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ON SOME PROPERTIES OF THE FUNCTION SPACE $\mathcal{M}$

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ABSTRACT. Let M be the vector space of all real S-measurable functions defined on a measure space $(X, \mathcal{S}, \mu)$. In this paper, we investigate some topological structure of $\mathcal{T}$ on $\mathcal{M}$. Indeed, $(M, \mathcal{T})$ becomes a topological vector space. Moreover, if μ is σ-finite, we can define a complete invariant metric on $\mathcal{M}$ which is compatible with the topology $\mathcal{T}$ on $\mathcal{M}$, and hence $(M, \mathcal{T})$ becomes a F-space.

1. Introduction

Let $(X, \mathcal{S}, \mu)$ be an arbitrary measure space. We consider the set of all real valued $\mathcal{S}$-measurable(or simply measurable) functions defined on $(X, \mathcal{S}, \mu)$ and identify μ-equivalent measurable functions. This means that we deal with a set $\mathcal{M} \equiv \mathcal{M}(X, \mathcal{S}, \mu)$ of real valued measurable functions which contains exactly one representative for each μ-equivalence class. Thus the set $\mathcal{M}$ is the set of all non μ-equivalent real valued measurable functions on $(X, \mathcal{S}, \mu)$. Also $\mathcal{M}$ is a vector space over the real field R under the pointwise addition and the pointwise scalar multiplication.

For $E \in \mathcal{S}$ with $\mu(E) < \infty$ and $f, g \in \mathcal{M}$, we define

$$d_E(f, g) = \int_E \frac{|f-g|}{1+|f-g|} d\mu.$$

Then one can easily see that d_E is an invariant pseudometric on $\mathcal{M}$.

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Now we shall give the topology $\mathcal{T}$ on $\mathcal{M}$ determined by a family of pseudometric on $\mathcal{M}$, $\mathcal{D} = \{d_E : E \in \mathcal{S}, \mu(E) < \infty\}$; that is, a subbasis for the topology is formed by the sets

$$B_E(f,\delta) = \{g \in \mathcal{M} : d_E(f,g) < \delta\},\ f \in \mathcal{M},\ \delta > 0,\ d_E \in \mathcal{D}.$$

This topology $\mathcal{T}$ on $\mathcal{M}$ will be called the topology of convergence in measure on the measurable subsets of X whose measure is finite.

Indeed, $(\mathcal{M}, \mathcal{T})$ becomes a topological vector space over R, and then the convergence of a sequence (f_n) to a function f in $\mathcal{M}$ relative to the topology $\mathcal{T}$ is equivalent to that of (f_n) to f with respect to d_E for every $d_E \in \mathcal{D}$.

Moreover, we show that if a measure space $(X, \mathcal{S}, \mu)$ is a σ- finite, one can define a complete invariant metric d on $\mathcal{M}$ which is compatible with the topology $\mathcal{T}$ on $\mathcal{M}$, and hence $(\mathcal{M}, \mathcal{T})$ becomes a F-space over R.

2. Topological structure $\mathcal{T}$ of $\mathcal{M}$

In this section we shall topologize the set $\mathcal{M}$ by a family of pseudometrics on $\mathcal{M}$. And then it will be seen that $\mathcal{M}$ is in fact a topological vector space over the real filed R. We also examine a relationship between the convergence of a sequence (f_n) in $\mathcal{M}$ with respect to the topology $\mathcal{T}$ on $\mathcal{M}$ and that of (f_n) in $\mathcal{M}$ with respect to pseudometric on $\mathcal{M}$ which induced $\mathcal{T}$.

DEFINITION 2.1. Let $\mathcal{D} = \{d_E : E \in \mathcal{S}, \mu(E) < \infty\}$ be the family of pseudometrics on E. Then we provide the topology $\mathcal{T}$ on $\mathcal{M}$ determined by $\mathcal{D}$; that is, a subbasis for the topology is formed by the sets

$$B_E(f,\epsilon) = \{g \in \mathcal{M} : d_E(f,g) < \epsilon\}, f \in \mathcal{M}, \epsilon > 0, d_E \in \mathcal{D}.$$

This topology $\mathcal{T}$ on $\mathcal{M}$ will be called the topology of convergence in measure on every measurable subsets of X whose measure is finite.

