Commun. Korean Math. Soc. **18** (2003), No. 4, pp. 703–712

THE STRONG CONSISTENCY OF THE ASYMMETRIC LEAST SQUARES ESTIMATORS IN NONLINEAR CENSORED REGRESSION MODELS

SEUNG HOE CHOI AND HAE KYUNG KIM

ABSTRACT. This paper deals with the strong consistency of the asymmetric least squares for the nonlinear censored regression models which includes dependent variables cut off midway by any of external conditions, and provide the sufficient conditions which ensure the strong consistency of proposed estimators of the censored regression models. One example is given to illustrate the application of the main result.

1. Introduction

Censored observation may arise naturally in time series if there is an upper or lower limit of detection. For example, in medicine the survival time of a patient can not be observed due to the patient was alive at the termination of the study, the patient withdrew alive during the study, or the patient died of causes other than those under study. In accordance with this, in case of the dependent variables are subject to censoring time various statistical properties derived from ordinary regression model are not available. Therefore, it is necessary that method to estimating and testing for censored regression model be suggested.

We consider in this paper the following nonlinear censored regression model

$$(1.1) \qquad y_t = \min\{c_t, f(x_t, \theta_o) + \epsilon_t\}, \quad t = 1, 2, \cdots, n,$$

where $x_t \in \Omega \subset R^q$ denotes the t-th fixed input value, the true parameter vector $\theta_o = (\theta_1, \cdots, \theta_p)$ is an interior point of parameter space $\Theta \subset R^p$, f is a real valued function on $R^q \times \Theta$, random errors ϵ_t are independent and identically distributed (i.i.d.) random variables with a finite second

Received May 30, 2002.
2000 Mathematics Subject Classification: 62J02, 62G05.
Key words and phrases: nonlinear censored regression model, asymmetric least square estimators, strong consistency.

moment and the distribution function $G(x)$ and y_t is the t-th dependent value which are censored from left at fixed censoring time c_t. Assume that the function $f(x, \theta_o)$ could be written in the form $f(x, \theta_o) = \theta_1 + \tilde{f}(x, (\theta_2, \cdots, \theta_p))$.

Unlike the ordinary regression model, in censored regression model we observe only the censored data (Z_t, δ_t, x_t) with $Z_t = \min\{y_t, c_t\}, \delta_t = I_{[y_t < c_t]}$ where c_t is censoring time and I is indicator function. Statistical analysis under nonlinear censored regression model (1.1) involves estimation of the parameter θ and test of hypotheses about its components by utilizing the data (Z_t, δ_t, x_t). As the important estimation method for censored regression model when the response function is linear, Powell [6] proposed regression quantiles estimators which provide a natural generalization of the notion of sample quantiles to the general regression model.

The Censored Regression Quantiles (CRQ) estimators, denoted by $\tilde{\theta}_n(\beta)$, are defined as the value of θ minimizing the following function

$$R_n(\theta; \beta) = \frac{1}{n} \sum_{t=1}^{n} \varphi_\beta(y_t - \min\{c_t, f(x_t, \theta)\}), \tag{1.2}$$

where the "check function"

$$\varphi_\beta(x) = \begin{cases} \beta x & \text{if} \quad x \geq 0 \\ (\beta - 1)x & \text{if} \quad x < 0 \end{cases}$$

and $0 < \beta < 1$. Especially, the Least Absolute Deviation (LAD) estimators are easily seen to be a special case of the β-th RQ estimators when $\beta = \frac{1}{2}$. Analysis of linear models using CRQ estimation has been published by many authors ([2, 3, 4, 6, 7]).

