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ALMOST P-SPACES

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ABSTRACT. In this paper, we have characterizations of almost P-spaces which are analogous characterizations of P-spaces and we will show that if X is an almost P-space such that it is C*-embedded in every almost P-space in which X is embedded, then $| \upsilon X - X | \leq 1$ and that if $| \upsilon X - X | \leq 1$ and υX is Lindelöf, then for any almost P-space Y in which X is dense embedded, then X is C*-embedded in Y.

1. Introduction

All spaces in this paper are Tychonoff. For the terminology not introduced in the paper, we refer to [3] and [7]. For any space X, βX (υX, resp.) denotes the Stone-Čech compactification (Hewitt realcompactification, resp.) of X.

The almost P-spaces studied here, introduced by Veksler ([8]), are by definition Tychonoff spaces in which every non-empty zero-set has non-empty interior. It is known that for any locally compact and realcompact space X, $\beta X - X$ is an almost P-space ([8]). Hence the class of almost P-spaces is a wide class of spaces which includes all P-spaces.

Hewitt ([4]) proved that a Tychonoff space is C*-embedded in every Tychonoff space in which it is embedded if and only if X is an almost compact space. C. E. Aull ([1]) has shown that a P-space X is C*-embedded in every P-space in which it is embedded if and only if X is an almost Lindelöf space, that is, given two disjoint zero-sets of X, at least one of them is Lindelöf and A. Dow and O. Förster ([2]) showed that an F-space X is C*-embedded in every F-space in which it is embedded if and only if X has no P-covers or X is an almost compact space.

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In this paper, we first have characterizations of almost P-spaces which are analogous characterizations of P-spaces ([3]) and show that every countable almost P-space is discrete and that if an almost P-space X is a countable set and a countably compact space, then X is finite. Finally, we show that (1) if X is an almost P-space such that it is C*-embedded in every almost P-space in which X is embedded, then $\mid \upsilon X - X \mid \leq 1$, (2) if $\mid \upsilon X - X \mid \leq 1$ and υX is Lindelöf, then for any almost P-space Y in which X is dense embedded, then X is C*-embedded in Y.

2. Almost P-spaces

As usual, C(X) ($C^*(X)$, resp.) will denote the set of all (bounded, resp.) continuous real-valued functions on a space X and for any $f \in C(X)$, $f^{-1}(0)$ ($X - f^{-1}(0)$, resp.), denoted by $Z(f)$, is called a *zero (co-zero, resp.)-set* in X.

DEFINITION 2.1. A space X is said to be an *almost P-space* if for any non-empty zero-set Z in X, $int_X(Z)$ is also non-empty.

Recall that a space X is called a *P-space* if every zero-set in X is open in X. Hence every P-space is an almost P-space. For any locally compact and realcompact space X, $\beta X - X$ is an almost P-space ([8]). Let R be the set of all real numbers with the usual topology. Then $\beta R - R$ is an almost P-space but not a P-space ([3]).

LEMMA 2.2. *Let X be an almost P-space, T a dense subspace of X and Z a zero-set in X. Then* $cl_X(int_X(Z)) = cl_X(int_T(Z \cap T))$.

PROOF. Note that $int_X(Z) \cap T \subseteq int_T(Z \cap T)$. Since T is dense in X, $cl_X(int_X(Z)) = cl_X(int_X(Z) \cap T)$ and hence $cl_X(int_X(Z)) \subseteq cl_X(int_T(Z \cap T))$.

Let $x \notin cl_X(int_X(Z))$. Then there is a zero-set neighborhood A of x in X with $int_X(Z) \cap int_X(A) = \emptyset$. Since X is an almost P-space, $Z \cap A = \emptyset$. Hence $Z \cap T \subseteq (X - A) \cap T$. Since $X - A$ is open in X, $int_T(Z \cap T) \subseteq (X - A) \cap T$ and hence $cl_X(int_T(Z \cap T)) \cap (X - cl_X((X - A) \cap T)) = cl_X(int_T(Z \cap T)) \cap int_X(A) = \emptyset$. Thus $x \notin cl_X(int_T(Z \cap T))$. □

DEFINITON 2.3. Let A be a commutative ring and $r \in A$. Then

(a) r is called a *regular element* in A if $ra = 0$ and $a \in A$ implies $a = 0$, and

(b) an ideal I of A is called *regular* if it contains a regular element in A.