We note that a basic open neighborhood of f in the topology $\mathcal{T}$ is of the form

$$\begin{aligned} U(f;\epsilon;d_{E_1},d_{E_2},\cdots,d_{E_n}) &= \{g \in \mathcal{M} : d_{E_k}(f,g) < \epsilon, k = 1,2,\cdots,n\} \\ &= \bigcap_{k=1}^{n} B_{E_k}(f,\epsilon) \end{aligned}$$

where $d_{E_1}, d_{E_2}, \cdots, d_{E_n} \in \mathcal{D}$ and $\epsilon > 0$.

EXAMPLE 2.2. (a) Let X be any non-empty set, and let $\mathcal{S} = \{\phi, X\}$. If we define a set function μ on $\mathcal{S}$ by

$$\mu(A) = \begin{cases} 0, & \text{if } A = \phi \\ \infty, & \text{if } A = X \end{cases}$$

then μ is a measure on $\mathcal{S}$. Hence $(X, \mathcal{S}, \mu)$ is a measure space. Clearly every $\mathcal{S}$-measurable functions on $(X, \mathcal{S})$ is a constant function. Thus

$$\mathcal{M} = \{f | f : (X, \mathcal{S}) \to R \text{ is a constant function}\}$$

and

$$\mathcal{D} = \{d_E : E \in \mathcal{S}, \mu(E) < \infty\} = \{d_\phi\}.$$

Since $d_\phi = \int_\phi \frac{|f-g|}{1+|f-g|} d\mu = 0$ for all $f, g \in \mathcal{M}$, it follows that

$$B_\phi(f, \epsilon) = \{g \in \mathcal{M} | d_\phi(f, g) < \epsilon\} = \mathcal{M}$$

where $f \in \mathcal{M}$ and $\epsilon > 0$.

Thus the topology $\mathcal{T}$ on $\mathcal{M}$ induced by $\mathcal{D} = \{d_\phi\}$ is $\{\phi, \mathcal{M}\}$. Therefore $(\mathcal{M}, \mathcal{T})$ is an indiscrete topological space.

(b) Let X and $\mathcal{S}$ be as in (a). Let μ be defined for $A \in \mathcal{S}$ by

$$\mu(A) = \begin{cases} 0, & \text{if } A = \phi \\ 1, & \text{if } A = X. \end{cases}$$

Then $(X, \mathcal{S}, \mu)$ is a finite measure space. By the same reason as in (a),

$$\mathcal{M} = \{f | f : (X, \mathcal{S}) \to R \text{ is a constant function}\}.$$

It is readily seen that $(\mathcal{M}, d_X)$ is a metric space. We observe that $f, g \in \mathcal{M}$ with $f \equiv \alpha$, $g \equiv \beta$,

$$\begin{aligned} d_X(f, g) &= \int_X \frac{|f-g|}{1+|f-g|} d\mu \\ &= \int_X \frac{|\alpha-\beta|}{1+|\alpha-\beta|} d\mu \\ &= \frac{|\alpha-\beta|}{1+|\alpha-\beta|} \mu(X) \qquad (1) \\ &= \frac{|\alpha-\beta|}{1+|\alpha-\beta|}. \qquad (2) \end{aligned}$$

Since $\rho(\alpha, \beta) = \dfrac{|\alpha-\beta|}{1+|\alpha-\beta|}$ for all $\alpha, \beta \in R$ is clearly a metric on R, it follows from (2) that $(\mathcal{M}, d_X)$ is isometric isomorphic to the metric space (R, ρ).

THEOREM 2.3. *The topological space $(\mathcal{M}, \mathcal{T})$ is topological vector space over R.*

PROOF. For any $f, g \in \mathcal{M}$ and $\lambda \in R$, since $f+g$ and λf are clearly measurable functions, we have $f+g \in \mathcal{M}$ and $\lambda f \in \mathcal{M}$. Thus $\mathcal{M}$ is a vector space over R.