Although RQ estimators is robust against outlier, it can be poor estimators in case of the distribution of the errors is similar to normal distribution, or has less variance than standard normal distribution and there is the difficulty of computing RQ estimation and the lack of adequate sampling theory of such estimators because object function of RQ estimation is not continuously differentiable. See Newey and Powell [8] and Billas et al. [2]. To improve these problem, Newey and Powell [8] consider replacing the "check function" of RQ estimators with the following "loss function"

$$\rho_\tau(x) = \begin{cases} \tau x^2 & \text{if} \quad x \geq 0 \\ (1 - \tau)x^2 & \text{if} \quad x < 0, \end{cases}$$

where $0 < \tau < 1$. Thus, in order to make up for the weak point of CRQ estimators there is need to suggest the new estimators based on the loss function $\rho_\tau(x)$ in censored model (1.1).

The Censored Asymmetric Least Squares (CALS) estimators of the true parameter θ_o based on (Z_t, δ_t, x_t), denoted by $\hat{\theta}_n(\tau)$, are a parameter which minimizes the objective function

$$L_n(\theta; \tau) = \frac{1}{n} \sum_{t=1}^{n} \rho_\tau(y_t - \min\{c_t, f(x_t, \theta)\}). \tag{1.3}$$

Since the loss function $\rho_\tau(x)$ rotates the square function $\frac{x^2}{2}$ by some angle in the clockwise direction, the Least Squares (LS) estimators is an obviously important special case of the ASL estimation. Efron [5] and Newey and Powell [8] investigated asymptotic behavior of ALS estimators in the in the ordinary linear regression model with i.i.d. random errors. Amemiya [1] proved the asymptotic properties of the maximum likelihood estimators for linear censored regression model.

The main purpose of this paper is to provide some simple sufficient conditions for the strong consistency of the CALS estimators in nonlinear regression model when dependent variables are subject to censoring time. Also, one example is given to illustrate the application of the main result.

2. Strong consistency of CALS estimators

In this section, we present sufficient conditions for strong consistency of the CALS estimators in model (1.1). To simplify the notations, we denote

$$f_t(\theta) = f(x_t, \theta), \nabla f_t(\theta) = \Big[(f_t)_i(\theta)\Big]_{(p\times 1)}, (f_t)_i(\theta) = \frac{\partial}{\partial \theta_i} f_t(\theta),$$

$$\nabla^2 f_t(\theta) = \Big[(f_t)_{ij}(\theta)\Big]_{(p\times p)}, \text{ and } (f_t)_{ij}(\theta) = \frac{\partial^2}{\partial \theta_i \partial \theta_j} f_t(\theta).$$

For the strong consistency of CALS estimators, we need the following assumptions in censored regression model (1.1):

Assumption A

A_1 : The parameter space Θ is a compact subspace of R^q and Ω is compact subset of R^q.

A_2 : For all t, the partial derivatives $\nabla f_t(\theta)$ and $\nabla^2 f_t(\theta)$ exist and $\nabla f_t(\theta)$ are continuous on $\Gamma \times \Theta$.

Now, in order to explain the relation between the τ given in (1.3) and the distribution function of $G(x)$ we introduce the expectiles which are determined by tail expectation. Newey and Powell [8] pointed out the expectiles, denoted by $u(\tau)$, summarize the distribution function in the similar way that the quantiles $\eta_\beta = G^{-1}(\beta)$ and suggested the following equation

$$\frac{\tau}{1-\tau} = \frac{\int_{-\infty}^{\mu(\tau)}(\mu(\tau) - x)dG(x)}{\int_{\mu(\tau)}^{\infty}(x - \mu(\tau))dG(x)}. \tag{2.1}$$

From (2.1) we get the relation between τ and $\mu(\tau)$ as the following

$$\begin{aligned}\tau &= \frac{\int_{-\infty}^{\mu(\tau)}(x - \mu(\tau))dG(x)}{\int_{-\infty}^{\mu(\tau)}(x - \mu(\tau))dG(x) - \int_{\mu(\tau)}^{\infty}(x - \mu(\tau))dG(x)} \\ &= \frac{\int_{-\infty}^{\mu(\tau)}|x - \mu(\tau)|dG(x)}{E|\epsilon - \mu(\tau)|}.\end{aligned}$$

Thus, in case of the density function of errors is symmetric about a we know that the CALS estimators is the same of LS estimators by choosing $\tau = \frac{1}{2}$ and $\mu(\tau) = a$. The next assumption gives the condition of the density function of random errors.

