Commun. Korean Math. Soc. 18 (2003), No. 4, pp. 661–667

ON A CASE OF ALL PAIRS OF POLYNOMIAL ZEROS BAD

SEON-HONG KIM

ABSTRACT. In this paper, we show, for a positive integer m and a large odd integer n, the polynomial equation

$$\prod_{k=0}^{n}(x-k^{1+\frac{1}{m}})+\prod_{k=n+1}^{2n+1}(x-k^{1+\frac{1}{m}})=0$$

has a real zero on

$$\left((n+1)^{1+\frac{1}{m}},\ (n+1)^{1+\frac{1}{m}}+\frac{1}{2}\right)$$

and

$$\left((n+1)^{1+\frac{1}{m}}+\frac{1}{2},\ (n+2)^{1+\frac{1}{m}}\right),$$

respectively.

1. Introduction

Given some (or all) information about the individual two polynomials, in particular about their factorizations, what can be said about the factorization of their sum? For various results and examples about this, see [3].

Let $P_A(x)$ and $P_B(x)$ be monic real polynomials with degree n having all zeros distinct and real. Then it is natural for us to be interested in the number of real zeros of $P_A(x)+P_B(x)$. It is an easy consequence of Fell [1] that, if the all zeros of $P_A(x)$ and $P_B(x)$ form good pairs (for definitions of good pairs and bad pairs, see Definition 1.1 of [2]), $P_A(x)+P_B(x)$ has all its zeros real. But there has been no neat result

Received November 5, 2002.
2000 Mathematics Subject Classification: Primary 30C15; Secondary 26C10.
Key words and phrases: zero, polynomial, bad pair, good pair.

for the case having bad pairs so far. In this paper, we consider a case of all pairs of polynomial zeros bad.

One can check, by computational search, an arithmetic progression $F = \{1, 2, \cdots, 2n\}$ of length ≤ 10 satisfies the following condition.

CONDITION 1. *Whenever* $F = A \cup B$, $A \cap B = \emptyset$, $|A| = |B| = n$, *the number of bad pairs of* F *is equal to the number of nonreal zeros of*

$$\prod_{a_i \in A} (x - a_i) + \prod_{b_j \in B} (x - b_j).$$

But an arithmetic progression F does not satisfy Condition 1. For example, there are (only) ten polynomials for the arithmetic progression of length 12 that do not satisfy Condition 1. However, for an odd integer $n \geq 3$, taking

$$A = \{0, 1, 2, \cdots, n\} \quad \text{and} \quad B = \{n+1, n+2, \cdots, 2n+1\}$$

in Condition 1 leads the conclusion of Condition 1, i.e., the polynomial equation

$$\prod_{k=0}^{n} (x - k) + \prod_{k=n+1}^{2n+1} (x - k) = 0$$

has all pairs bad and all its zeros nonreal on $\Re x = n + 1/2$. Hence we might conjecture that, for an odd integer n and large positive integer m, all zeros of

$$\prod_{k=0}^{n} (x - k^{1+\frac{1}{m}}) + \prod_{k=n+1}^{2n+1} (x - k^{1+\frac{1}{m}}) = 0. \tag{1}$$

lie very near on $\Re x = n + 1/2$ and all nonreal. Here we observe that all pairs of the zeros of each summand of (1) are still bad. However, in this paper, we show that (1) has two real zeros near $(n+1)^{1+1/m} + 1/2$. More precisely,

THEOREM 2. *Let* n *be a large odd integer. Then for a positive integer* m, *the polynomial*

$$f(x) := \prod_{k=0}^{n} (x - k^{1+\frac{1}{m}}) + \prod_{k=n+1}^{2n+1} (x - k^{1+\frac{1}{m}})$$

has a real zero on intervals

$$\left((n+1)^{1+\frac{1}{m}}, \, (n+1)^{1+\frac{1}{m}} + \frac{1}{2} \right) \text{ and } \left((n+1)^{1+\frac{1}{m}} + \frac{1}{2}, \, (n+2)^{1+\frac{1}{m}} \right),$$

respectively.

