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ON THE SPECTRUM OF THE RHALY OPERATORS ON bv_0

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ABSTRACT. In 1989, Rhaly [4] determined the spectrum of Rhaly operator R_a on the Hilbert space ℓ_2. In this paper Authors determine the spectrum of the Rhaly matrix R_a as an operator on the space bv_0 with assumption $0 < L = \lim_n (n+1)a_n < \infty$.

1. Introduction

Given a scalar sequence of $a = (a_n)$, a Rhaly matrix $R_a = (a_{nk})$ is the lower triangular matrix where $a_{nk} = a_n$ for $k \leq n$ and $a_{nk} = 0$ for n, k otherwise.

c_0, bv, bv_0 and bs; will denote the space of null sequence, sequences x such that $\sum_k \mid x_k - x_{k-1} \mid < \infty$, $bv_0 = bv \cap c_0$, sequences x such that $\sup_{n \geq 0} \mid \sum_{k=0}^{n} x_k \mid < \infty$ respectively. From [5, formula 119], $A : bv_0 \to bv_0$ if and only if

$$\lim_n a_{nk} = 0 \quad \text{for all } k \tag{1.1}$$

and

$$\|A\|_{bv_0} := \sup_N \sum_n \left| \sum_{k=0}^{N} (a_{nk} - a_{n-1,k}) \right| < \infty. \tag{1.2}$$

For $a = (\frac{1}{n+1})$, the spectra of the Cesàro matrix on bv_0 are studied by Okutoyi [2]. In [4] taking $L = \lim_n (n+1)a_n$ Rhaly showed that R_a is a bounded operator on the Hilbert space ℓ_2 of square summable sequences, and he also determined its spectrum as

$$(\sigma(R_a, \ell_2) = \{ \lambda \ : \mid \lambda - L \mid < L\} \ \cup \ \{ a_n \ : \ n = 0, 1, 2, \ldots\}).$$

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In this paper, we assume that (a) $L = \lim_n (n+1)a_n$ exists, finite, and nonzero, (b) $a_n > 0$ for all n, (c) (a_n) is monotone decreasing sequence, (d) $S := \{ a_n \ : \ n = 0, 1, 2, \ \dots \}$. We represent the set of eigenvalues of the Rhaly matrix R_a and the spectrum of R_a on the Banach space bv_0 by $\pi_0(R_a, bv_0)$ and $\sigma(R_a, bv_0)$, respectively. Under the above conditions, the purpose of this study is to determine the spectrum of Rhaly operator R_a as an operator on the Banach space bv_0.

2. Matrix operators on bv_0

Now, it will be shown that $R_a \in B(bv_0)$ under the above conditions.

THEOREM 2.1. *If* $\{(n+1)a_n\}$ *monotone and* $\lim_n (n+1)a_n = L < \infty$, $R_a \in B(bv_0)$.

PROOF. From [5, Formula 9], $R_a \in B(bv_0)$ if and only if $\lim_n a_{nk} = 0$ for all k, and

$$\|R_a\|_{bv_0} := \sup_i R_i < \infty$$

where $\sum_n \mid \sum_{k=0}^{i} (a_{nk} - a_{n-1,k}) \mid < \infty$. So we have

$$\begin{aligned}
R_i &:= \sum_{n=0}^{\infty} \mid \sum_{k=0}^{i} (a_{nk} - a_{n-1,k}) \mid \\
&= \sum_{n=0}^{i} \mid \sum_{k=0}^{n} (a_{nk} - a_{n-1,k}) \mid + \sum_{n=i+1}^{\infty} \mid \sum_{k=0}^{i} (a_{nk} - a_{n-1,k}) \mid \\
&= a_0 + \sum_{n=1}^{i} \mid \left(\sum_{k=0}^{n} (a_{nk} - a_{n-1,k}) \right) + a_n \mid + \sum_{n=i+1}^{\infty} \mid \sum_{k=0}^{i} (a_n - a_{n-1}) \mid \\
&= a_0 + \sum_{n=1}^{i} \mid [(n+1)a_n - n a_{n-1}] + \sum_{n=i+1}^{\infty} (i+1)(a_{n-1} - a_n) \mid \\
&= a_0 + \sum_{n=1}^{i} \mid (n+1)a_n - n a_{n-1} \mid + (i+1)a_i - (i+1) \lim_{m \to \infty} a_m \\
&= a_0 + \sum_{n=1}^{i} \mid (n+1)a_n - n a_{n-1} \mid + (i+1)a_i.
\end{aligned}$$