For any space X and $f \in C(X)$, f is a regular element in C(X) if and only if $X - Z(f)$ is dense in X. Recall that a subspace Y of a space X is *C (C^*, resp.)-embedded in X* if for any $f \in C(X)$ ($C^*(X)$, resp.), there is $g \in C(X)$ ($C^*(X)$, resp.) such that $g|_X = f$ and that Y is *z-embedded in X* if for any zero-set A in Y, there is a zero-set B in X with $A = B \cap Y$.

THEOREM 2.4. *For any space X, the following are equivalent:*
(1) *X is an almost P-space,*
(2) *υX is an almost P-space,*
(3) *every dense z-embedded subspace of X is C-embedded in X,*
(4) *every dense cozero-set in X is C-embedded in X,*
(5) *every regular element in C(X) has the inverse element in C(X),*
(6) *C(X) has no proper regular ideal, and*
(7) *every zero-set in X is a regular closed set in X.*

PROOF. (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6) and (7) $\Rightarrow$ (1) are trivial. Moreover, (1) $\Leftrightarrow$ (2) is also trivial, because for any zero-set Z in X, $\mathrm{cl}_{\upsilon X}(Z)$ is a zero-set in υX ([3]).

(1) $\Rightarrow$ (3) Let T be a dense z-embedded subspace of X and A_1, A_2 zero-sets in T with $\mathrm{int}_T(A_1) \cap \mathrm{int}_T(A_2) = \emptyset$. Then there are zero-sets B_1, B_2 in X with $A_i = B_i \cap T$ $(i = 1, 2)$. Since T is dense in X, by Lemma 2.2, $\mathrm{cl}_X(\mathrm{int}_X(B_1 \cap B_2)) = \mathrm{cl}_X(\mathrm{int}_T(B_1 \cap B_2 \cap T)) = \mathrm{cl}_X(\mathrm{int}_T(A_1) \cap \mathrm{int}_T(A_2)) = \emptyset$. Since X is an almost P-space, $B_1 \cap B_2 = \emptyset$. By Urysohn's extension theorem, T is C^*-embedded in X. Let Z be a zero-set in X with $T \cap Z = \emptyset$. Since T is dense in X and X is an almost P-space, $Z = \emptyset$ and so T and Z are completely separated in X. Thus T is C-embedded in X .

(4) $\Rightarrow$ (5) Let f be a regular element in C(X), then $X - Z(f)$ is a dense cozero-set in X and hence $X - Z(f)$ is C-embedded in X. Hence $X - Z(f)$ and $Z(f)$ are completely separated in X. Since $X - Z(f)$ is dense in X, $Z(f) = \emptyset$. Thus f has the inverse element in C(X).

(6) $\Rightarrow$ (7) Let $f \in C(X)$ such that $0 \leq f(x)$ for all $x \in X$. Suppose that $\mathrm{cl}_X(\mathrm{int}_X(Z(f))) \neq Z(f)$. Pick $x \in Z(f) - \mathrm{cl}_X(\mathrm{int}_X(Z(f)))$. Then there is $g \in C(X)$ such that $0 \leq g(y)$ for all $y \in X$, $Z(g)$ is a neighborhood of x in X and $\mathrm{int}_X(Z(f) \cap Z(g)) = \emptyset$. Since $\mathrm{int}_X(Z(f) \cap Z(g)) = \mathrm{int}_X(Z(f+g)) = \emptyset$, $f + g$ is a regular element in C(X). Let I be the principal ideal of C(X) generated by $f + g$. Then $I = C(X)$ and hence $\underline{1} = (f + g)h$ for some $h \in C(X)$, where $\underline{1}(y) = 1$ for all $y \in X$. This is a contradiction.□

COROLLARY 2.5. *Let X and Y be spaces such that C(X) and C(Y)* *are ring isomorphic. If X is an almost P-space, then Y is also an almost P-space.*

We recall that a space X is called *pseudocompact* if $C(X) = C(C)^*$, equivalently, $\beta X = \upsilon X$.

