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CONFORMAL DEFORMATIONS ON WARPED PRODUCT MANIFOLDS

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ABSTRACT. In this paper, when N is a compact Riemannian manifold, using warped products, we discuss the conformal deformation of a warped product metric on $M = [a, \infty) \times_f N$ with specific warping functions.

I. Introduction

In [6, 7, 8], M.C. Leung has studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function. He has studied the uniqueness of positive solution to the equation

$$\Delta_{g_0} u(x) + d_n u(x) = d_n u(x)^{\frac{n+2}{n-2}}, \tag{1.1}$$

where Δ_{g_0} is the Laplacian operator for an $n-$ dimensional Riemannian manifold (N, g_0) and $d_n = \frac{n-2}{4(n-1)}$. Equation (1.1) is derived from the conformal deformation of a Riemannian metric (cf. [1, 4, 5, 8]).

Similarly, let (N, g_0) be a compact Riemannian $n-$dimensional manifold with constant scalar curvature. We consider the $(n+1)$-dimensional Riemannian warped manifold $M = [a, \infty) \times_f N$ with the metric $g = -dt^2 + f(t)^2 g_0$, where f is a positive function on $[a, \infty)$. Let $u(t, x)$ be a positive smooth function on M and let g have a scalar curvature equal to $r(t, x)$. If the conformal metric $g_c = u(t, x)^{\frac{4}{n-1}} g$ has a scalar curvature

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$R(t,x)$, which is an arbitrary smooth function in $C^\infty(M)$, then $u(t,x)$ satisfies equation

$$\frac{4n}{n-1}\square_g u(t,x) - r(t,x)u(t,x) + R(t,x)u(t,x)^{\frac{n+3}{n-1}} = 0, \tag{1.2}$$

where $\square_g$ is the d'Alembertian for a Lorentzian warped manifold $M = [a,\infty)\times_f N$.

In [6, 7], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric. And also in [3], authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces a constant scalar curvature when the fiber manifold has a constant scalar curvature.

Ironically, even though there exist some obstructions of positive or zero scalar curvature on a Riemannian manifold, results of [3], say, Theorem 3.1, Theorem 3.5 and Theorem 3.7 of [3] show that there exists no obstruction of positive scalar curvature on a Lorentzian warped product manifold, but there may also exist some obstructions of negative or zero scalar curvature.

In this paper, when N is a compact Riemannian manifold with a constant scalar curvature, we discuss the method of using conformal deformations to construct timelike or null future complete Lorentzian metrics on $M = [a,\infty)\times_f N$ with specific scalar curvatures.

2. Main results

In this section, we let (N, g_0) be a compact Riemannian n-dimensional manifold with $n \geq 3$ and without boundary. The following proposition is well known(cf. Theorem 5.4 in [2]).

PROPOSITION 2.1. *Let $M = [a,\infty)\times_f N$ have a Lorentzian warped product metric $g = -dt^2 + f(t)^2 g_0$. Then the d'Alembertian $\square_g$ is given by*

$$\square_g = -\frac{\partial^2}{\partial t^2} - \frac{nf'(t)}{f(t)}\frac{\partial}{\partial t} + \frac{1}{f(t)^2}\Delta_x,$$

where Δ_x is the Laplacian on fiber manifold N.

By Proposition 2.1, equation (1.2) is changed into the following equation

$$\begin{aligned} (2.1) \qquad & \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{nf'(t)}{f(t)}\frac{\partial u(t,x)}{\partial t} - \frac{1}{f(t)^2}\Delta_x u(t,x) \\ & + \frac{n-1}{4n} r(t,x)u(t,x) - \frac{n-1}{4n} R(t,x)u(t,x)^{\frac{n+3}{n-1}} = 0. \end{aligned}$$

A positive solution to equation (1.2) or (2.1) is called future timelike (null-like, nonspacelike) complete if the conformal metric $g_c = u^{\frac{4}{n-1}} g$ is a future timelike (null-like, nonspacelike, respectively) complete Lorentzian metric on M. In this section we discuss whether the complete positive solution of equation (2.1) exists for some prescribed smooth function $R(t,x)$.

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian $n-$manifold without boundary. Every compact C^∞ manifold admits a metric of a constant scalar curvature, so we assume that N admits a constant scalar curvature. Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [3], we have the following proposition.

PROPOSITION 2.2. *If the scalar curvature of the fiber manifold N is arbitrary constant, then there exists a nonconstant warping function $f(t)$ on $[a,\infty)$ such that $f(t) \to \infty$ as $t \to \infty$, $|\frac{f'(t)}{f(t)}| \leq$ constant and the resulting Lorentzian warped product metric on $[a,\infty) \times_f N$ produces a positive constant scalar curvature.*

Proposition 2.2 implies that in Lorentzian warped product there is no obstruction of metric with positive scalar curvature. However, the results of [3] show that there may exist some obstructions about the Lorentzian warped product metric with negative or zero scalar curvature even when the fiber manifold has constant scalar curvature.