i) If $\{(n+1)a_n\}$ is monotone increasing, then we have

$$sup_i R_i = 2a_0 = \| \ R_a \ \| \ .$$

ii) If $\{(n+1)a_n\}$ is monotone decreasing, then we obtain

$$sup_i R_i = \lim_{i\to\infty} 2(i+1)a_i = 2L = \| R_a \| .$$

□

For example, if $a_n = \frac{n+2}{(n+1)^2}$, then $(n+1)a_n < na_{n-1}$ and if $a_n = \sin\frac{1}{(n+1)}$ then $(n+1)a_n > na_{n-1}$.

THEOREM 2.2. *If $\{(n+1)a_n\}$ monotone and $\lim_n(n+1)a_n = L < \infty$, then*

$$S \cap (2L, \infty) \subseteq \pi_0(R_a, bv_0). \tag{2.1}$$

PROOF. Since $bv_0 \subseteq c_0$, the proof is trivial. □

LEMMA 2.3. *Let $\{(n+1)a_n\}$ monotone and $\lim_n(n+1)a_n = L < \infty$. Then R_a^* which is the adjoint operator of R_a is transpose of the matrix R_a on bv_0 and $R_a^* \in B(bv_0^* \cong bs)$.*

ispat. bv_0 is an AK-space and $bv_0^* \cong bs$ [6, p.110]. Hence from [6, p.266] we have

$$R_a^* = R_a^t = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_2 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.2}$$

Since bv_0 is a Banach space, $\| R_a \|_{bv_0} = \| R_a^* \|_{bv_0^*} = \| R_a^t \|_{bs}$. Thus, from Theorem 2.1, $R_a^t \in B(bv_0^*)$. □

LEMMA 2.4. *Let $0 < L = \lim_n(n+1)a_n < \infty$ and*

$$Z_n := \prod_{\vartheta=0}^{n} \left(1 - \frac{a_\vartheta}{\lambda}\right), \ \lambda \neq 0, \ \lambda \in C.$$

Then the partial sums of $\sum_{\vartheta=1}^{\infty} Z_n$ are bounded if and only if $LRe\frac{1}{\lambda} \geq 1, \lambda \neq L$.

PROOF. We show the proof of the Lemma as proved in [2, Lemma 1.6]. The series $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ is uniformly convergent in every subinterval of $| u |< 1$. Hence $\ln(1-u) = -u + O(u^2)$, uniformly in $| u |< 1/2$,

$u \in C$. Since $a_\vartheta \to 0$, for a given $\lambda \neq 0$ there exist ϑ_0 such that $\frac{a_\vartheta}{\lambda} \leq \frac{1}{2}$ for $\vartheta > \vartheta_0$,

$$\begin{aligned}\ln Z_n &= \ln \textstyle\prod_{\vartheta=0}^{n}(1 - \frac{a_\vartheta}{\lambda}) = \sum_{\vartheta=0}^{n} \ln(1 - \frac{a_\vartheta}{\lambda}) \\ &= C + \textstyle\sum_{\vartheta=\vartheta_0}^{n} \ln(1 - \frac{a_\vartheta}{\lambda}) \\ &= C + \textstyle\sum_{\vartheta=\vartheta_0}^{n}(-\frac{a_\vartheta}{\lambda} + O\left(\frac{a_\vartheta^2}{|\lambda|^2}\right)) \\ &= C - \frac{L}{\lambda}\textstyle\sum_{\vartheta=\vartheta_0}^{n} \frac{1}{\vartheta+1} + \frac{L^2}{|\lambda|^2}\sum_{\vartheta=\vartheta_0}^{n} O\left(a_\vartheta^2\right),\end{aligned}$$