COROLLARY 2.6. *Let X be a space. Then βX is an almost P-space if and only if X is a pseudocompact, almost P-space.*

Recall that a space X is called *basically disconnected* if for any zero-set Z in X, $\text{int}_X(Z)$ is closed in X.

COROLLARY 2.7. *A space is a P-space if and only if it is a basically disconnected almost P-space.*

Every space X has the minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ and if X is a weakly Lindelöf space, then $\Lambda\beta X$ and $\beta\Lambda X$ are homeomorphic ([5]) and hence every $Z(X)^{\#}$-filter on a weakly Lindelöf space X with the countable meet property is a z-filter on X with the countable intersection property ([6]). Note that a space X is weakly Lindelöf if and only if every $Z(X)^{\#}$-filter with the countable meet property is fixed. Using these and the above theorem, we have the following:

COROLLARY 2.8. *Every weakly Lindelöf almost P-space is Lindelöf.*

PROPOSITION 2.9. *Let X be an almost P-space. Then*
(1) *if X is a countable set, then X is discrete, and*
(2) *if X is a countable set and a countably compact space, then X is finite.*

PROOF. (1) Suppose that X is a countable set and $x \in X$. For any $y \in X$ with $x \neq y$, there is a zero-set neighborhood Z_y of x in X such that $y \notin Z_y$. Let $Z = \cap\{Z_y : y \in X - \{x\}\}$. Then Z is a zero-set in X and $(X - \{x\}) \cap Z = \emptyset$. Since X is an almost P-space and $Z = \{x\}$, $\text{int}_X(Z) = \{x\}$ and so X is a discrete space.
(2) By (1), it holds. □

3. Absolute C*-embedding of almost P-spaces

It is known that a space X is C*-embedded in every space in which it is embedded if and only if X is an almost compact space, that is, for any two disjoint zero-sets in X, at least one of them is compact, equivalently, $|\beta X - X| \leq 1$ ([4]).

THEOREM 3.1. *Let X be an almost P-space.*

(1) *Suppose that for any almost P-space Y in which X is embedded, X is C^*-embedded. Then* $|\upsilon X - X| \leq 1$.

(2) *Suppose that for any disjoint zero-sets A and B in X, A or B is Lindelöf. Then for any P-space Y in which X is embedded, X is C^*-embedded in Y.*

PROOF. (1) Suppose that $|\upsilon X - X| \geq 2$. Pick $p, t \in \upsilon X - X$ with $p \neq t$. Then there are disjoint zero-sets A, B in X with $p \in cl_{\upsilon X}(A)$ and $t \in cl_{\upsilon X}(B)$. Let $R = \{(x,x) : x \in \upsilon X\} \cup \{(p,t),(t,p)\}$. Then R is an equivalence relation on υX. Let Y denote the quotient space $\upsilon X/R$ and $q : \upsilon X \longrightarrow Y$ be the quotient map. Then Y is a Tychonoff space and X is embedded in Y. Let $[x]$ be the equivalence class containing t. Suppose that $[x] \notin cl_Y(X)$. Then there is a zero-set neighborhood Z of x in Y with $Z \cap X = \emptyset$ and hence $q^{-1}(Z) \cap X = \emptyset$. Since $q^{-1}(Z)$ is a zero-set in υX, $q^{-1}(Z) = \emptyset$ and hence $Z = \emptyset$. This is a contradiction. Thus X is dense in Y.

Let C be a zero-set in Y with $C \neq \emptyset$. Then $q^{-1}(C)$ is a non-empty zero-set in υX and so $q^{-1}(C) \cap X = C \cap X$ is a non-empty zero-set in X. Since X is an almost P-space, $int_X(C \cap X) \neq \emptyset$ and hence $cl_X(int_X(C \cap X)) = cl_X(int_Y(C) \cap X) \neq \emptyset$. Thus $int_Y(C) \neq \emptyset$ and therefore Y is an almost P-space. Since $q(p) = q(t) \in cl_Y(A) \cap cl_Y(B)$, X is not C^*-embedded in Y.

