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NEGATIVE DIFFERENCE POSETS AND FUZZY IMPLICATION ALGEBRAS

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ABSTRACT. The notion of negative difference on a poset is introduced, and the interrelations between FI-algebras and posets with negative difference are discussed.

1. Introduction

"Implication" is a basic logic connective in various logic systems. "$p \longrightarrow q$" means "if p then q" for two propositions p and q, where "$\longrightarrow$" is called an implication operator. There are lots of papers concerning with this operator, especially on approximate reasoning, and most research is about semantic. In 1990, W. M. Wu [4] introduced the notion of fuzzy implication algebras, and investigated several properties. In [2], Z. W. Li and C. Y. Zheng introduced the notion of distributive (resp. regular, commutative) fuzzy implication algebra, and investigated the relations between such fuzzy implication algebras and MV-algebras. F. Kôpka and F. Chovanec [1] introduced the notion of difference on a poset. In this paper, we introduce the concept of negative difference on a poset, and investigate the interrelations between FI-algebras and posets with negative difference.

2. Preliminaries

A nonempty set X together with a binary operation $\longrightarrow$ and a zero element 0 is said to be a *fuzzy implication algebra* (*FI-algebra*, for short) if the following axioms are satisfied for all $x, y, z \in X$:

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(I1) $x \to (y \to z) = y \to (x \to z)$,
(I2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(I3) $x \to x = 1$,
(I4) $x \to y = y \to x = 1 \Rightarrow x = y$,
(I5) $0 \to x = 1$,

where $1 = 0 \to 0$. We can define a partial ordering $\preceq$ on an FI-algebra X by $x \preceq y$ if and only if $x \to y = 1$. An FI-algebra X is said to be *commutative* if $x+y = y+x$ for all $x, y \in X$, where $x+y := (x \to y) \to y$. A subset S of an FI-algebra X is called a *subalgebra* of X if $x \to y \in S$ for all $x, y \in S$.

In an FI-algebra X, the following hold for all $x, y, z \in X$:

(p1) $x \to 1 = 1$ and $1 \to x = x$,
(p2) $y \to z \preceq (x \to y) \to (x \to z)$,
(p3) $x \preceq y \Rightarrow z \to x \preceq z \to y,\ y \to z \preceq x \to z$,
(p4) $x \preceq y \to z \Rightarrow y \preceq x \to z$,
(p5) $x \preceq y \to x$,
(p6) $x \preceq (x \to y) \to y$,
(p7) $x \preceq y \Rightarrow y = (x \to y) \to y$.

EXAMPLE 2.1. (1) Let $X = [0, 1]$ and define a binary operation $\to$ on X by

$$x \to y = \sup\{z \in X \mid \min\{x, z\} \leq y\},\ \forall x, y \in X.$$

Then $(X, \to, 0)$ is an FI-algebra (see [4]).

(2) Let $X = \{0, 1\}$ and define a binary operation $\to$ on X by

$$x \to y = \max\{1 - x, y\},\ \forall x, y \in X.$$

Then $(X, \to, 0)$ is an FI-algebra (see [4]).

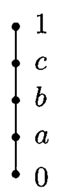
(3) Let $X = [0, 1]$ and define a binary operation $\to$ on X by

$$x \to y = \min\{1, 1 - x + y\},\ \forall x, y \in X.$$

Then $(X, \to, 0)$ is an FI-algebra (see [4]).

(4) Let $X = \{0, a, b, c, 1\}$ be a set with the following Cayley table and Hasse diagram:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1



Then $(X, \to, 0)$ is an FI-algebra.

3. Negative differences

DEFINITION 3.1. Let $(P;\leq)$ be a nonempty partially ordered set (poset). A partial binary operation $\ominus$ is called a *negative difference* on P if the element $a\ominus b$ is defined in P if and only if $a\leq b$, and the following axioms hold for all $a,b,c\in P$:

(D1) $a\leq a\ominus b$.
(D2) $a\ominus(a\ominus b)=b$.
(D3) $a\leq b\leq c\Rightarrow a\ominus c\leq a\ominus b$ and $(a\ominus c)\ominus(a\ominus b)=b\ominus c$.

EXAMPLE 3.2. Let $\mathbb{R}^-$ be the set of all non-positive real numbers. The negative difference $a-b$ of real numbers, $a,b\in\mathbb{R}^-$, $a\leq b$, satisfies the conditions (D1) – (D3).

