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ON DENJOY-MCSHANE-STIELTJES INTEGRAL

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ABSTRACT. In this paper we introduce the concepts of the McShane-Stieltjes integral and the Denjoy-McShane-Stieltjes integral for Banach-valued functions and give a characterization of the McShane-Stieltjes integrability and investigate some properties of the Denjoy-McShane-Stieltjes integral.

1. Introduction

The McShane integral of real-valued functions is a generalization of the Riemann integral and equivalent to the Lebesgue integral. R. A. Gordon [5] and D. H. Fremlin, J. Mendoza [1] studied the McShane integral of Banach-valued functions. The Denjoy integral of real-valued functions is an extension of the Lebesgue integral of real-valued functions. In [3], R. A. Gordon defined the Denjoy-Dunford, Denjoy-Pettis and Denjoy-Bochner integrals of functions mapping an interval $[a, b]$ into a Banach space X and studied some properties of those integrals. In [7], D. H. Lee and J. M. Park studied the Denjoy extension of the McShane integral of functions mapping an interval $[a, b]$ into a Banach space X. In [8], we introduced the Denjoy-Stieltjes integral of real-valued functions which is an extension of the Denjoy integral.

In this paper we introduce the concepts of the McShane-Stieltjes integral and the Denjoy-McShane-Stieltjes integral which are generalizations of the McShane integral and the Denjoy-McShane integral respectively and give a characterization of the McShane-Stieltjes integrability and investigate some properties of the Denjoy-McShane-Stieltjes integral.

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2. Preliminaries

Unless otherwise stated, we always assume that X and Y are real Banach spaces with duals X^* and Y^*.

DEFINITION 2.1 [5]. A McShane partition of $[a,b]$ is a finite collection $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ such that $\{[c_i,d_i] : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[a,b]$ covering $[a,b]$ and $t_i \in [a,b]$ for each $i \leq n$. A gauge on $[a,b]$ is a function $\delta : [a,b] \to (0,\infty)$. A McShane partition $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ is subordinate to a gauge δ if $[c_i,d_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for every $i \leq n$. If $f : [a,b] \to X$ and if $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a,b]$, we will denote $f(\mathcal{P})$ for $\sum_{i=1}^n f(t_i)(d_i - c_i)$. A function $f : [a,b] \to X$ is McShane integrable on $[a,b]$, with McShane integral z, if for each $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that $\|f(\mathcal{P}) - z\| < \varepsilon$ whenever $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a,b]$ subordinate to δ.

DEFINITION 2.2 [3]. Let $F : [a,b] \to X$ and let $t \in (a,b)$. A vector z in X is the approximate derivative of F at t if there exists a measurable set $E \subset [a,b]$ that has t as a point of density such that $\lim_{\substack{s\to t \\ s\in E}} \dfrac{F(s)-F(t)}{s-t} = z$. We will write $F'_{ap}(t) = z$.

A function $f : [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$ if there exists an ACG function $F : [a,b] \to \mathbb{R}$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$. The function f is Denjoy integrable on a set $E \subset [a,b]$ if $f\chi_E$ is Denjoy integrable on $[a,b]$.

DEFINITION 2.3 [8]. Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function and let $E \subset [a,b]$.

(a) The function F is BV with respect to α on E if $V(F,\alpha,E) = \sup\left\{\sum_{i=1}^n \|F(d_i) - F(c_i)\| \dfrac{\alpha(d_i)-\alpha(c_i)}{d_i - c_i}\right\}$ is finite where the supremum is taken over all finite collections $\{[c_i,d_i] : 1 \leq i \leq n\}$ of non-overlapping intervals that have endpoints in E.

(b) The function F is AC with respect to α on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^n \|F(d_i) - F(c_i)\| < \epsilon$ whenever $\{[c_i,d_i] : 1 \leq i \leq n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}[\alpha(d_i) - \alpha(c_i)] < \delta$.

(c) The function F is BVG with respect to α on E if E can be expressed as a countable union of sets on each of which F is BV with respect to α.

(d) The function F is ACG with respect to α on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC with respect to α.

DEFINITION 2.4 [8]. Let $F : [a,b] \to X$, $t \in (a,b)$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. A vector $z \in X$ is the approximate derivative of F with respect to α at t if there exists a measurable set $E \subset [a,b]$ that has t as a point of density such that $\lim_{\substack{s\to t\\ s\in E}} \frac{F(s) - F(t)}{\alpha(s) - \alpha(t)} = z$. We will write $F'_{\alpha,ap}(t) = z$.