where $t_\vartheta = O(\frac{1}{\vartheta^2})$. Now since $t_\vartheta = O(\frac{1}{\vartheta^2})$,

$$\sum_{\vartheta=\vartheta_0}^{n} t_\vartheta = \sum_{\vartheta=\vartheta_0}^{\infty} t_\vartheta - \sum_{\vartheta=n+1}^{\infty} t_\vartheta = C + O(\frac{1}{n}).$$

Since if $C_n = \sum_{\vartheta=0}^{n} \frac{1}{\vartheta+1} - \log n$, then

$$\begin{aligned}C_{n+1} - C_n &= \frac{1}{2+n} - \log\frac{n+1}{n} = \frac{1}{2+n} - \log(1 + \frac{1}{n}) \\ &= \frac{1}{2+n} - \frac{1}{n} + O(\frac{1}{n^2})\end{aligned}$$

we have

$$\sum_{\vartheta=\vartheta_0}^{n} \frac{1}{\vartheta+1} = C + \log n + O(\frac{1}{n}).$$

So

$$\begin{aligned}C_{n+1} &= C_0 + \sum_{\vartheta=0}^{n}(C_{\vartheta+1} - C_\vartheta) \\ &= C_0 + \sum_{\vartheta=0}^{\infty}(C_{\vartheta+1} - C_\vartheta) \sum_{\vartheta=n+1}^{\infty}(C_{\vartheta+1} - C_\vartheta) \\ &= C + O(\frac{1}{n}).\end{aligned}$$

Hence as $n \to \infty$,

$$\log Z_n = C - \frac{L}{n}\log n + O(\frac{1}{n}),$$

that is

$$\begin{aligned} Z_n &= exp(C - \frac{L}{n}\log n + O(\frac{1}{n})) \\ &= exp(C)n^{-\frac{L}{\lambda}}(1 + O(\frac{1}{n})) \\ &= An^{\frac{L}{\lambda}}O(n^{-LRe\frac{1}{\lambda}-1})). \end{aligned}$$

If $L\lambda \neq 1$, $LRe(\frac{1}{\lambda}) \geq 1$, then $s_n = \sum_{k=1}^{n} k^{-\frac{L}{\lambda}}$ are bounded and $\sum_{n=1}^{\infty} n^{-LRe\frac{1}{\lambda}-1} < \infty$. So that the partial sums of $\sum_n Z_n$ are bounded. If $0 < LRe(\frac{1}{\lambda}) < 1$ or $L\lambda = 1$, then the partial sums of $\sum_{n=1}^{\infty} n^{-LRe\frac{1}{\lambda}}$ are unbounded, but still we have $\sum_{n=1}^{\infty} n^{-LRe\frac{1}{\lambda}-1} < \infty$. If $0 < LRe\frac{1}{\lambda} \leq 0$ then

$$\sum_{n=1}^{N} n^{-\frac{L}{\lambda}} \asymp \frac{N^{1-\frac{L}{\lambda}}}{1 - \frac{L}{\lambda}} \tag{2.3}$$

where $a_n \asymp b_n$ means that there exist $m, M \in R^+$ such that $mb_n < a_n < Mb_n$.

Using (2.3), we see that the partial sums of $\sum_{n=1}^{\infty} n^{-\frac{L}{\lambda}}$ are unbounded although $\sum_{n=1}^{\infty} n^{-LRe\frac{1}{\lambda}-1} < \infty$ and hence we obtain that the partial sums of $\sum_n Z_n$ are bounded if and only if $LRe\frac{1}{\lambda} \geq 1$.