Now it remains only to show that the vector operations are continuous. First, we show that the addition $+$ is continuous. Let $f_0, g_0 \in \mathcal{M}$ and $\epsilon > 0$, and consider the open neighborhood $U(f_0; g_0; \epsilon; d_{E_1}, d_{E_2}, \cdots, d_{E_n})$ of $f_0 + g_0$ in $\mathcal{T}$. If U denotes the open neighborhood

$$U(f_0; \frac{\epsilon}{2}; d_{E_1}, d_{E_2}, \cdots, d_{E_n}) \times U(g_0; \frac{\epsilon}{2}; d_{E_1}, d_{E_2}, \cdots, d_{E_n})$$

in the product topology on $\mathcal{M} \times \mathcal{M}$, then clearly $(f,g) \in U$ implies that

$$\begin{aligned} d_{E_k}(f+g, f_0+g_0) &= \int_{E_k} \frac{|(f+g)-(f_0+g_0)|}{1+|(f+g)-(f_0+g_0)|} d\mu \\ &\le \int_{E_k} \frac{|f-f_0|+|g-g_0|}{1+|f-f_0|+|g-g_0|} d\mu \\ &\le \int_{E_k} \frac{|f-f_0|}{1+|f-f_0|} d\mu + \int_{E_k} \frac{|g-g_0|}{1+|g-g_0|} d\mu \\ &= d_{E_k}(f, f_0) + d_{E_k}(g, g_0) \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon (k = 1, 2, \cdots, n). \end{aligned}$$

This shows that addition is continuous. Next we prove that scalar multiplication is continuous. Let $f_0 \in \mathcal{M}$ and $\lambda_0 \in R$ be fixed. For any $d_E \in \mathcal{D}$,

$$\begin{aligned} d_E(\lambda f, \lambda_0 f_0) &\le d_E(\lambda f, \lambda f_0) + d_E(\lambda f_0, \lambda_0 f_0) \\ &= \int_E \frac{|\lambda f - \lambda f_0|}{1+|\lambda f - \lambda f_0|} d\mu + \int_E \frac{|\lambda f_0 - \lambda_0 f_0|}{1+|\lambda f_0 - \lambda_0 f_0|} d\mu \\ &= \int_E \frac{|\lambda||f-f_0|}{1+|\lambda||f-f_0|} d\mu + \int_E \frac{|\lambda-\lambda_0||f_0|}{1+|\lambda-\lambda_0||f_0|} d\mu \\ &\le (1+|\lambda_0|) \int_E \frac{|f-f_0|}{1+|f-f_0|} d\mu + \int_E \frac{|\lambda-\lambda_0||f_0|}{1+|\lambda-\lambda_0||f_0|} d\mu \\ &= (1+|\lambda_0|) d_E(f, f_0) + d_E(|\lambda-\lambda_0| f_0, 0). \end{aligned} \tag{3}$$

Provided $|\lambda - \lambda_0| < 1$. Now we see that Lebesgue Dominated Convergence Theorem [1, p.44] implies

$$\text{(4)} \qquad \lim_{\delta \to 0} \int_E \frac{\delta |f_0|}{1 + \delta |f_0|} = \lim_{\delta \to 0} d_E(\delta f_0, 0) = 0.$$

Let $\epsilon > 0$. For any $d_{E_1}, d_{E_2}, \cdots, d_{E_n}$ in $\mathcal{D}$ it follows from (4) that there exist positive real numbers $\delta_1, \delta_2, \cdots, \delta_n$ in $(0,1)$ such that $0 < \delta < \delta_k$ implies $|d_{E_k}(\delta f_0, 0)| < \frac{\epsilon}{2}$.