A function $f : [a,b] \to \mathbb{R}$ is Denjoy-Stieltjes integrable with respect to α on $[a,b]$ if there exists an ACG function $F : [a,b] \to \mathbb{R}$ with respect to α such that $F'_{\alpha,ap} = f$ almost everywhere on $[a,b]$. The function f is Denjoy-Stieltjes integrable with respect to α on a set $E \subset [a,b]$ if $f\chi_E$ is Denjoy-Stieltjes integrable with respect to α on $[a,b]$.

We note that $F'_{ap}(t) = F'_{\alpha,ap}(t)\alpha'(t)$ for each $t \in (a,b)$.

DEFINITION 2.5 [7]. A function $f : [a,b] \to X$ is Denjoy-McShane integrable on $[a,b]$ if there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is approximately differentiable almost everywhere on $[a,b]$ and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a,b]$.

3. McShane-Stieltjes integral

In this section we introduce the concept of the McShane-Stieltjes integral and give a characterization of the McShane-Stieltjes integrability.

Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If $f : [a,b] \to X$ and if $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ is a McShane partition of $[a,b]$, we will denote $f_\alpha(\mathcal{P})$ for $\sum_{i=1}^{n} f(t_i)\,[\alpha(d_i) - \alpha(c_i)]$.

DEFINITION 3.1. Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A function $f : [a,b] \to X$ is McShane-Stieltjes integrable with respect to

α on $[a,b]$, with McShane-Stieltjes integral z, if for each $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that $\|f_\alpha(\mathcal{P}) - z\| < \varepsilon$ whenever $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ is a McShane partition of $[a,b]$ subordinate to δ. The function f is McShane-Stieltjes integrable with respect to α on a set $E \subset [a,b]$ if $f\chi_E$ is McShane-Stieltjes integrable with respect to α on $[a,b]$.

REMARK. From the definition of the McShane-Stieltjes integral we can easily obtain the following:

Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A function $f : [a,b] \to X$ is McShane-Stieltjes integrable with respect to α on $[a,b]$ if and only if for each $\epsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that $\|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)\| < \epsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are McShane partitions of $[a,b]$ subordinate to δ.

THEOREM 3.2. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $f : [a,b] \to X$ be a bounded function. Then f is McShane-Stieltjes integrable with respect to α on $[a,b]$ if and only if $\alpha' f$ is McShane integrable on $[a,b]$.*

PROOF. Since $f : [a,b] \to X$ is a bounded function, there exists $M > 0$ such that $\|f(x)\| \le M$ for all $x \in [a,b]$. Continuity of α' on $[a,b]$ implies uniform continuity on $[a,b]$. Hence for each $\epsilon > 0$ there exists $\eta > 0$ such that

$$x, y \in [a,b],\ |x-y| < \eta \Rightarrow |\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{3M(b-a)}.$$

Choose a gauge δ_1 on $[a,b]$ with $\delta_1(x) < \eta$ for all $x \in [a,b]$. Let $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ be a McShane partition of $[a,b]$ subordinate to δ_1. Then by the Mean Value Theorem, there exists $x_i \in (c_i,d_i)$ such that $\alpha(d_i) - \alpha(c_i) = \alpha'(x_i)(d_i - c_i)$ for $1 \le i \le n$. Since $|t_i - x_i| < \delta_1(t_i) < \eta$ for $1 \le i \le n$, $|\alpha'(t_i) - \alpha'(x_i)| < \dfrac{\epsilon}{3M(b-a)}$ for $1 \le i \le n$. Hence we have

$$\begin{aligned}
&\|f_\alpha(\mathcal{P}) - (\alpha' f)(\mathcal{P})\| \\
&= \|\sum_{i=1}^{n} f(t_i)[\alpha(d_i) - \alpha(c_i)] - \sum_{i=1}^{n} \alpha'(t_i) f(t_i)(d_i - c_i)\| \\
&= \|\sum_{i=1}^{n} f(t_i)[\alpha'(x_i) - \alpha'(t_i)](d_i - c_i)\|
\end{aligned}$$

$$\leq \sum_{i=1}^{n} \|f(t_i)\| \, |\alpha'(x_i) - \alpha'(t_i)|(d_i - c_i)$$
$$< \sum_{i=1}^{n} M \frac{\epsilon}{3M(b-a)}(d_i - c_i) = \frac{\epsilon}{3}$$

whenever $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a, b]$ subordinate to δ_1.