PROPOSITION 3.3. *Let $(P,\leq)$ be a poset with a negative difference $\ominus$ and let $a,b,c,d\in P$. Then the following assertions are true.*

(i) *If $a\leq b\leq c$, then $a\ominus c\leq b\ominus c$ and $(a\ominus c)\ominus(b\ominus c)=a\ominus b$.*
(ii) *If $a\leq b$ and $a\ominus b\leq c$, then $a\ominus c\leq b$ and $(a\ominus b)\ominus c=(a\ominus c)\ominus b$.*
(iii) *If $a\leq b\leq c$, then $a\ominus(b\ominus c)\leq c$ and $(a\ominus(b\ominus c))\ominus c=a\ominus b$.*
(iv) *If $a\leq b$ and $a\leq c$, then $a\ominus b=a\ominus c$ if and only if $b=c$.*
(v) *If $a\leq c\leq d$ and $a\leq b\leq d$, then $a\ominus c=b\ominus d$ if and only if $a\ominus b=c\ominus d$.*

PROOF. (i) Assume that $a\leq b\leq c$. Since $a\ominus c\leq a\ominus b$ by (D3), it follows from (D1) – (D3) that $a\ominus c\leq(a\ominus c)\ominus(a\ominus b)=b\ominus c$ and

$$(a\ominus c)\ominus(b\ominus c)=(a\ominus c)\ominus((a\ominus c)\ominus(a\ominus b))=a\ominus b.$$

(ii) Suppose that $a\leq b$ and $a\ominus b\leq c$. Then $a\leq a\ominus b\leq c$, and so $a\ominus c\leq a\ominus(a\ominus b)=b$ by (D3) and (D2). Since $a\ominus c\leq(a\ominus b)\ominus c$ by (i), we have

$$(a\ominus c)\ominus((a\ominus b)\ominus c)=a\ominus(a\ominus b)=b,$$

which implies that

$$(a\ominus c)\ominus b=(a\ominus c)\ominus((a\ominus c)\ominus((a\ominus b)\ominus c))=(a\ominus b)\ominus c.$$

(iii) Assume that $a\leq b\leq c$. According to (i) and (D1), we get $a\leq a\ominus c\leq b\ominus c$. It follows from (D3) and (D2) that $a\ominus(b\ominus c)\leq a\ominus(a\ominus c)=c$. Using (i) and (ii), we obtain

$$(a\ominus(b\ominus c))\ominus c=(a\ominus c)\ominus(b\ominus c)=a\ominus b.$$

(iv) If $a\ominus b=a\ominus c$, then $c=a\ominus(a\ominus c)=a\ominus(a\ominus b)=b$. The converse assertion is obvious.

(v) Assume that $a \le c \le d$ and $a \le b \le d$. If $a \ominus c = b \ominus d$, then

$$a \ominus b = (a \ominus d) \ominus (b \ominus d) = (a \ominus d) \ominus (a \ominus c) = c \ominus d$$

by (i) and (D3). Conversely, if $a \ominus b = c \ominus d$, then

$$a \ominus c = (a \ominus d) \ominus (c \ominus d) = (a \ominus d) \ominus (a \ominus b) = b \ominus d$$

by (i) and (D3). This completes the proof. □

Let w be an element of a poset $(P; \le)$. The set

$$T(w) := \{a \in P \mid w \le a\}$$

is called a *terminal segment* of w. Obviously, $w \ominus w \in T(w)$ by (D1).

A poset $(P; \le)$ is said to be *directed upwards* if for any $x, y \in P$, there is an element $z \in P$ such that $x \le z$ and $y \le z$.

PROPOSITION 3.4. *Let $(P; \le)$ be a poset with a negative difference $\ominus$ and let $w \in P$. Then for any $a \in T(w)$, we have*

(i) *$w \ominus a \in T(w)$.*
(ii) *$w \ominus w$ is the greatest element in $T(w)$.*
(iii) *$a \ominus (w \ominus w) = a$.*
(iv) *$a \ominus a = w \ominus w$.*
(v) *$(P; \le)$ has the greatest element if and only if $(P; \le)$ is directed upwards.*
(vi) *If $(P; \le)$ is directed upwards, then $u \ominus u$ is the greatest element of P for any $u \in P$.*

PROOF. Let $a \in T(w)$. According to (D1), we have $w \ominus a \in T(w)$, which proves (i). Since $w \le w \le w \ominus a$, it follows from (D2) and (D3) that $a = w \ominus (w \ominus a) \le w \ominus w$. Hence $w \ominus w$ is the greatest element in $T(w)$. Since $w \le a \le a \ominus a$, we conclude that $a \ominus a \le w \ominus w$ because $w \ominus w$ is the greatest element in $T(w)$. Using (D1) – (D3), we get $a \le a \ominus (w \ominus w) \le a \ominus (a \ominus a) = a$, which implies (iii). From (iii) and (D2), we obtain

$$a \ominus a = a \ominus (a \ominus (w \ominus w)) = w \ominus w.$$

Hence (iv) is valid. (v) is trivial. Assume that P is directed upwards. Denote $e_u := u \ominus u$ for every $u \in P$. We claim that $a \le e_u$ for any $a \in P$. Since P is directed upwards, there is an element $z \in P$ such that $e_u \le z$ and $e_a \le z$. Since $z \in T(u) \cap T(a)$, it follows from (ii) that $z \le u \ominus u = e_u \le z$ and $z \le a \ominus a = e_a \le z$ so that $a \le e_a = e_u$. This completes the proof. □

DEFINITION 3.5. Let $(P, \leq)$ be a poset with a negative difference $\ominus$. If there exists the least element, say e, in P, we say that P is a *negative difference poset* (briefly, *ND-poset*).