Therefore we may assume that the Lorentzian warped manifold $M = [a,\infty) \times_f N$ admits a positive constant scalar curvature $r(t,x) = c > 0$ and the warping function $f(t)$ with $f(t) \to \infty$ as $t \to \infty$ and $|\frac{f'(t)}{f(t)}| \leq$ *constant*.

If $u(t,x) = u(t)$ is a positive function with only variable t, then equation (2.1) becomes

$$(2.2) \qquad u''(t) + \frac{nf'(t)}{f(t)}u'(t) + \frac{n-1}{4n}cu(t) - \frac{n-1}{4n}R(t,x)u(t)^{\frac{n+3}{n-1}} = 0.$$

REMARK 2.3. Theorem 5.5 in [9] implies that all timelike geodesics are future complete on $[a, +\infty) \times_{f(t)} N$ if and only if $\int_{t_0}^{+\infty} \frac{f(t)}{\sqrt{1+f(t)^2}} dt = +\infty$ for some $t_0 \in [a, \infty)$ and that all null geodesics are future complete if and only if $\int_{t_0}^{+\infty} f(t)dt = +\infty$ for some $t_0 \in [a, \infty)$ (cf. Theorem 4.1 and Remark 4.2 in [2]).

For the existence theorem of the solution, we use the method of upper solution and lower solution.

THEOREM 2.4. *Suppose that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function and that there exist positive constants t_0 and C_0 such that $|\frac{f'(t)}{f(t)}| \leq C_0$ for all $t > t_0$. Assume that for $t > t_0$ there exist an upper solution u_+ and a lower solution u_- such that $0 < u_- < u_+$. Then there exists a solution u of (2.2) such that $0 < u_- < u < u_+$ for $t > t_0$.*

PROOF. We have only to show that there exist an upper solution $\tilde{u}_+(t)$ and a lower solution $\tilde{u}_-(t)$ such that for all $t \in [a, \infty)$ $\tilde{u}_-(t) < \tilde{u}_+(t)$. Since $R(t) \in C^\infty([a,\infty))$ is a positive function , there exist positive constants b_1, b_2 such that $b_1 \leq R(t) \leq b_2$ for $t \in [a, t_0]$. Since $\frac{n+3}{n-1} > 1$ and $R(t)$ is a bounded function, the constant $c_1 = (\frac{c}{b_1})^{\frac{n-1}{4}}$ is an upper solution for $t \in [a, t_0]$. Then put $\tilde{u}_+(t) = c_1$ for $t \in [a, t_0]$ and $\tilde{u}_+(t) = u_+(t)$ for $t > t_0$, which is our desired (weak) upper solution such that $c_1 \leq \tilde{u}_+(t)$ for all $t \in [a, t_0]$. Put $\tilde{u}_-(t) = c_1 e^{-\alpha t}$ for $t \in [a, t_0]$ and some large positive α, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $t > t_0$. Then, for $t \in [a, t_0]$,

$$\begin{aligned}
& \frac{4n}{n-1}(u_-''(t) + \frac{nf'(t)}{f(t)} u_-'(t)) + cu_-(t) - R(t)u_-(t)^{\frac{n+3}{n-1}} \\
\geq \;& \frac{4n}{n-1}(u_-''(t) + \frac{nf'(t)}{f(t)} u_-'(t)) + cu_-(t) - b_2 u_-(t)^{\frac{n+3}{n-1}} \\
= \;& \frac{4n}{n-1} c_1 e^{-\alpha t}(\alpha^2 - \alpha \frac{f'(t)}{f(t)} + \frac{n-1}{4n} c - \frac{n-1}{4n} b_2 (c_1 e^{-\alpha t})^{\frac{4}{n-1}}) \\
\geq \;& 0
\end{aligned}$$

for large α. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in [a, \infty)$, $0 < \tilde{u}_-(t) < \tilde{u}_+(t)$. Therefore there exists a solution $u(t)$ such that $0 < \tilde{u}_-(t) \leq u(t) \leq \tilde{u}_+(t)$. □

THEOREM 2.5. *Let (M, g) be a future nonspacelike complete Lorentzian manifold with scalar curvature equal to c. Suppose that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function and that there exist positive* *constants t_0 and C_0 such that $|\frac{f'(t)}{f(t)}| \le C_0$ for all $t > t_0$. If $R(t)$ is not rapidly increasing and not rapidly decreasing such that $e^{-kt} \le R(t) \le e^{kt}$ for some positive constant k and for all large t, then there exists a positive solution to equation (2.2).*

PROOF. By Theorem 2.4, for large t, we have only to show that there exist an upper solution $u_+(t)$ and a lower solution $u_-(t)$ such that $u_-(t) < u_+(t)$.