2. Proof of the result

Main purpose of this section is to prove Theorem 2. For the proof of Theorem 2, we need Euler-Maclaurin summation formula and an integral inequality, Lemma 4 below.

THEOREM 3. **(Euler-Maclaurin summation formula)** *Let f have continuous function on $[a,b]$ with a piecewise continuous derivative there. Then*

$$\sum_{a\le n\le b} f(n) = \int_a^b f(t)\,dt + \int_a^b \left(\{t\}-\frac{1}{2}\right) f'(t)\,dt - \left(\{t\}-\frac{1}{2}\right) f(t)|_{a-}^{b},$$

where $\{t\} = t - [t]$.

LEMMA 4. *Let m be a fixed positive integer and $p>0$. Then for a positive integer n,*

$$\int_{n+\frac{1}{2}}^{2n} \log\left|t^{1+\frac{1}{m}} - n^{1+\frac{1}{m}}\right|^p dt > \int_0^{n-\frac{1}{2}} \log\left|n^{1+\frac{1}{m}} - t^{1+\frac{1}{m}}\right|^p dt.$$

PROOF. Let m a fixed positive integer and $p>0$. Put $\alpha = 1+1/m$. By appropriate change of variable

$$I_1 =: \int_{n+\frac{1}{2}}^{2n} \log|t^\alpha - n^\alpha|^p\,dt = \int_{\frac{1}{2}}^{n} \log|(n+u)^\alpha - n^\alpha|^p\,du,$$

and

$$I_2 =: \int_0^{n-\frac{1}{2}} \log|n^\alpha - t^\alpha|^p\,dt = \int_{\frac{1}{2}}^{n} \log|n^\alpha - (n-u)^\alpha|^p\,du.$$

Thus $I_1 > I_2$ becomes

$$\int_{\frac{1}{2}}^{n} \log\left|\frac{(n+u)^\alpha - n^\alpha}{n^\alpha - (n-u)^\alpha}\right|^p du > 0,$$

and it suffices to show that the integrand is positive, i.e. $(n+u)^\alpha - n^\alpha > n^\alpha - (n-u)^\alpha$. Note that if $\alpha > 1$ and $x \ge -1, x \ne 0$, then

$$(1+x)^\alpha > 1 + \alpha x.$$

So, for $\frac{1}{2} \le u \le n$,

$$\left(1+\frac{u}{n}\right)^{\alpha} + \left(1-\frac{u}{n}\right)^{\alpha} > 2,$$

i.e.,

$$(n+u)^{\alpha} + (n-u)^{\alpha} > 2n^{\alpha}.$$

This completes the proof. □

It follows from sign changes that the next lemma proves Theorem 2.

LEMMA 5. *Let m be a fixed positive integer. Then for n large odd integer and*

$$x = \frac{1}{2} + (n+1)^{1+\frac{1}{m}},$$

we have

$$\prod_{k=0}^{n} (x - k^{1+\frac{1}{m}}) < \left| \prod_{k=n+1}^{2n+1} (x - k^{1+\frac{1}{m}}) \right|.$$

PROOF. Let m be a fixed positive integer. It is enough to show that, for n large even integer and

$$x = \frac{1}{2} + n^{1+\frac{1}{m}},$$

we have

$$\sum_{k=0}^{n-1} \log (x - k^{1+\frac{1}{m}}) < \sum_{k=n}^{2n-1} \log |x - k^{1+\frac{1}{m}}|. \tag{2}$$

For convenience, we define

$$g(t) = \log |x - t^{1+\frac{1}{m}}| = \log \left| \frac{1}{2} + n^{1+\frac{1}{m}} - t^{1+\frac{1}{m}} \right|.$$