If f is McShane-Stieltjes integrable with respect to α on $[a, b]$, then there exists a gauge δ_2 on $[a, b]$ such that $\|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)\| < \epsilon/3$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are McShane partitions of $[a, b]$ subordinate to δ_2. Define δ on $[a, b]$ by $\delta(x) = min\{\delta_1(x), \delta_2(x)\}$ for $x \in [a, b]$. Then δ is a gauge on $[a, b]$ and

$$\|(\alpha' f)(\mathcal{P}_1) - (\alpha' f)(\mathcal{P}_2)\| < \|(\alpha' f)(\mathcal{P}_1) - f_\alpha(\mathcal{P}_1)\| + \|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)\| + \|f_\alpha(\mathcal{P}_2) - (\alpha' f)(\mathcal{P}_2)\| < \epsilon$$

whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are McShane partitions of $[a, b]$ subordinate to δ. Hence $\alpha' f$ is McShane integrable on $[a, b]$ by [5, Theorem 3].

Conversely, if $\alpha' f$ is McShane integrable on $[a, b]$, then by [5, Theorem 3] for each $\epsilon > 0$ there exists a gauge δ_3 on $[a, b]$ such that $\|(\alpha' f)(\mathcal{P}_1) - (\alpha' f)(\mathcal{P}_2)\| < \epsilon/3$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are McShane partitions of $[a, b]$ subordinate to δ_3. Define δ on $[a, b]$ by $\delta(x) = min\{\delta_1(x), \delta_3(x)\}$ for $x \in [a, b]$. Then δ is a gauge on $[a, b]$ and

$$\|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)\| \leq \|f_\alpha(\mathcal{P}_1) - (\alpha' f)(\mathcal{P}_1)\| + \|(\alpha' f)(\mathcal{P}_1) - (\alpha' f)(\mathcal{P}_2)\| + \|(\alpha' f)(\mathcal{P}_2) - f_\alpha(\mathcal{P}_2)\| < \epsilon$$

whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are McShane partitions of $[a, b]$ subordinate to δ. Hence f is McShane-Stieltjes integrable with respect to α on $[a, b]$. □

THEOREM 3.3. *Let $\alpha : [a, b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a, b])$. If $f : [a, b] \to X$ is McShane-Stieltjes integrable with respect to α on $[a, b]$ and $T : X \to Y$ is a bounded linear operator, then $T \circ f : [a, b] \to Y$ is McShane-Stieltjes integrable with respect to α on $[a, b]$ and $(MS)\int_a^b T \circ f d\alpha = T((MS)\int_a^b f d\alpha)$.*

PROOF. If $T = 0$, then it is clear. Suppose that $T \neq 0$. If $f : [a, b] \to X$ is McShane-Stieltjes integrable with respect to α on $[a, b]$, then for each $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that $\|f_\alpha(\mathcal{P}) -$

$(MS)\int_a^b fd\alpha\| < \epsilon/\|T\|$ whenever $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ is a McShane partition of $[a, b]$ subordinate to δ. Hence we have

$$\begin{aligned}
&\|(T\circ f)_\alpha(\mathcal{P}) - T((MS)\int_a^b fd\alpha)\| \\
&= \|\sum_{i=1}^{n}(T\circ f)(t_i)[\alpha(d_i)-\alpha(c_i)] - T((MS)\int_a^b fd\alpha)\| \\
&= \|T(\sum_{i=1}^{n} f(t_i)[\alpha(d_i)-\alpha(c_i)] - (MS)\int_a^b fd\alpha)\| \\
&\le \|T\|\|\sum_{i=1}^{n} f(t_i)[\alpha(d_i)-\alpha(c_i)] - (MS)\int_a^b fd\alpha\| \\
&= \|T\|\|f_\alpha(\mathcal{P}) - (MS)\int_a^b fd\alpha\| \\
&< \|T\|\frac{\epsilon}{\|T\|} = \epsilon
\end{aligned}$$

whenever $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ is a McShane partition of $[a, b]$ subordinate to δ. Hence $T\circ f : [a, b] \to Y$ is McShane-Stieltjes integrable with respect to α on $[a, b]$ and $(MS)\int_a^b T\circ fd\alpha = T((MS)\int_a^b fd\alpha)$.□

4. Denjoy-McShane-Stieltjes integral

In this section we introduce the concept of the Denjoy-McShane-Stieltjes integral and investigate some properties of this integral.