Put $u_+(t) = e^{\alpha t}$ and $u_-(t) = e^{-\alpha t}$ for some large positive constant, which will be determined later. Then

$$\begin{aligned}
&\frac{4n}{n-1}(u_+''(t) + \frac{nf'(t)}{f(t)}u_+'(t)) + cu_+(t) - R(t)u_+(t)^{\frac{n+3}{n-1}} \\
\le\ & \frac{4n}{n-1}(u_+''(t) + \frac{nf'(t)}{f(t)}u_+'(t)) + cu_+(t) - e^{-kt}u_+(t)^{\frac{n+3}{n-1}} \\
=\ & \frac{4n}{n-1}e^{\alpha t}(\alpha^2 + \alpha\frac{nf'(t)}{f(t)} + \frac{n-1}{4n}c - \frac{n-1}{4n}e^{(\alpha\frac{4}{n-1}-k)t})) \\
\le\ & 0
\end{aligned}$$

for large $\alpha > 0$. And similarly

$$\begin{aligned}
&\frac{4n}{n-1}(u_-''(t) + \frac{nf'(t)}{f(t)}u_-'(t)) + cu_-(t) - R(t)u_-(t)^{\frac{n+3}{n-1}} \\
\ge\ & \frac{4n}{n-1}(u_-''(t) + \frac{nf'(t)}{f(t)}u_-'(t)) + cu_-(t) - e^{kt}u_-(t)^{\frac{n+3}{n-1}} \\
=\ & \frac{4n}{n-1}e^{-\alpha t}(\alpha^2 - \alpha\frac{nf'(t)}{f(t)} + \frac{n-1}{4n}c - \frac{n-1}{4n}e^{(k-\alpha\frac{4}{n-1})t}) \\
\ge\ & 0
\end{aligned}$$

for large $\alpha > 0$. □

COROLLARY 2.6. *Let (M, g) be a future nonspacelike complete Lorentzian manifold with scalar curvature equal to c. Suppose that $R(t, x) = R(t) \in C^\infty([a, \infty))$ is a positive function and that there exist positive constants t_0 and C_0 such that $|\frac{f'(t)}{f(t)}| \le C_0$ for all $t > t_0$. If $R(t)$ is decreasing not rapidly such that $e^{-k_1 t} \le R(t) \le t^{-k_2}$ for some positive constants k_1 and k_2 and for all large t, then there exists a conformal deformation such that the resulting metric is nonspacelike future complete.*

PROOF. Let the upper solution be the same one with Theorem 2.5. And we can choose α such that $0 < \alpha < \frac{n-1}{4}k_2$. Then we put $u_-(t) = t^\alpha$, which is our desired (weak) lower solution. In fact,

$$\begin{aligned}
& \frac{4n}{n-1}(u_-''(t) + \frac{nf'(t)}{f(t)}u_-'(t)) + cu_-(t) - R(t)u_-(t)^{\frac{n+3}{n-1}} \\
\geq\ & \frac{4n}{n-1}(u_-''(t) + \frac{nf'(t)}{f(t)}u_-'(t)) + cu_-(t) - t^{-k_2}u_-(t)^{\frac{n+3}{n-1}} \\
=\ & \frac{4n}{n-1}t^{\alpha}(\alpha(\alpha-1)t^{-2} + \alpha\frac{nf'(t)}{f(t)}t^{-1} + \frac{n-1}{4n}c - \frac{n-1}{4n}t^{-k_2+\frac{4\alpha}{n-1}}) \\
\geq\ & 0
\end{aligned}$$

for large t. Therefore

$$\begin{aligned}
& \int_{t_0}^{\infty}[\frac{(u(t)^{\frac{2}{n-1}}f(t))^2}{1+(u(t)^{\frac{2}{n-1}}f(t))^2}]^{\frac{1}{2}}u(t)^{\frac{2}{n-1}}dt \\
\geq\ & \int_{t_0}^{\infty}[\frac{(u_-(t)^{\frac{2}{n-1}}f(t))^2}{1+(u_-(t)^{\frac{2}{n-1}}f(t))^2}]^{\frac{1}{2}}u_-(t)^{\frac{2}{n-1}}dt \\
\geq\ & \int_{t_0}^{\infty}[\frac{t^{\frac{2\alpha}{n-1}}f(t)}{\sqrt{1+t^{\frac{4\alpha}{n-1}}f(t)^2}}]t^{\frac{2\alpha}{n-1}}dt \\
\geq\ & \int_{t_0}^{\infty}[\frac{f(t)}{\sqrt{1+f(t)^2}}]dt \longrightarrow \infty
\end{aligned}$$

and

$$\begin{aligned}
\int_{t_0}^{\infty} u(t)^{\frac{2}{n-1}}f(t)u(t)^{\frac{2}{n-1}}dt \ &\geq \ \int_{t_0}^{\infty} u_-(t)^{\frac{2}{n-1}}f(t)u_-(t)^{\frac{2}{n-1}}dt \\
&\geq \ \int_{t_0}^{\infty} t^{\alpha}f(t)t^{\alpha}dt \geq \int_{t_0}^{\infty} f(t)dt \longrightarrow \infty,
\end{aligned}$$

which, by Remark 2.3, implies that the resulting warped product metric is a future nonspacelike complete one. □

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