Assumption B

B_1 : The density function $g(x)$ of random errors is continuous on R and strictly positive at some finite real number $u(\tau)$ which is depend on τ.

Meanwhile, to derive the another objective function of CALS estimators let

$$S_n(\theta;\tau) = \frac{1}{n}\sum_{t=1}^{n}\rho_\tau(y_t - \min\{c_t, f(x_t,\theta) - \mu(\tau)\}). \tag{2.2}$$

Then, the value which minimizing (2.2) is equivalent to $\hat{\theta}_n(\tau) + (\mu(\tau), 0, \cdots, 0)$. The following theorem gives uniform convergence of the objective function of CALS estimators

$$Q_n(\theta : \tau) = S_n(\theta : \tau) - S_n(\theta_o : \tau).$$

THEOREM 1. *Suppose that Assumption A and B hold for the censored model* (1.1). *Then we have*

$$Q_n(\theta : \tau) - E[Q_n(\theta : \tau)] = o_p(1),$$

where $o_p(1)$ *stands for convergence in probability.*

PROOF. For the proof, let

$$A_t(\tau,\theta,\theta_o)=\rho_\tau(y_t-\min\{c_t,f_t(\theta)\}-\mu(\tau))-\rho_\tau(y_t-\min\{c_t,f_t(\theta_o)\}-\mu(\tau)).$$

Then, the convexity of the check function $\rho_\tau(x)$ implies that $A_t(\tau,\theta,\theta_o)$ is less than

$$2\rho_\tau(\min\{c_t,f_t(\theta_o)\}-\min\{c_t,f_t(\theta)\})+\rho_\tau(y_t-\min\{c_t,f_t(\theta_o)\}-\mu(\tau)).$$

On the other hand, by simple calculation we obtain that

$$\rho_\tau(\min\{c_t,f_t(\theta_o)\}-\min\{c_t,f_t(\theta)\})$$

is equal to

$$\begin{cases} \rho_\tau(f_t(\theta_o)-f_t(\theta)), & \text{on } \Omega_1=\{x\in\Omega: f_t(\theta)<c_t, f_t(\theta_o)<c_t\} \\ \rho_\tau(c_t-f_t(\theta)), & \text{on } \Omega_2=\{x\in\Omega: f_t(\theta)<c_t\le f_t(\theta_o)\} \\ \rho_\tau(f_t(\theta_o)-c_t), & \text{on } \Omega_3=\{x\in\Omega: f_t(\theta_o)<c_t\le f_t(\theta)\} \\ 0, & \text{on } \Omega_4=\{x\in\Omega: f_t(\theta_o)\ge c_t, f_t(\theta)\ge c_t\} \end{cases}$$

So, from the above result we have

$$\rho_\tau(\min\{c_t,f_t(\theta_o)\}-\min\{c_t,f_t(\theta)\})\le\rho_\tau(f_t(\theta_o)-f_t(\theta))$$

for $t\in\Omega_2\cup\Omega_3$. Thus, for all t we get

$$\rho_\tau(\min\{c_t,f_t(\theta_o)\}-\min\{c_t,f_t(\theta)\})\le\tau_{\max}(f_t(\theta_o)-f_t(\theta))^2$$

where $\tau_{\max}=\max\{\tau,1-\tau\}$.