Let $\delta_0 = \min\{\delta_1, \delta_2, \cdots, \delta_n\}$, then $0 < \delta < \delta_0$ implies $|d_{E_k}(\delta_0, 0)| < \frac{\epsilon}{2}$ for all $k = 1, 2, \cdots, n$. Now consider the open neighborhood $U(\lambda_0 f_0; \epsilon; d_{E_1}, d_{E_2}, \cdots, d_{E_n})$ of $\lambda_0 f_0$ in $\mathcal{T}$.

If U denotes the open neighborhood

$$\{\lambda \in R : |\lambda - \lambda_0| < \delta_0\} \times U(f_0; \frac{\epsilon}{2}(1 + |\lambda_0|); d_{E_1}, d_{E_2}, \cdots, d_{E_n})$$

in the product topology on $R \times \mathcal{M}$, then $\lambda f \in U$ and (3) imply that

$$\begin{aligned} d_{E_k}(\lambda f, \lambda_0 f_0) &\le (1 + |\lambda_0|) d_E(f, f_0) + d_{E_k}(|\lambda - \lambda_0| f_0, 0) \\ &< (1 + |\lambda_0|)\ \epsilon/2\ (1 + |\lambda_0|) + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for every $k = 1, 2, \cdots, n$.

This proves that scalar multiplication is continuous. □

THEOREM 2.4. *A sequence (f_n) in $\mathcal{M}$ converges to $f \in \mathcal{M}$ in the topology $\mathcal{T}$ if and only if for any $d_E \in \mathcal{D}$, $d_E(f_n, f) \to 0$ as $n \to \infty$.*

PROOF. (Necessity) Let $\epsilon > 0$ be given. Then for each $d_E \in \mathcal{D}$, the neighborhood $U(f; \epsilon; d_E)$ is an open neighborhood of f in $\mathcal{T}$. Since (f_n) converges to f in $(\mathcal{M}, \mathcal{T})$, there exists some N such that if $n > N$, then $f_n \in U(f; \epsilon; d_E)$, that is $d_E(f_n, f) < \epsilon$. Thus $\lim_{n \to \infty} d_E(f_n, f) = 0$.

(Sufficiency) Let U be an open set containing f in the topology $\mathcal{T}$. Then by the definition of $\mathcal{T}$, there exist $d_{E_1}, d_{E_2}, \cdots, d_{E_n} \in \mathcal{D}$ such that

$$U(f; \epsilon; d_{E_1}, d_{E_2}, \cdots, d_{E_n}) \subset U.$$

Since $\lim_{n \to \infty} d_E(f_n, f) = 0$ for all $d_E \in \mathcal{D}$, for each $d_{E_1}, d_{E_2}, \cdots, d_{E_n}$, there exist some $N_k, k = 1, 2, \cdots, n$ such that if $n > N_k, k = 1, 2, \cdots, n$ then $d_{E_k}(f_n, f) < \epsilon$.

Now let $N = \max\{N_1, N_2, \cdots, N_n\}$, then for all $n > N, d_{E_k}(f_n, f) < \epsilon$ for all $k = 1, 2, \cdots, n$. Thus $f_n \in U(f; \epsilon; d_{E_1}, d_{E_2}, \cdots, d_{E_n})$ for all $n > N$. Hence (f_n) converges to f in the topology $\mathcal{T}$. □

3. Metrization of topological vector space $\mathcal{M}$

Until further notice, $(X, \mathcal{S}, \mu)$ will be an arbitrary σ-finite measure space, and $\{E_n\}$ is an increasing sequence of subsets of X in $\mathcal{S}$ such that $\bigcup_{n=1}^{\infty} E_n = X$ and $\mu(E_n) < \infty$ for all $n \geq 1$. In this section, we investigate some topological structures of the function space $\mathcal{M}$. Indeed, we shall show that it is possible to define a complete invariant metric on $\mathcal{M}$ which is compatible with the topology. For any two functions $f, g \in \mathcal{M}$, let $d : \mathcal{M} \times \mathcal{M} \to R$ be defined by

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-\infty} \frac{d_{E_n}(f,g)}{1 + d_{E_n}(f,g)}$$

where $d_{E_n}(f,g) = \int_{E_n} \frac{|f-g|}{1+|f-g|} d\mu$, $n = 1, 2, \cdots$. Then it easily follows that d is an invariant metric on $\mathcal{M}$.