DEFINITION 4.1. Let $\alpha : [a, b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a, b])$. A function $f : [a, b] \to X$ is Denjoy-McShane-Stieltjes integrable with respect to α on $[a, b]$ if there exists a continuous function $F : [a, b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG with respect to α on $[a, b]$ and

(ii) for each $x^* \in X^*$ x^*F is approximately differentiable with respect to α almost everywhere on $[a, b]$ and $(x^*F)'_{\alpha,ap} = x^*f$ almost everywhere on $[a, b]$.

THEOREM 4.2. *Let $\alpha : [a, b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a, b])$. Then $f : [a, b] \to X$ is Denjoy-McShane-Stieltjes integrable with respect to α on $[a, b]$ if and only if $\alpha' f$ is Denjoy-McShane integrable on $[a, b]$.*

PROOF. If $f : [a,b] \to X$ is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$, then there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG with respect to α on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is approximately differentiable with respect to α almost everywhere on $[a,b]$ and $(x^*F)'_{\alpha,ap} = x^*f$ almost everywhere on $[a,b]$.

From [8, Theorem 3.6] and Definition 2.4 we have

(i) for each $x^* \in X^*$ x^*F is ACG on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is approximately differentiable almost everywhere on $[a,b]$ and $(x^*F)'_{ap} = (x^*F)'_{\alpha,ap}\alpha' = (x^*f)\alpha' = x^*(\alpha' f)$ almost everywhere on $[a,b]$.

Hence $\alpha' f$ is Denjoy-McShane integrable on $[a,b]$.

Conversely, if $\alpha' f : [a,b] \to X$ is Denjoy-McShane integrable on $[a,b]$, then there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is approximately differentiable almost everywhere on $[a,b]$ and $(x^*F)'_{ap} = x^*(\alpha' f)$ almost everywhere on $[a,b]$.

From [8, Theorem 3.6] and Definition 2.4 we have

(i) for each $x^* \in X^*$ x^*F is ACG with respect to α on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is approximately differentiable with respect to α almost everywhere on $[a,b]$ and $(x^*F)'_{\alpha,ap} = \dfrac{1}{\alpha'}(x^*F)'_{ap} = \dfrac{1}{\alpha'}x^*(\alpha' f) = x^*f$ almost everywhere on $[a,b]$.

Hence f is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$. □

The following three corollaries are obtained from Theorem 4.2 and [7, Theorem 3.2, 3.3, 3.4].

COROLLARY 4.3. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $f : [a,b] \to X$. If $\alpha' f$ is McShane integrable on $[a,b]$, then f is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$.*

COROLLARY 4.4. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $f : [a,b] \to X$. If $\alpha' f$ is Denjoy-Bochner integrable on $[a,b]$, then f is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$.*

COROLLARY 4.5. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $f : [a,b] \to X$. If f is Denjoy-McShane-*

Stieltjes integrable with respect to α on $[a,b]$, then $\alpha' f$ is Denjoy-Pettis integrable on $[a,b]$.

THEOREM 4.6. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. If $f : [a,b] \to X$ is a bounded McShane-Stieltjes integrable with respect to α on $[a,b]$, then f is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$.*

PROOF. If $f : [a,b] \to X$ is a bounded McShane-Stieltjes integrable with respect to α on $[a,b]$, then by Theorem 3.2 $\alpha' f$ is McShane integrable on $[a,b]$. By [7, Theorem 3.2], $\alpha' f$ is Denjoy-McShane integrable on $[a,b]$. By Theorem 4.2, f is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$. □

THEOREM 4.7. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. If $f : [a,b] \to X$ is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$ and $T : X \to Y$ is a bounded linear* operator, *then $T \circ f : [a,b] \to Y$ is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$.*

PROOF. If $f : [a,b] \to X$ is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$, then there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG with respect to α on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is approximately differentiable with respect to α almost everywhere on $[a,b]$ and $(x^*F)'_{\alpha,ap} = x^*f$ almost everywhere on $[a,b]$.