Also, Assumption A and application of Mean Value Theorem and Hölder's inequality yield that there exist a finite $M_{t1}(\tau,\theta,\theta_o)$ such that

$$\rho_\tau(f_t(\theta_o)-f_t(\theta))\le\tau_{\max}\|\nabla f_t(\theta_o^*)\|\|\theta-\theta_o\|\le M_{t1}(\tau,\theta,\theta_o)$$

where $\theta_o^*=\lambda\theta+(1-\lambda)\theta_o, 0<\lambda<1$ and $\|\cdot\|$ denotes Euclidean norm. By similar method as before, we know

$$\rho_\tau(y_t-\min\{c_t,f_t(\theta_o)\}-\mu(\tau))\le\tau_\beta(\epsilon_t-\mu(\tau))^2.$$

While, since second moment of ϵ_t and $\mu(\tau)$ are finite, there exist a finite $M_{t2}(\tau,\theta_o)$ such that

$$\rho_\tau(y_t-\min\{c_t,f_t(\theta_o)\}-\mu(\tau))\le M_{t2}(\tau,\theta_o).$$

Thus, we get

$$A_t(\tau,\theta,\theta_o)\le 2M_{t1}(\tau,\theta,\theta_o)+M_{t2}(\tau,\theta_o).$$

Moreover, Chebyshev's inequality gives

$$P\left[|Q_n(\theta;\tau)-EQ_n(\theta;\tau)|>\varepsilon\right]\le\frac{\max\limits_{1\le t\le n} Var[A_t(\tau,\theta,\theta_o)]}{n\epsilon^2}.$$

Since $Var[A_t(\tau, \theta, \theta_o)]$ is finite due to Assumption A, the proof is completed. □

Let P_x be a probability measure on R^q and we add the following assumptions to discuss the strong consistency of CALS estimators.

Assumption C

C_1 : $P_x[x_t \in \Omega : f_t(\theta) \neq f_t(\theta_o)] > 0$ if $\theta \neq \theta_o$.

C_2 : The ratio of the number of elements of the set $U_n(\theta_o) = \{t : f_t(\theta_o) < c_t\}$ to the number of the observations in model (1.1), denoted by $\frac{u_n(\theta_o)}{n}$, converges to $u(\theta_o), 0 < u(\theta_o) < 1$.

C_3: The matrix $V_n(\theta_o) = \frac{1}{n} \sum_{t \in U_n(\theta_o)} \nabla^T f_t(\theta_o) \nabla f_t(\theta_o)$ converges to a positive definite matrix $V(\theta_o)$ as $n \to \infty$.

REMARK. In an actual experiment, it is an important point how many of the value of dependent variables can be observed within the censoring time. Since it can be $u(\theta_o) = 1$ in case that the experiment is observed or recorded in sufficient time, Assumption C_3 is transformed to a condition often referred in ordinary nonlinear regression model, and the condition $u(\theta_o) = 0$ which happen in case that the value of dependent variables that is observable in a fixed time is extremely small may be unsuitable to the regression model. Hence, as suggested in Assumption C the ratio of the number of the dependent variables within censoring time to the total number of observation should maintain a fixed ratio.

The next result deals with the strong consistency of CALS estimators.

THEOREM 2. *For the nonlinear censored regression model* (1.1), *suppose that Assumption A, B and C are fulfilled. Then the CALS estimators $\hat{\theta}_n(\tau)$ is strongly consistent for $\theta_o(\tau)$, denoted by*

$$\hat{\theta}_n(\tau) \stackrel{a.s}{\to} \theta_o(\tau)$$

where $\theta_o(\tau) = (\theta_1 - \mu(\tau), \theta_2, \cdots, \theta_p)$.

PROOF. To prove the theorem it suffices to show that

$$\lim_{n \to \infty} \inf_{\|\theta - \theta_o\| > \delta} \{Q_n(\theta; \tau)\} > 0 \ a.e.$$

for any $\delta > 0$.