THEOREM 2.5. *If $\{(n+1)a_n\}$ monotone and $0 < L = \lim_n (n+1)a_n < \infty$, then $S \cup \left(\{\lambda : \mid \lambda - \frac{L}{2} \mid \leq \frac{L}{2}\} - \{0\}\right) \subset \pi_0(R_a^*, bv_0^* \cong bs)$.*

PROOF. If $R_a^* x = \lambda x$, then

$$\lambda a_n^{-1} x_{n+1} = (\lambda a_n^{-1} - 1)x_n. \tag{2.4}$$

Hence $0 \in \pi_0(R_a^*, bs)$ (because if $\lambda = 0$, then $x = \theta$). From (4), we have

$$x_{n+1} \;=\; (1 - \frac{a_n}{\lambda})x_n. \tag{2.5}$$

If $\lambda = a_m$, $\lambda \in \pi_0(R_a^*, bs)$ (because for $n \geq m+1,\ x_n = 0$). From (2.5), we have

$$x_n \;=\; \prod_{j=0}^{n-1}(1 - \frac{a_j}{\lambda})x_0. \tag{2.6}$$

From Lemma 2.4 the other λ's have the properties $\alpha L \geq 1$. Hence we obtain

$$S \cup \left(\{\lambda : \mid \lambda - \frac{L}{2} \mid \leq \frac{L}{2}\} - \{0\}\right) \subset \pi_0(R_a^*, bv_0^* \cong bs).$$

LEMMA 2.6. *Let $T_\lambda = \lambda I - R_a$. Then T_λ^{-1} is given by*

$$T_\lambda^{-1} = (b_{nk}) = \begin{cases} \frac{1}{\lambda - a_n}, & k=n \\ \frac{a_n}{\lambda^2 \prod_{j=k}^{n}\left(1-\frac{a_j}{\lambda}\right)}, & k < n \\ 0, & otherwise. \end{cases} \tag{2.7}$$

PROOF. If $T_\lambda x = y$, then we have

$$x_0 = \frac{1}{\lambda - a_0} y_0$$

$$x_1 = \frac{1}{\lambda - a_1} y_1 + \frac{a_1}{(\lambda - a_1)(\lambda - a_0)} y_0$$

$$x_2 = \frac{1}{\lambda - a_2} y_2 + \frac{a_1}{(\lambda - a_2)(\lambda - a_1)} y_1 + \frac{a_2 \lambda}{(\lambda - a_2)(\lambda - a_1)(\lambda - a_0)} y_1$$

$$\vdots \; \vdots \; \vdots$$

$$x_n = \frac{1}{\lambda - a_n} y_n + \frac{a_n}{(\lambda - a_n)(\lambda - a_{n-1})} y_{n-1} + \frac{a_n \lambda}{(\lambda - a_n)(\lambda - a_{n-1})(\lambda - a_{n-2})} y_{n-2} + \cdots + \frac{a_n \lambda^{n-2}}{\prod_{k=1}^{n}} (1 - \frac{a_k}{\lambda}) y_1 + \frac{a_n \lambda^{n-1}}{\prod_{k=0}^{n}} (1 - \frac{a_k}{\lambda}) y_0$$

$$\vdots \; \vdots \; \vdots$$

Therefore $T_\lambda^{-1} = (b_{nk})$ is given by (2.7). □

LEMMA 2.7. *If $Re\frac{1}{\lambda} = \alpha$, then*

$$\prod_{k=0}^{N-1} \mid 1 - \frac{a_k}{\lambda} \mid \simeq \frac{1}{N^{\alpha L}} \tag{2.8}$$

as $N \longrightarrow \infty$. We use the notation $a_n \simeq b_n$ in the sense that $\left(\frac{a_n}{b_n}\right)$, $\left(\frac{b_n}{a_n}\right)$ are both bounded.