Let $G = T \circ F$. Then $G : [a,b] \to Y$ is a continuous function such that

(i) for each $y^* \in Y^*$ $y^*G = y^*(T \circ F) = (y^*T)F$ is ACG with respect to α on $[a,b]$ since $y^*T \in X^*$, and

(ii) for each $y^* \in Y^*$ $y^*G = y^*(T \circ F) = (y^*T)F$ is approximately differentiable with respect to α almost everywhere on $[a,b]$ and $(y^*G)'_{\alpha,ap} = (y^*(T \circ F))'_{\alpha,ap} = ((y^*T)F)'_{\alpha,ap} = (y^*T)f = y^*(T \circ f)$ almost everywhere on $[a,b]$ since $y^*T \in X^*$.

Hence $T \circ f : [a,b] \to Y$ is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$. □

DEFINITION 4.8 [9]. Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$.

(a) A function $f : [a,b] \to X$ is Denjoy-Stieltjes-Dunford integrable with respect to α on $[a,b]$ if for each $x^* \in X^*$ x^*f is Denjoy-Stieltjes integrable with respect to α on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector $x_I^{**} \in X^{**}$ such that $x_I^{**}(x^*) = (DS)\int_I x^*f\,d\alpha$ for all $x^* \in X^*$.

(b) A function $f : [a,b] \to X$ is Denjoy-Stieltjes-Pettis integrable with respect to α on $[a,b]$ if f is Denjoy-Stieltjes-Dunford integrable with respect to α on $[a,b]$ and if $x_I^{**} \in X$ for every interval I in $[a,b]$.

(c) A function $f : [a,b] \to X$ is Denjoy-Stieltjes-Bochner integrable with respect to α on $[a,b]$ if there exists an ACG function $F : [a,b] \to X$ with respect to α such that F is approximately differentiable with respect to α almost everywhere on $[a,b]$ and $F'_{\alpha,ap} = f$ almost everywhere on $[a,b]$.

A function $f : [a,b] \to X$ is integrable in one of the above senses on a set $E \subset [a,b]$ if $f\chi_E$ is integrable in that sense on $[a,b]$.

THEOREM 4.9. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. If $f : [a,b] \to X$ is Denjoy-Stieltjes-Bochner integrable with respect to α on $[a,b]$, then $f : [a,b] \to X$ is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$.*

PROOF. If $f : [a,b] \to X$ is Denjoy-Stieltjes-Bochner integrable with respect to α on $[a,b]$, then there exists an ACG function $F : [a,b] \to X$ with respect to α such that F is approximately differentiable with respect to α almost everywhere on $[a,b]$ and $F'_{\alpha,ap} = f$ almost everywhere on $[a,b]$. It is easy to show that for each $x^* \in X^*$ x^*F is ACG with respect to α on $[a,b]$ and x^*F is approximately differentiable with respect to α almost everywhere on $[a,b]$ and $(x^*F)'_{\alpha,ap} = x^*f$ almost everywhere on $[a,b]$. Hence f is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$. □

THEOREM 4.10. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. If $f : [a,b] \to X$ is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$, then $f : [a,b] \to X$ is Denjoy-Stieltjes-Pettis integrable with respect to α on $[a,b]$.*

PROOF. Suppose that $f : [a,b] \to X$ is Denjoy-McShane-Stieltjes integrable with respect to α on $[a,b]$. Let $F(t) = (DMS)\displaystyle\int_a^t f d\alpha$. Since x^*F is ACG with respect to α on $[a,b]$ and $(x^*F)'_{\alpha,ap} = x^*f$ almost everywhere on $[a,b]$ for each $x^* \in X^*$, x^*f is Denjoy-Stieltjes integrable with respect to α on $[a,b]$ for each $x^* \in X^*$. For every interval $[c,d]$ in

$[a, b]$ and $x^* \in X^*$, we have

$$\begin{aligned} x^*(F(d) - F(c)) &= x^*F(d) - x^*F(c) \\ &= (DS)\int_a^d x^* f d\alpha - (DS)\int_a^c x^* f d\alpha \\ &= (DS)\int_c^d x^* f d\alpha. \end{aligned}$$

Since $F(d) - F(c) \in X$, f is Denjoy-Stieltjes-Pettis integrable with respect to α on $[a, b]$. □

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