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NORM OF THE COMPOSITION OPERATOR MAPPING BLOCH SPACE INTO HARDY OR BERGMAN SPACE

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Abstract. Let $1 \leq p < \infty$ and $\alpha > -1$. If f is a holomorphic self-map of the open unit disc U of $\mathbb{C}$ with $f(0) = 0$, then the quantity

$$\int_U \left\{ \frac{|f'(z)|}{1-|f(z)|^2} \right\}^p (1-|z|)^{\alpha+p} dxdy$$

is equivalent to the operator norm of the composition operator $C_f : \mathcal{B} \to A^{p,\alpha}$ defined by $C_f h = h \circ f - h(0)$, where $\mathcal{B}$ and $A^{p,\alpha}$ are the Bloch space and the weighted Bergman space on U respectively.

1. Introduction

Consider holomorphic mappings f of the unit ball of $\mathbb{C}^n$ into the unit disc U of $\mathbb{C}$. It is said that f has the pull-back property if $h \circ f \in BMOA$ whenever h belongs to the Bloch space $\mathcal{B}$ on U. Since the pull-back property was first studied for monomials in [1], there have been several examples and conditions for f to have the pull-back property ([1], [2], [7]). When $n = 1$, if f is a function of Yamashita's hyperbolic $BMOA$ class then the composition operator $\mathcal{C}_f$ defined by $\mathcal{C}_f(h) = h \circ f$ takes $\mathcal{B}$ into $BMOA$ ([6], [7]). In view of a known parallelism between the Hardy space H^p and the Yamashita hyperbolic Hardy class H^p_σ, the first author gave a necessary and sufficient condition for $\mathcal{C}_f$ to take $\mathcal{B}$ into H^{2p} ([6]).

We, in this paper, restrict ourselves to $n = 1$ and give a quantity equivalent to the operator norm $||\mathcal{C}_f||$ of the composition operator $\mathcal{C}_f$ that takes $\mathcal{B}$ boundedly into the weighted Bergman space $A^{p,\alpha}$.

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THEOREM 1. *Let $f : U \to U$ be a holomorphic function with $f(0) = 0$. For $1 \le p < \infty$ and $-1 < \alpha < \infty$, the bounded operator $C_f^0 : \mathcal{B} \to A^{p,\alpha}$ defined by $C_f^0 h = h \circ f - h(0)$ has its operator norm equivalent to the quantity*

$$\left\{ \int_U (1-|z|)^{\alpha+p} \Big(\frac{|f'(z)|}{1-|f(z)|^2} \Big)^p dxdy \right\}^{1/p}. \tag{1.1}$$

By the lemma of Schwarz-Pick, it is easy to see that (1.1) remains bounded for any holomorphic self map f of U. What Theorem 1 expresses is that there are positive constants C_1 and C_2 independent of f such that

$$C_1 \|C_f^0\| \le (1.1) \le C_2 \|C_f^0\|.$$

COROLLARY 2. *Let $f : U \to U$ be a holomorphic function. For $1 \le p < \infty$ and $-1 < \alpha < \infty$, the bounded operator $C_f^0 : \mathcal{B} \to A^{p,\alpha}$ defined by $C_f^0 h = h \circ \varphi_{f(0)} \circ f - h(0)$ has its operator norm equivalent to the quantity* (1.1).

2. Preliminaries

We introduce a few facts that we need in the sequel, most of which are well known.

The group of automorphisms of U will be denoted by $\mathcal{M}$. It is known that it consists of functions of the form $e^{i\beta}\varphi_a$, where β is a real number and

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in U.$$

For $1 \le p < \infty$ and for f subharmonic in U, we set

$$||f||_p := \sup_r \Big(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \Big)^{1/p}.$$

Then the class $H^p = H^p(U)$ consists of those f holomorphic in U for which $||f||_p < \infty$.

The Yamashita hyperbolic Hardy class H_σ^p is defined as the set of those holomorphic self-maps f of U for which $||\sigma(f)||_p < \infty$, where $\sigma(z)$ denotes the hyperbolic distance of z and 0 in U, namely,

$$\sigma(z) = \frac{1}{2} \log \frac{1+|z|}{1-|z|}.$$

We set, following Yamashita,

$$\lambda(f) = \log \frac{1}{1-|f|^2} \quad \text{and} \quad f^\sharp = \frac{|f'|}{1-|f|^2}$$

for holomorphic self-maps f of U. It is obvious that $f \in H^p_\sigma$ if and only if $||\lambda(f)||_p < \infty$ and that $f^\sharp$ is $\mathcal{M}$-invariant in the sense that $f^\sharp = (\varphi \circ f)^\sharp$ for any $\varphi \in \mathcal{M}$.