First, Theorem 1 implies that

$$Q_n(\theta : \tau) = \frac{1}{n} \sum_{t=1}^{n} E\big[A_t(\tau, \theta, \theta_o)\big] + o_p(1).$$

Let

$$B_n(\theta:\tau) = \frac{1}{n}\sum_{t=1}^{n} E\big[A_t(\tau,\theta,\theta_o)\big].$$

Since $A_t(\tau,\theta,\theta_o)$ is independent of θ in case of $c_t \geq f_t(\theta)$, we have that

$$\nabla B_n(\theta:\tau) = \frac{-1}{n}\sum_{t\in U_n(\theta)} \rho_\tau^{'}(y_t - f_t(\theta) - \mu(\tau))\nabla f_t(\theta)$$

and

$$\nabla^2 B_n(\theta:\tau) = \frac{1}{n}\sum_{t\in U_n(\theta)} [\rho_\tau^{''}(y_t - f_t(\theta) - \mu(\tau))\nabla f_t(\theta)\nabla^T f_t(\theta) - \rho_\tau^{'}(y_t - f_t(\theta) - \mu(\tau))\nabla^2 f_t(\theta)]$$

where $\rho_\tau^{'}(x)$ and $\rho_\tau^{''}(x)$ denotes first and second derivative of $\rho_\tau(x)$, respectively.

Note that in case of t belong to $U_n(\theta_o)$ we get

$$y_t = f_t(\theta_o) + \eta_t$$

where $\eta_t = \epsilon_t I_{[\eta_t < a_t(\theta_o)]}$ and $a_t(\theta_o) = c_t - f_t(\theta_o)$.

Define $h_t(x)$ by

$$h_t(x) = \begin{cases} \frac{g(x)}{\int_{-\infty}^{a_t(\theta_o)} g(x)dx}, & x \leq a_t(\theta_o) \\ 0, & x > a_t(\theta_o). \end{cases}$$

Then, $H_t(k) = \int_{-\infty}^{k} h_t(\lambda)d\lambda$ is the distribution function of η_t. Moreover, by means of the relation of τ and $\mu(\tau)$ and Chebyshev's inequality we obtain $\nabla B_n(\theta_o:\tau) = 0$ and $\nabla^2 B_n(\theta_o:\tau)$ is greater than

$$2\tau_{\min}\min_{1\leq t\leq n}\int_{\mu(\tau)}^{a_t(\theta_o)} dH_t(\lambda)\frac{1}{n}\sum_{t\in U_n(\theta_o)} f_t(\theta_o)\nabla^T f_t(\theta_o)$$

where $\tau_{\min} = \min\{1-\tau,\tau\}$. Hence, since the Hessan matrix $\nabla^2 B_n(\theta_o:\tau)$ is positive definite for sufficiently large n, $B_n(\theta:\tau)$ attains a local minimum at θ_o.

Suppose that there exist $\theta_n^*(\tau)$ in $\{\theta : \|\theta-\theta_o\| \geq \delta\}\cap\Theta$ such that $B_n(\theta_n^*(\tau):\tau) \leq B_n(\theta_o:\tau)$ for sufficiently large n. If $d_t(\theta) = f_t(\theta) - f_t(\theta_o) > 0$, by some tedious algebra the expectation of $A_t(\tau,\theta,\theta_o)$ is greater than

$$\begin{cases} F_{t1}(\tau:\theta), & \text{if } t\in\Omega_1, a_t(\theta_o)\geq \mu(\tau) \\ F_{t2}(\tau:\theta), & \text{if } t\in\Omega_1, a_t(\theta_o)< \mu(\tau) \\ F_{t3}(\tau:\theta), & \text{if } t\in\Omega_3, a_t(\theta_o)\geq \mu(\tau) \\ F_{t4}(\tau:\theta), & \text{if } t\in\Omega_4, a_t(\theta_0)< \mu(\tau) \\ 0, & \text{if } t\in\Omega_5 \end{cases}$$