PROOF. See [7]. □

THEOREM 2.8. *If $\{(n+1)a_n\}$ monotone and $0 < L = \lim_n (n+1)a_n < \infty$, then*

$$\sigma(R_a, bv_0) = \left\{ \lambda : \mid \lambda - \frac{L}{2} \mid \leq \frac{L}{2} \right\} \cup S.$$

PROOF. By Theorem 2.5, and Lemma 2.7, we have

$$\left\{ \lambda \ : | \ \lambda - \frac{L}{2} \ | < \ \frac{L}{2} \right\} \ \cup \ S$$
$$\subseteq \pi_0(R_a^*, bv_0 \cong bs) \ \subseteq \ \sigma(R_a^*, bv_0^*) \ = \ \sigma(R_a, bv_0).$$

Hence

$$\left\{ \lambda \ : | \ \lambda - \frac{L}{2} \ | \leq \ \frac{L}{2} \right\} \ \cup \ S \ \subseteq \sigma(R_a, bv_0).$$

To complete the proof let us show that,

$$\sigma(R_a, bv_0) \ \subseteq \left\{ \lambda \ : | \ \lambda - \frac{L}{2} \ | \leq \ \frac{L}{2} \right\} \ \cup \ S.$$

Now, let $| \ \lambda - \frac{L}{2} \ | > \ \frac{L}{2}$, (*which means* $\alpha L < 1$) and $\lambda \neq a_m$ $(m = 0, 1, 2, \ldots)$. We prove that the matrix $T^{-1} = (b_{nk})$ given by Lemma 2.6 satisfies the properties in equations (1.1) and (1.2).

Since $\alpha L < 1 \Leftrightarrow | \ \lambda - \frac{L}{2} \ | > \frac{L}{2}$ and $\alpha L < 1$ and $\lambda \neq a_m$ $(m = 0, 1, 2, \ldots)$, then

$$\lim_{n\to\infty} b_{nk} = \ \lim_{n\to\infty} \frac{a_n | \ \lambda \ |^2 \prod_{j=k}^{n} | \ 1 - \frac{a_j}{\lambda} \ |}{=} \ \lim_{n\to\infty} \frac{(n+1)^{\alpha L} a_n n^{\alpha L - 1}}{(k-1)^{\alpha L}} \ = 0.$$

for every k. Hence (1.1) is satisfied.

Now lets show that the equation (1.2) is satisfied. Let

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^{N} (b_{nm} - b_{n-1,m}) \right| = \sum_1 + \sum_2 + \sum_3$$

where

$$\sum_1 = \ \sum_{n=0}^{N} \left| \sum_{m=0}^{n} b_{nm} - \sum_{m=0}^{n-1} b_{n-1,m}) \right|, \ 0 \leq n \leq N$$
$$\sum_2 = \ \left| \sum_{m=0}^{N+1} b_{N+1,m} - b_{N+1,N+1} + \sum_{m=0}^{N+1} b_{N,m} \right|, \ n = N+1$$
$$\sum_3 = \ \sum_{n=N+2}^{\infty} \ \left| \sum_{m=0}^{N} (b_{nm} - b_{n-1,m}) \right|, \ N+2 \leq n \leq \infty.$$

Using the Lemma 2.7, it can be shown that $\sum_i = O(1)$ for $i = \ 1, \ 2, \ 3.$ □

This agrees, with the result obtained by Okutoyi [2], for the special case $R_a = C_1$.

For the other special cases of spectrums of Rhaly matrices R_a we give the following examples.

EXAMPLE 1. If $a = (\frac{n+3}{n^2+1})$ then

$$\pi_0(R_a^*, bs) = \{ \lambda : | \lambda - \frac{1}{2} | < \frac{1}{2} \} \cup \{ 1, 2, 3 \},$$

$$\pi_0(R_a, bv_0) = \emptyset$$

and

$$\sigma(R_a, bv_0) = \{ \lambda : | \lambda - \frac{1}{2} | \leq \frac{1}{2} \} \cup \{ 2, 3 \}.$$

EXAMPLE 2. If $a = (\sin \frac{1}{n+1})$ then

$$\pi_0(R_a^*, bs) = \{ \lambda : | \lambda - \frac{1}{2} | < \frac{1}{2} \} \cup \{ 1 \},$$

and

$$\sigma(R_a, bv_0) = \{ \lambda : | \lambda - \frac{1}{2} | \leq \frac{1}{2} \}.$$

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