(2) Let Y be a P-space in which X is embedded. Let A and B be disjoint zero-sets in X. We may assume that A is Lindelöf. Then there is a zero-set Z in Y with $B \subseteq Z$ and $A \cap Z = \emptyset$. Since Z is open in Y, $cl_Y(A) \cap Z = \emptyset$. Thus X is C^*-embedded in Y. □

Let X be a dense subspace of a space T. It is well-known that $\upsilon X = \upsilon T$ if and only if for any $p \in T$, there is a unique z-ultrafilter $\mathcal{A}^p$ in X with the countable intersection property such that $\mathcal{A}^p$ converges to p in T, that is, for any neighborhood V of p in T, there is $Z \in \mathcal{A}^p$ with $Z \subseteq V$ ([3]). Using this, we have the following:

THEOREM 3.2. *Let X be a subspace of a space T such that υX is a Lindelöf space and $|\upsilon X - X| \leq 1$. Then $\upsilon X \subseteq \upsilon T$.*

PROOF. Let $j_X : X \hookrightarrow \upsilon T$ be the inclusion map. Then there is a continuous map $f : \upsilon X \longrightarrow \upsilon T$ with $f|_X = j_X$. Let $\{p\} = \upsilon X - X$, $f(p) = t$ and $Y = X \cup \{t\}$. Take any $y \in Y$. If $y \in X$, then $\mathcal{A}_y = \{A :$ A is a zero-set in X with $y \in cl_{\upsilon X}(A)\}$ is the unique z-ultrafilter on X with the countable intersection property such that $\mathcal{A}_y$ converges to y in Y ([3]).

Suppose that $y = t$. Let $\mathcal{A}^p = \{A : A$ is a zero-set in X with $p \in \mathrm{cl}_{\upsilon X}(A)\}$. Then $\mathcal{A}^p$ is a z-ultrafilter on X with the countable intersection property and for any $A \in \mathcal{A}^p$, $t \in \mathrm{cl}_Y(f(A)) = \mathrm{cl}_Y(A)$. Hence $t \in \cap\{\mathrm{cl}_Y(A) : A \in \mathcal{A}^p\}$. Let V be a neighborhood of t in Y. Then $f^{-1}(V)$ is a neighborhood of p in υX. Since $\mathcal{A}^p$ is convergent to p in υX, there is A in $\mathcal{A}^p$ such that $A \subseteq f^{-1}(V)$ and hence $f(A) = A \subseteq V$. Hence $\mathcal{A}^p$ is convergent to t in Y. Let $\mathcal{A}$ be a z-ultrafilter on X with the countable intersection property such that $\mathcal{A}$ converges to t in Y. Suppose that $\mathcal{A}^p \neq \mathcal{A}$. Then there are disjoint zero-sets C, D in X such that $C \in \mathcal{A}^p$ and $D \in \mathcal{A}$. Since $p \notin \mathrm{cl}_{\upsilon X}(D)$, $p \notin \cap\{\mathrm{cl}_{\upsilon X}(A) : A \in \mathcal{A}\}$. Since $\mathcal{A}$ has the countable intersection property, $\cap\ \{\mathrm{cl}_{\upsilon X}(A) : A \in \mathcal{A}\} \neq \emptyset$ and since $\upsilon X - X = \{p\}$, $\cap\{A : A \in \mathcal{A}\} \neq \emptyset$. Pick $x \in \cap\{A : A \in \mathcal{A}\}$. Then $\mathcal{A} = \mathcal{A}_x = \{A \in Z(X) : x \in A\}$. Hence $\mathcal{A}_x$ is convergent to x in Y. Since $x \neq t$, this is a contradiction and so $\mathcal{A}^p = \mathcal{A}$. By the above fact, $\upsilon X \subseteq \upsilon Y$ and since υX is Lindelöf, Y is a realcompact space. Hence $\upsilon X \subseteq \upsilon T$.

Corollary 3.3. *Let X be an almost P-space such that υX is Lindelöf. Then $| \upsilon X - X | \leq 1$ if and only if for any almost P-space Y in which X is dense embedded, X is C*-embedded in Y.*

Proof. Suppose that $| \upsilon X - X | \leq 1$. Let Y be an almost P-space in which X is dense embedded. By Theorem 3.2, $\upsilon X \subseteq \upsilon Y$. Since υX is Lindelöf, υX is a dense z-embedded subspace in υY and since Y is an almost P-space, by Theorem 2.4, υX is C*-embedded in υY. □

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