where

$$\begin{aligned} F_{t1}(\tau:\theta) = {} & \tau\int_{\mu(\tau)}^{\mu(\tau)+d_t(\theta)} (\mu(\tau)-\lambda)(\lambda-\mu(\tau)-2d_t(\theta))dG_t(\lambda) \\ & +\tau\int_{a_t(\theta_o)}^{\infty} 2d_t(\theta)(\lambda-a_t(\theta_o))dG_t(\lambda), \\ F_{t2}(\tau:\theta) = {} & 2d_t(\theta)\{(1-\tau)\int_{-\infty}^{a_t(\theta_o)}(\mu(\tau)-\lambda) \\ & +\tau\int_{-a_t(\theta_o)}^{\infty}(\mu(\tau)-a_t(\theta_o))\}dG_t(\lambda), \\ F_{t3}(\tau:\theta) = {} & \tau\int_{\mu(\tau)}^{\mu(\tau)+a_t(\theta_o)}(\lambda-\mu(\tau))(2a_t(\theta_o)-\lambda+\mu(\tau))dG_t(\lambda) \\ & +\tau 2a_t(\theta_o)\int_{a_t(\theta_o)}^{\infty}(\lambda-a_t(\theta_o))dG_t(\lambda), \end{aligned}$$

and

$$\begin{aligned} F_{t4}(\tau:\theta) = {} & 2(1-\tau)a_t(\theta_o)\{\int_{-\infty}^{a_t(\theta_o)}(\mu(\tau)-\lambda) \\ & +\int_{a_t(\theta_o)}^{\infty}(\mu(\tau)-a_t(\theta_o))\}dG_t(\lambda). \end{aligned}$$

All of integrals in this expression are strictly positive by Assumption B. So, we can choose a positive constant $\eta_{ti}(\tau:\theta)$ such that

$$F_{ti}(\tau:\theta)\geq \eta_{ti}(\tau:\theta)$$

for each i. Likewise if $d_t(\theta)<0$, we have a similar result. Thus, the proceeding results show that

$$B_n(\theta_n^*(\tau):\tau)\geq \min_{1\leq t\leq n}\eta_t(\tau:\theta_n^*(\tau))w_t(\tau),$$

where $\eta_t(\tau:\theta)=\min_{1\leq i\leq 4}\eta_{ti}(\tau:\theta)$ and $w_t(\tau)=P_x[x_t\in\Gamma: f_t(\theta)\neq f_t(\theta_o)]$.

Hence this is a contradiction the fact $S_n(\theta_n^*(\tau):\tau)\leq S_n(\theta_o:\tau)$ for sufficiently large n. The proof is thus complete. □

To illustrate the application of the main result we consider with the following example.

EXAMPLE 1. Let $p(x)$ be a differential function from R^q to R^+ and k be fixed real number. Consider the response function

$$f_t(\theta) = \theta_1 + \theta_2 p(x_t)^{\theta_3}.$$

Assume that input variable x_t chose as random sample from some distribution with the distribution function $H(x)$ and the true parameter $\theta_o = (\theta_1, \theta_2, \theta_3)$ belong to $\Theta = [0, a_1] \times \{k\} \times [0, a_3]$, where a_1 and a_3 are finite real number. Then, by simple calculation we obtain

$$\begin{aligned} V_n(\theta_o) &= \frac{1}{n} \sum_{t \in U_n(\theta_o)} \nabla f_t(\theta_o) \nabla f_t^T(\theta_o) \\ &= \begin{bmatrix} \frac{u_n(\theta_o)}{n} & \frac{1}{n} \sum_{t \in U_n(\theta_o)} kp(x_t)^{\theta_3} \ln p(x_t) \\ \frac{1}{n} \sum_{t \in U_n(\theta_o)} kp(x_t)^{\theta_3} \ln p(x_t) & \frac{1}{n} \sum_{t \in U_n(\theta_o)} \left(kp(x_t)^{\theta_3} \ln p(x_t)\right)^2 \end{bmatrix}. \end{aligned}$$