The Bloch space $\mathcal{B}$ consists of holomorphic functions h in U for which

$$\sup_{z\in U} |h'(z)|(1-|z|^2) < \infty.$$

This is a Banach space, if the norm $||h||_{\mathcal{B}}$ of $h \in \mathcal{B}$ is defined to be the sum of $|h(0)|$ and the left side of above inequality. A pair of Bloch functions $h_j, j = 1, 2$ are constructed such that

$$(2.1) \qquad (1-|z|^2)(|h_1'(z)| + |h_2'(z)|) \geq 1, \; z \in U$$

([7]). Then it follows that

$$(2.2) \qquad \frac{1}{1-|f|^2} \leq |h_1' \circ f| + |h_2' \circ f| \leq \frac{C}{1-|f|^2}$$

for holomorphic self-maps f, where $C = 2 \max(||h_1||_{\mathcal{B}}, \; ||h_2||_{\mathcal{B}})$. For $h \in \mathcal{B}$, it follows from Schwarz-Pick's Lemma ([5]) that

$$(2.3) \qquad |(h \circ f)'(z)| \leq ||h||_{\mathcal{B}} f^\sharp(z) \leq ||h||_{\mathcal{B}} \frac{1}{1-|z|^2}, \quad z \in U.$$

For $-1 < \alpha < \infty$ and $0 < p < \infty$, let $A^{p,\alpha}$ denote the weighted Bergman space of holomorphic functions on U, that is,

$$A^{p,\alpha} = \Big\{ f \text{ holomorphic on } U : ||f||_{A^{p,\alpha}} \equiv \Big(\int_U |f(z)|^p (1-|z|)^\alpha dxdy \Big)^{1/p} < \infty \Big\}.$$

We note that H^p is the limiting space of $A^{p,\alpha}$ as $\alpha \to -1$.

For h holomorphic in U, g-function of Paley defined by

$$g(\theta) := g(h)(\theta) = \Big(\int_0^1 |h'(re^{i\theta})|^2 (1-r) \; dr \Big)^{1/2}, \quad 0 \leq \theta < 2\pi,$$

satisfies

$$||g(h)||_{L^p} \sim ||h||_p \quad \text{if} \quad h(0) = 0, \tag{2.4}$$

for $1 \leq p < \infty$ (see [4] and [8]). Here and after $\psi \sim \phi$ means the equivalence of two quantities in the sense that either both sides are zeroes or the quotient ψ/ϕ lies between two positive constants depending only on p.

The hyperbolic version of g-function is defined as

$$g_\sigma(\theta) := g_\sigma(f)(\theta) = \int_0^1 \left(f^\sharp(re^{i\theta})\right)^2 (1-r)\, dr, \quad 0 \leq \theta < 2\pi,$$

and then for $1 \leq p < \infty$ it is satisfied that

$$||\lambda(f)||_p \sim ||g_\sigma(f)||_{L^p} \quad \text{if} \quad f(0) = 0. \tag{2.5}$$

See [6].

3. Proof of the results

For functions holomorphic in U and for $0 < p < \infty$, $0 \leq r < 1$, $M_p(r, f)$ is defined as usual by

$$M_p(r, f) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p}.$$

For simplicity, we denote $\psi \lesssim \phi$ meaning that either $\psi \sim \phi$ or the quotient ψ/ϕ is bounded by a positive constant depending only on p.

Lemma. *Let f be holomorphic in U. Then, for $1 \leq p < \infty$ and $-1 < \alpha < \infty$,*

$$\int_0^1 (1-r)^\alpha M_p^p(r, f)\, dr \sim \int_0^1 (1-r)^{\alpha+p} M_p^p(r, f')\, dr + |f(0)|^p. \tag{3.1}$$

Proof. Applying the same process as in the proof of [3, Theorem 5.6] to $1 \leq p < \infty$, we can obtain

$$\int_0^1 (1-r)^\alpha M_p^p(r, f)\, dr \lesssim \int_0^1 (1-r)^{\alpha+p} M_p^p(r, f')\, dr + |f(0)|^p.$$