EXAMPLE 3.6. Let $P = [-1, 0]$. For any two numbers $a, b \in P$ with $a \leq b$, we define $a \ominus b = a - b$, where "$-$" is the usual subtraction. Then $(P; \leq, \ominus)$ is an ND-poset with -1 as the least element.

PROPOSITION 3.7. *Every ND-poset P contains the greatest element* 1, *and* $1 = e \ominus e$.

PROOF. Let $a \in P$. Then $e \ominus a$ is defined in P because $e \leq a$, and $e \leq e \leq e \ominus a$ by (D1). It follows from (D2) and (D3) that $a = e \ominus (e \ominus a) \leq e \ominus e$, i.e., $e \ominus e$ is the greatest element in P, and we denote it by 1. □

PROPOSITION 3.8. *Let* $(X, \rightarrow, 0)$ *be an FI-algebra. We define a partial binary operation* $\ominus$ *on X such that, for all* $x, y \in X$, $x \ominus y$ *is defined if and only if* $x \preceq y$, *and in this case*

$$x \ominus y := y \rightarrow x.$$

Then

(i) $x \preceq x \ominus y$ *if* $x \preceq y$.
(ii) $y \preceq x \ominus (x \ominus y)$ *if* $x \preceq y$.
(iii) $x \ominus z \preceq x \ominus y$ *if* $x \preceq y \preceq z$.
(iv) $y \ominus z \preceq (x \ominus z) \ominus (x \ominus y)$ *if* $x \preceq y \preceq z$.

PROOF. (i) Using (I1), (I3) and (p1), we have that

$$x \rightarrow (y \rightarrow x) = y \rightarrow (x \rightarrow x) = y \rightarrow 1 = 1, \text{ i.e., } x \preceq y \rightarrow x.$$

Hence $x \preceq x \ominus y$ if $x \preceq y$.

(ii) According to (p6) and (i), we obtain

$$y \preceq (y \rightarrow x) \rightarrow x = x \ominus (x \ominus y) \text{ if } x \preceq y.$$

(iii) Assume that $x \preceq y \preceq z$. By (p3), $y \preceq z$ implies $z \rightarrow x \preceq y \rightarrow x$, and so $x \ominus z \preceq x \ominus y$.

(iv) If $x \preceq y \preceq z$, then (iii) and (I2) entail that $y \ominus z \preceq (x \ominus z) \ominus (x \ominus y)$. This completes the proof. □

THEOREM 3.9. *If X is a commutative FI-algebra, then the partial binary operation* $\ominus$ *on X described in Proposition 3.8 is a negative difference on* $(X; \preceq)$.

Proof. Assume that X is a commutative FI-algebra. Note that if $x \preceq y \preceq z$, then

$$\begin{aligned} 1 &= x \to y = ((x \to y) \to y) \to y \\ &\preceq (z \to ((x \to y) \to y)) \to (z \to y) \\ &= (z \to (x + y)) \to (z \to y) \\ &= (z \to (y + x)) \to (z \to y) \\ &= (z \to ((y \to x) \to x)) \to (z \to y) \\ &= ((y \to x) \to (z \to x)) \to (z \to y) \\ &= ((x \ominus z) \ominus (x \ominus y)) \to (y \ominus z) \end{aligned}$$

so that $((x \ominus z) \ominus (x \ominus y)) \to (y \ominus z) = 1$, i.e., $(x \ominus z) \ominus (x \ominus y) \preceq y \ominus z$. It follows from Proposition 3.8(iv) and (I4) that $(x \ominus z) \ominus (x \ominus y) = y \ominus z$. Now, since $y \preceq x \ominus (x \ominus y)$ if $x \preceq y$, it is sufficient to show that $x \ominus (x \ominus y) \preceq y$ whenever $x \preceq y$. Using (p1), (I3) and the commutativity, we have

$$\begin{aligned} (x \ominus (x \ominus y)) \to y &= ((y \to x) \to x) \to y = (y + x) \to y \\ &= (x + y) \to y = ((x \to y) \to y) \to y \\ &= (1 \to y) \to y) = y \to y = 1, \end{aligned}$$

which means $x \ominus (x \ominus y) \preceq y$. Hence the partial binary operation $\ominus$ on X is a negative difference on $(X; \preceq)$. □

Corollary 3.10. *Every commutative FI-algebra is an ND-poset.*

Proof. It is straightforward. □

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