Also, Assumption C implies that $V_n(\theta_o)$ converges to

$$u(\theta_o) \begin{bmatrix} \int dH(x) & \int kp(x)^{\theta_3} \ln p(x)\, dH(x) \\ \int kp(x)^{\theta_3} \ln p(x)\, dH(x) & \int \left(kp(x)^{\theta_3} \ln p(x)\right)^2 dH(x). \end{bmatrix}.$$

Moreover, for non-zero vector $\alpha = (\alpha_1, \alpha_2) \neq (0, 0)$ we have

$$\alpha V(\theta) \alpha^T = u(\theta_o) \int \left(\alpha_1 + \alpha_2 kp(x)^{\theta_3} \ln p(x)\right)^2 dH(x) > 0.$$

Hence, under some conditions we can obtain the strong consistency of CALS estimators. □

Let k be fixed real number and a_1 and a_3 be finite real number. Consider the response function

$$f_t(\theta) = \theta_1 + \theta_2 x_t^{\theta_3}, x_t > 0.$$

Then, note that

$$\begin{aligned} V_n(\theta_o) &= \frac{1}{n} \sum_{t \in U_n(\theta_o)} \nabla f_t(\theta_o) \nabla f_t^T(\theta_o) \\ &= \begin{bmatrix} \frac{u_n(\theta_o)}{n} & \frac{1}{n} \sum_{t \in U_n(\theta_o)} kx_t^{\theta_3} \ln p(x_t) \\ \frac{1}{n} \sum_{t \in U_n(\theta_o)} kx_t^{\theta_3} \ln x_t & \frac{1}{n} \sum_{t \in U_n(\theta_o)} \left(kx_t^{\theta_3} \ln x_t\right)^2 \end{bmatrix}. \end{aligned}$$

Also, Assumption C implies that $V_n(\theta_o)$ converges to

$$u(\theta_o)\begin{bmatrix} \int dH(x) & \int kx^{\theta_3}\ln p(x)\, dH(x) \\ \int kx^{\theta_3}\ln x\, dH(x) & \int \left(kx^{\theta_3}\ln x\right)^2 dH(x) \end{bmatrix}.$$

Moreover, for non-zero vector $\alpha = (\alpha_1, \alpha_2) \neq (0,0)$ we have

$$\alpha V(\theta)\alpha^T = u(\theta_o)\int \left(\alpha_1 + \alpha_2 kp(x)^{\theta_3}\ln p(x)\right)^2 dH(x) > 0.$$

Hence, under some conditions we can obtain the strong consistency of CALS estimators. □

References

[1] T. Amemiya, *Regression analysis when the dependent variable is truncated normal*, Econometrica **41** (1973), 997–1016.

[2] Y. Billas, S. Chen, and Z. Ying, *Simple resampling methods for censored regression quantiles*, Journal of Econometrics **99** (2000), 373–386.

[3] X. R. Chen and Y. Wu, *Consistency of L_1 Estimates in Censored Linear Regression Models*, Communication in Statistics **23** (1994), 1849–1858.

[4] S. H. Choi and H. Kim, *The Strong Consistency of the L_1-norm Estimators in Censored Nonlinear Regression Models*, Bull. Korean Math. Soc. **34** (1998), 573–581.

[5] B. Efron, *Regression Percentiles using Asymmetric Squared Error Loss*, Statistica Sinica **1** (1991), 93–125.

[6] J. L. Powell, *Least Absolute Deviation Estimation for the Censored Regression Model*, Journal of Econometrics **25** (1984), 303–325.

[7] ———, *Censored regression quantiles*, Journal of Econometric **32** (1986), 143–155.

[8] W. K. Newey and L. Powell, *Asymmetric least squares estimation and testing*, Econometrica **55** (1987), 819–847.

Seung Hoe Choi
Department of General Studies
Hankuk Aviation University
Koyang 411, Korea
E-mail: shchoi@mail.hangkong.ac.kr

Hae Kyung Kim
Department of Mathematics
Yonsei University
Seoul 120-749, Korea
E-mail: abc1217@yonsei.ac.kr