \oplus M_2$ is also p-injective.*

PROOF. It is routine. □

THEOREM 2. *Let R be a semiprime ring. If R contains a finitely generated p-injective maximal left ideal, then R is a left p-injective ring.*

PROOF. Let M be a finitely generated p-injective maximal left ideal of R. Then by [5, Lemma 1.2], $R = M \oplus L$, where L is a minimal left ideal of R. So $M = Re$ for some $e = e^2 \in R$.

If $ML = 0$, then $M = \ell(L)$ since M is maximal. So M is a two-sided ideal of R. Since R is semiprime, e is a central idempotent. For each $a \in M$, let $a = re$ for some $r \in R$, then $a = re = re^2 = ere = ea$. Hence by [1, Proposition 2.2], $(R/M)_R$ is flat and so ${}_R(R/M)$ is p-injective by [4, Proposition 1.4]. So ${}_RL$ is p-injective.

Now suppose that $ML \neq 0$. Then there exists $u \in L$ such that $Mu \neq 0$, whence $L = Mu$. Let $f : M \longrightarrow L$ be the map defined by $f(x) = xu$ for each $x \in M$. Since f is an epimorphism and L is projective, $M \cong \ker f \oplus (M/\ker f) \cong \ker f \oplus L$. Hence ${}_R L$ is p-injective. In any case, ${}_R L$ is p-injective. By Lemma 1, R is left p-injective. □

As an application of Theorem 2, we have the following result.

THEOREM 3. *For a ring R, the following statements are equivalent:*

(1) *R is a von Neumann regular ring with nonzero socle.*
(2) *R is a semiprime left p.p.-ring containing a finitely generated p-injective maximal left ideal.*

PROOF. (1)⇒(2): Suppose that R is a von Neumann regular ring with nonzero socle. Obviously, R is a semiprime left p.p.-ring. If every maximal left ideal of R is essential, then the socle of R is contained in the Jacobson radical of R. Since R has a nonzero socle, there exists a maximal left ideal M of R which is not essential. Therefore M is a direct summand of R. Note that R is von Neumann regular if and only if every left R-module is p-injective. Hence M is p-injective.

(2)⇒(1): Let M be a finitely generated p-injective maximal left ideal of R. Then by [5, Lemma 1.2], $R = M \oplus L$, where L is a minimal left ideal of R. So the socle of R is nonzero. Also by Theorem 2, R is left p-injective. Since R is a left p.p.-ring, R is von Neumann regular. □

The following result can be compared with a result in [6, Theorem 11].

COROLLARY 4. *For a ring R, the following statements are equivalent:*

(1) *R is a semisimple Artinian ring.*
(2) *R is a semiprime left p.p. left Noetherian ring containing a p-injective maximal left ideal.*
(3) *R is a semiprime left p.p.-ring containing a finitely generated p-injective maximal left ideal and satisfies the ACC on left annihilators.*

PROOF. (1)⇒(2) and (2)⇒(3) are clear. (3)⇒(1): By Theorem 3, R is von Neumann regular. It is a well-known fact that von Neumann regular ring satisfying the ACC on left annihilators is semisimple Artinian. □

The following example shows that the condition "R is semiprime" is not superfluous in Theorem 2, Theorem 3 and Corollary 4.

EXAMPLE 5. There exists a left hereditary ring (and so left p.p.) containing a finitely generated p-injective maximal left ideal which is not von Neumann regular.

Let $\mathbb{Z}_2$ be the ring of integers modulo 2. We consider the ring $R = \begin{bmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$. Then R is left hereditary by [2, Corollary 4.9] and so it is a left p.p.-ring. Let $M = \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$. Then M is a maximal left ideal of R which is finitely generated. Moreover, ${}_RM$ is p-injective by [8, Theorem 3]. However R is not von Neumann regular. Moreover, R is not left p-injective. For, if R is left p-injective, then R is von Neumann regular because R is a left p.p.-ring, which is a contradiction.

Recall that a ring R is called to be *abelian* if every idempotent element of R is central. As a parallel result to Theorem 3, we obtain the following result.

PROPOSITION 6. *For a ring R, the following statements are equivalent:*

(1) *R is a strongly regular ring with nonzero socle.*
(2) *R is an abelian left p.p.-ring containing a finitely generated p-injective maximal left ideal.*

PROOF. (1)⇒(2): Note that strongly regular ring is always abelian. So we are done by Theorem 3.

(2)⇒(1): By the same method in the proof of Theorem 2, we have R is a left p-injective ring with nonzero left socle. So R is von Neumann regular with nonzero socle. Since R is abelian, R is strongly regular. □

PROPOSITION 7. *For a ring R, the following statements are equivalent:*

(1) *R is a von Neumann regular ring with nonzero socle.*
(2) *R is a left p.p.-ring containing a finitely generated p-injective maximal left ideal which is von Neumann regular.*

PROOF. Obviously, (1) implies (2). Assume (2). Let M be a finitely generated p-injective maximal left ideal of R which is von Neumann regular. Then $R = M \oplus L$ for some left ideal L of R. So $M = Re$ for some idempotent e in R. Now L is a minimal left ideal of R. So R has a nonzero socle.

If $ML = 0$, then $M = \ell(L)$ is a two-sided ideal of R. Since M is von Neumann regular, for any $u \in M$, there exists $c \in R$ such that $u = ucu$.

Now $u \in Mu$ which implies that $(R/M)_R$ is flat, whence ${}_R(R/M)$ is p-injective. Hence ${}_RL$ is p-injective.

If $ML \neq 0$, then by the same method in the proof of Theorem 2, ${}_RL$ is also p-injective. Therefore R is left p-injective and hence R is von Neumann regular. □

REMARK. In Theorem 3 and Proposition 7, the condition "finitely generated" is necessary because there is a von Neumann regular ring with zero socle.

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