THEOREM 3.1. *The function space $(\mathcal{M}, d)$ is a complete metric space. The metric topology $\mathcal{T}_d$ on $(\mathcal{M}, d)$ determined by d coincides with the topology $\mathcal{T}_1$ determined by a family of pseudometric, $\{d_{E_n} : n = 1, 2, \cdots\}$. Consequently $\lim_{n\to\infty} d(f_n, f) = 0$ if and only if $\lim_{n\to\infty} d_E(f_n, f) = 0$ for all $n = 1, 2, \cdots$.*

PROOF. Let (f_n) be a Cauchy sequence in $(\mathcal{M}, d)$. Then $d(f_n, f) \to 0$ as $m, n \to \infty$. For any $k \geq 1$, we note that $d_{E_k}(f_m, f_n) \leq 2^k d(f_m, f_n)$ for all $m, n = 1, 2, \cdots$. Thus $d_{E_k}(f_m, f_n) \to 0$ for every k as $m, n \to \infty$, so that (f_n) converges in $\mathcal{M}$ as E_k to a function $f \in \mathcal{M}$. Since

$$\sum_{i=1}^{k} 2^{-i} \frac{d_i(f_n, f)}{1 + d_i(f_n, f)}$$

converges uniformly in n, it follows from the iterated limit theorem [2, p.143] that

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(f_n, f)}{1 + d_k(f_n, f)} &= \lim_{n\to\infty} \lim_{k\to\infty} \sum_{i=1}^{k} 2^{-i} \frac{d_i(f_n, f)}{1 + d_i(f_n, f)} \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} 2^{-i} \lim_{n\to\infty} \frac{d_i(f_n, f)}{1 + d_i(f_n, f)} \\
&= 0.
\end{aligned}$$

Hence $\lim_{n\to\infty} d(f_n f) = 0$. Therefore d is a complete metric on $\mathcal{M}$.

Now we shall show that $\mathcal{T}_d = \mathcal{T}_1$. To show that $\mathcal{T}_d \subset \mathcal{T}_1$, it suffices to show that for any $f \in \mathcal{M}$ and for any subbasic open neighborhood of f relative to $\mathcal{T}_d$ of the form $B(f,\epsilon) = \{g \in \mathcal{M} | d(f,g) < \epsilon\}$, there exists a sufficiently large positive integer m such that

$$B_{E_m}(f, 1/2^m) = \{g | d_{E_m}(f,g) < 1/2^m\} \subset B(f,\epsilon).$$

Choose a positive integer k such that $1/2^k < \epsilon$. If $g \in B_{E_m}(f, 1/2^m)$, then $d_{E_k}(f,g) < 1/2^m$, and hence

$$d_{E_1}(f,g) \leq d_{E_2}(f,g) \leq \cdots \leq d_{E_m}(f,g) < 1/2^m.$$

Moreover, since

$$\frac{d_{E_i}(f,g)}{1 + d_{E_i}(f,g)} \leq d_{E_i}(f,g) \text{ for every } i = 1, 2, \cdots ,$$

we see that

$$\begin{aligned} d(f,g) &= \sum_{i=1}^{\infty} \frac{d_{E_i}(f,g)}{2^i(1 + d_{E_i}(f,g))} \\ &= \sum_{i=1}^{m} \frac{d_{E_i}(f,g)}{2^i(1 + d_{E_i}(f,g))} + \sum_{i=m+1}^{\infty} \frac{d_{E_i}(f,g)}{2^i(1 + d_{E_i}(f,g))} \\ &\leq \frac{1}{2^m}(\sum_{i=1}^{m} \frac{1}{2^i} + \sum_{i=m+1}^{\infty} \frac{1}{2^i}) \\ &< \frac{1}{2^m}(\sum_{i=1}^{\infty} \frac{1}{2^i} + \sum_{i=1}^{\infty} \frac{1}{2^i}) \\ &= \frac{1}{2^{m-1}}. \end{aligned}$$