Now, by Theorem 3, the left side of (2) equals

$$\begin{aligned} &\int_0^{n-\frac{1}{2}} \log (x - t^{1+\frac{1}{m}})\, dt + \int_0^{n-\frac{1}{2}} \left(\{t\} - \frac{1}{2}\right) g'(t)\, dt \\ &- \left[\left(\{t\} - \frac{1}{2}\right) \log (x - t^{1+\frac{1}{m}})\right]_{0-}^{n-\frac{1}{2}} \\ =& \int_0^{n-\frac{1}{2}} \log (x - t^{1+\frac{1}{m}})\, dt + \int_0^{n-\frac{1}{2}} \left(\{t\} - \frac{1}{2}\right) g'(t)\, dt - \frac{1}{2} \log x, \end{aligned}$$

and the right side of (2) equals

$$\log \frac{1}{2} - \log \left((2n)^{1+\frac{1}{m}} - x\right) + \sum_{k=n+1}^{2n} \log (k^{1+\frac{1}{m}} - x)$$
$$= \log \frac{1}{2} - \log \left((2n)^{1+\frac{1}{m}} - x\right) + \int_{n+\frac{1}{2}}^{2n} \log (t^{1+\frac{1}{m}} - x)\, dt$$
$$+ \int_{n+\frac{1}{2}}^{2n} \left(\{t\} - \frac{1}{2}\right) g'(t)\, dt + \frac{1}{2} \log ((2n)^{1+\frac{1}{m}} - x).$$

To show the inequality (2), we need an integral inequality:

$$\int_0^{n-\frac{1}{2}} \log (x - t^{1+\frac{1}{m}})\, dt < \int_{n+\frac{1}{2}}^{2n} \log (t^{1+\frac{1}{m}} - x)\, dt.$$

This can be easily verified from n large and Lemma 4. On the other hand, the second integral of the above equality can be rewritten by changing variable as following:

$$\int_{n+\frac{1}{2}}^{2n} \left(\{t\} - \frac{1}{2}\right) g'(t)\, dt = \int_0^{n-\frac{1}{2}} \left(\{t + \frac{1}{2}\} - \frac{1}{2}\right) g'(n + t + \frac{1}{2})\, dt.$$

Hence it is enough to show that

$$- \log \left((2n)^{1+\frac{1}{m}} - x\right) + \frac{1}{2} \log ((2n)^{1+\frac{1}{m}} - x) + \log \frac{1}{2} + \frac{1}{2} \log x$$
$$+ \int_0^{n-\frac{1}{2}} \left(\{t + \frac{1}{2}\} - \frac{1}{2}\right) g'(n + t + \frac{1}{2})\, dt$$
$$- \int_0^{n-\frac{1}{2}} \left(\{t\} - \frac{1}{2}\right) g'(t)\, dt > 0.$$

Now we let

$$T(n) = \int_0^{n-\frac{1}{2}} \left(\{t + \frac{1}{2}\} - \frac{1}{2}\right) g'(n+t+\frac{1}{2})\, dt - \int_0^{n-\frac{1}{2}} \left(\{t\} - \frac{1}{2}\right) g'(t)\, dt.$$

Then

$$T(n) = \frac{1}{2} \int_0^{n-\frac{1}{2}} \left(g'(t) - g'(n + t + \frac{1}{2})\right)\, dt$$
$$+ \int_0^{n-\frac{1}{2}} \{t + \frac{1}{2}\} g'(n + t + \frac{1}{2}) - \{t\} g'(t)\, dt$$
$$= \frac{1}{2} \left[g(t) - g(n + t + \frac{1}{2})\right]_0^{n-\frac{1}{2}} + N(n),$$

where $N(n) = \int_0^{n-\frac{1}{2}} \{t+\frac{1}{2}\} g'(n+t+\frac{1}{2}) - \{t\} g'(t)\, dt$. Now we estimate $N(n)$. For this, we set

$$A(n) = \left\{ t \in [0, n - \frac{1}{2}) : 0 \le \{t\} < \frac{1}{2} \right\},$$
$$B(n) = \left\{ t \in [0, n - \frac{1}{2}) : \frac{1}{2} \le \{t\} < 1 \right\}.$$