Conversely, when $\rho = \frac{1}{2}(1+r)$ we see in the proof of [3, Theorem 5.5]

$$M_p(r,f') \le \frac{M_p(\rho,f)}{\rho^2 - r^2}.$$

If we integrate both sides of this inequality with respect to dr after multiplying them by $(1-r)^{\alpha+p}$, then we obtain

$$\int_0^1 (1-r)^{\alpha+p} M_p^p(r,f')\, dr \lesssim \int_0^1 (1-r)^{\alpha} M_p^p\Big(\frac{1+r}{2}, f\Big)\, dr,$$

whence a change of variable completes the proof. □

Proof of Theorem 1. We show that

$$||C_f^0|| \sim (1.1).$$

By (3.1) and (2.3), we have

$$\begin{aligned} ||C_f^0|| &= \sup_{\substack{h\in\mathcal{B}\\ ||h||_{\mathcal{B}}\le 1}} \Big\{\int_U (1-|z|)^{\alpha}\Big|(h\circ f)(z) - h(0)\Big|^p dxdy\Big\}^{1/p} \\ &\sim \sup_{\substack{h\in\mathcal{B}\\ ||h||_{\mathcal{B}}\le 1}} \Big\{\int_U (1-|z|)^{\alpha+p} |(h\circ f)'(z)|^p dxdy\Big\}^{1/p} \\ &\le \Big\{\int_U (1-|z|)^{\alpha+p} \Big(f^\sharp(z)\Big)^p dxdy\Big\}^{1/p}. \end{aligned}$$

Conversely, using Minkowski's inequality with those h_j, $j=1,2$, of (2.1) and using (3.1), we obtain

$$\begin{aligned} &\Big\{\int_U (1-|z|)^{\alpha+p}\Big(f^\sharp(z)\Big)^p dxdy\Big\}^{1/p} \\ &\le \Big\{\int_U (1-|z|)^{\alpha+p}\Big(\sum_{j=1}^{2} |(h_j\circ f)'(z)|\Big)^p dxdy\Big\}^{1/p} \\ &\le \sum_{j=1}^{2}\Big\{\int_U (1-|z|)^{\alpha+p} |(h_j\circ f)'(z)|^p dxdy\Big\}^{1/p} \\ &\le \sum_{j=1}^{2}\Big\{\int_U (1-|z|)^{\alpha}\Big|(h_j\circ f)(z) - h_j(0)\Big|^p dxdy\Big\}^{1/p}. \end{aligned} \tag{3.2}$$

Since

$$\left\{\int_U (1-|z|)^\alpha \Big|(h_j\circ f)(z)-h_j(0)\Big|^p dxdy\right\}^{1/p}$$
$$\le \|h_j\|_{\mathcal{B}} \sup_{\substack{h\in\mathcal{B}\\ \|h\|_{\mathcal{B}}\le 1}} \left\{\int_U (1-|z|)^\alpha \Big|(h\circ f)(z)-h(0)\Big|^p dxdy\right\}^{1/p}, \quad j=1,2,$$

from (3.2) we have

$$\begin{aligned}
&\left\{\int_U (1-|z|)^{\alpha+p}\Big(f^\sharp(z)\Big)^p dxdy\right\}^{1/p}\\
&\lesssim \sup_{\substack{h\in\mathcal{B}\\ \|h\|_{\mathcal{B}}\le 1}} \left\{\int_U (1-|z|)^\alpha \Big|(h\circ f)(z)-h(0)\Big|^p dxdy\right\}^{1/p}\\
&= \sup_{\substack{h\in\mathcal{B}\\ \|h\|_{\mathcal{B}}\le 1}} \|h\circ f-h(0)\|_{A^{p,\alpha}}\\
&= \|\mathcal{C}_f^0\|.
\end{aligned}$$

□

PROOF OF COROLLARY 2. The result follows from $\mathcal{M}$-invariance of $f^\sharp$ and Theorem 1. □

4. Limiting case H^p

As is well-known, we may regard $A^{p,-1} = H^p$. Using (2.1), (2.2), (2.3), (2.4) and (2.5), a method similar to Proof of Theorem 1 gives that the the quantity $\|\lambda(f)\|_p^{1/2}$ is equivalent to the norm of Bloch-H^p pullback operator. This fact will be discussed extensively in the coming paper of the first author.

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