Now let $m = k + 1$, then $d(f,g) < 1/2^k$, and hence

$$B_{E_{k+1}}(f, 1/2^{k+1}) \subset B(f, 1/2^k) \subset B(f,\epsilon).$$

This implies that $\mathcal{T}_d \subset \mathcal{T}_1$. Next, to show that $\mathcal{T}_1 \subset \mathcal{T}_d$, it is enough to show that for any $f \in \mathcal{M}$ and for any subbasic open neighborhood of f relative to $\mathcal{T}_1$ of the form

$$B_{E_m}(f,\epsilon) = \{g \in \mathcal{M} | d_{E_m}(f,g) < \epsilon\},$$

there exists a sufficiently large positive integer l such that

$$B(f,1/2^l) \subset B_{E_m}(f,\epsilon).$$

Choose a positive integer k such that $1/2^k < \epsilon$. If $g \in B(f,1/2^l)$ then

$$d(f,g) = \sum_{i=1}^{\infty} \frac{d_{E_i}(f,g)}{2^i(1+d_{E_i}(f,g))} < \frac{1}{2^l},$$

and hence we have

$$\frac{d_{E_m}(f,g)}{2^m(1+d_{E_m}(f,g))} < \frac{1}{2^l}.$$

This inequality can be solved for $d_{E_m}(f,g)$. Consequently, we obtain $d_{E_m}(f,g) < \frac{1}{2^{l-m}-1}$. Now let $l = k+m+1$, then $d_{E_m}(f,g) < \frac{1}{2^{k+1}-1} < \frac{1}{2^k}$ and hence

$$B(f,\frac{1}{2^{k+m+1}}) \subset B_{E_m}(f,1/2^k) \subset B_{E_m}(f,\epsilon).$$

This implies that $\mathcal{T}_1 \subset \mathcal{T}_d$. □

DEFINITION 3.2. Let X be a topological vector space with topology $\mathcal{T}$. X is called a F-space if the topology $\mathcal{T}$ coincides with the metric topology determined by a complete invariant metric d.

THEOREM 3.3. *The metric topology $\mathcal{T}_d$ on $\mathcal{M}$ coincides with the topology $\mathcal{T}$ in $\mathcal{M}$ convergence on each measurable subset of X whose measure is finite. Consequently the topological vector space $(\mathcal{M},\mathcal{T})$ becomes a F-space.*

PROOF. We recall that the topology $\mathcal{T}$ on $\mathcal{M}$ is topology determined by

$$\mathcal{D} = \{d_E : E \in \mathcal{S}, \mu(E) < \infty\}.$$

Since $\{d_{E_n} : n = 1,2,\cdots\} \subset \mathcal{D}$, it follows that $\mathcal{T}_d \subset \mathcal{T}$. Now we show that $\mathcal{T} \subset \mathcal{T}_d$. For this purpose, it is enough to show that for any $f \in \mathcal{M}$ and for any subbasic open neighborhood relative to $\mathcal{T}$ of f of the form $B_E(f,\delta)$, there exists a subbasic neighborhood relative to $\mathcal{T}_d$ of f, $B_n(f,\epsilon) = \{g : d_{E_n}(f,g) < \epsilon\}$ such that $B_{E_n}(f,\epsilon) \subset B_{E_n}(f,\delta)$. Since $E \in \mathcal{S}$ and $\mu(E) < \infty$, we can sufficiently large n such that $\mu(E) < \mu(E_n)$. Hence we have

$$\int_E \frac{|f-g|}{1+|f-g|} d\mu \leq \int_{E_n} \frac{|f-g|}{1+|f-g|} d\mu$$

so that $B_{E_n}(f,\delta) \subset B_E(f,\delta)$. Therefore we have $\mathcal{T} = \mathcal{T}_d$. As we have just shown above, $\mathcal{T}$ is induced by a complete invariant metric d. Therefore $(\mathcal{M}, \mathcal{T})$ is a F-space. □

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