Then it is obvious that

$$\left\{ t + \frac{1}{2} \right\} = \begin{cases} \{t\} + \frac{1}{2}, & t \in A(n), \\ \{t\} - \frac{1}{2}, & t \in B(n). \end{cases}$$

And it follow from $g'(t) < 0$, $g'(n+t+\frac{1}{2}) > 0$ on $[0, n-\frac{1}{2}]$ that

$$\begin{aligned} N(n) = & \int_{A(N)} \{t\} \left(g'(n+t+\frac{1}{2}) - g'(t) \right) dt \\ & + \int_{B(N)} \{t\} \left(g'(n+t+\frac{1}{2}) - g'(t) \right) dt \\ & + \frac{1}{2} \left(\int_{A(n)} g'(n+t+\frac{1}{2})\, dt - \int_{B(n)} g'(n+t+\frac{1}{2})\, dt \right) \\ \ge & \frac{1}{2} \int_{B(n)} \left(g'(n+t+\frac{1}{2}) - g'(t)\, dt \right) \\ & + \frac{1}{2} \left(\int_{A(n)} g'(n+t+\frac{1}{2})\, dt - \int_{B(n)} g'(n+t+\frac{1}{2})\, dt \right) \\ = & \frac{1}{2} \left(\int_{A(n)} g'(n+t+\frac{1}{2})\, dt - \int_{B(n)} g'(t)\, dt \right) \\ > & 0. \end{aligned}$$

Hence it remains to show that

$$-\frac{1}{2} \log \left((2n)^{1+\frac{1}{m}} - x \right) + \log \frac{1}{2} + \frac{1}{2} \log x + \frac{1}{2} \left[g(t) - g(n+t+\frac{1}{2}) \right]_0^{n-\frac{1}{2}} > 0.$$

This is easy because the left side of the inequality equals

$$\log \frac{1}{2} + \frac{1}{2} \left(\log \left(x - (n-\frac{1}{2})^{1+\frac{1}{m}} \right) - \log \left((n+\frac{1}{2})^{1+\frac{1}{m}} - x \right) \right),$$

which is positive for sufficiently large even integer n. In all, the inequality (2) is proved. $\square$

From Theorem 2, we let

$$f_1(x) = \prod_{k=0}^{n} (x - k^{1+\frac{1}{m}}), \qquad f_2(x) = \prod_{k=n+1}^{2n+1} (x - k^{1+\frac{1}{m}}).$$

From n odd and sign changes, both $f_1(x)$ and $f_2(x)$ are positive on the interval

$$\left(n^{1+\frac{1}{m}},\ (n+1)^{1+\frac{1}{m}}\right) \cup \left((n+2)^{1+\frac{1}{m}},\ (n+3)^{1+\frac{1}{m}}\right).$$

But on the interval

$$\left((n+1)^{1+\frac{1}{m}},\ (n+2)^{1+\frac{1}{m}}\right),$$

we have $f_1(x) > 0$ and $f_2(x) < 0$. Now Lemma 5 asserts that $f(x) = f_1(x) + f_2(x) < 0$ for

$$x = \frac{1}{2} + (n+1)^{1+\frac{1}{m}},$$

which completes the proof of Theorem 2.

References

[1] H. J. Fell, *On the zeros of convex combinations of polynomials*, Pacific J. Math. **89** (1980), 43–50.

[2] S.-H. Kim, *Bad pairs of polynomial zeros*, Commun. Korean Math. Soc. **15** (2000), no. 4, 697–706.

[3] ———, *Factorization of Sums of Polynomials*, Acta Appl. Math. **73** (2002), no. 3, 275–284.

Department of Mathematics
College of Natural Science
Chosun University
Gwangju 501-759, Korea
E-mail: shkim17@mail.chosun.ac.kr