Commun. Korean Math. Soc. **18** (2003), No. 4, pp. 713–723

LIMIT OF SOLUTIONS OF A SPDE DRIVEN BY MARTINGALE MEASURE WITH REFLECTION

NHANSOOK CHO AND YOUNGMEE KWON

ABSTRACT. We study a limit problem of reflected solutions of parabolic stochastic partial differential equations driven by martingale measures. The existence of solutions is found in an extension of the work with respect to white noise by Donati-Martin and Pardoux [4]. We show that if a certain sequence of driving martingale measures converges, the corresponding solutions also converge in the Skorohod topology.

1. Introduction

In this paper, we study a limit problem of reflected solutions of parabolic stochastic partial differential equations(SPDEs) driven by martingale measures. C. Donald-Martin and E. Pardoux [4] studied reflected solutions of parabolic SPDEs driven by a space time white noise and their results are the motivation of this research. Readers may refer recent studies and short history on this subject to [4].

We briefly introduce the main result of [4]. Suppose that u_0 is a positive continuous function on $[0,1]$ which vanishes at 0 and 1. Let W be a usual space-time white noise then there is a continuous process, u on $[0,1]\times R_+$ and a random measure η on $[0,1]\times R_+$ which satisfy

(1) u is a continuous process on $[0,1]\times R_+$.; $u(x,t)$ is $\mathcal{F}_t-$ measurable and $u(x,t)\geq 0$ a.s.

(2) $\eta((0,1)\times\{t\})=0, \int_0^t\int_0^1 x(1-x)\eta(dx,ds)<\infty$, for all $t\geq 0$, and for any measurable mapping $\phi:[0,1]\times R_+\to R_+, \int_0^t\int_0^1\phi(x,s)\eta(dx,ds)$ is $\mathcal{F}_t$-measurable.

Received October 7, 2002.

2000 Mathematics Subject Classification: Primary 60F05; Secondary 60H15.

Key words and phrases: reflected solutions, SPDE, martingale measure, Skorohod topology.

This research was financially supported by Engineering Research Center of Hansung University in the year of 2003.

(3) (u, η) solves the parabolic SPDE.

(1.1)
$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t)) = \sigma(u(x,t))\dot{W}(x,t) + \eta(x,t)$$
$$u(\cdot,0) = u_0; u(0,t) = u(1,t) = 0,$$

in the following sense ($\langle \cdot, \cdot \rangle$ denotes the scalar product in $L_2[0,1]$): for any $t \in R_+$, $\phi \in C^2[0,1]$ with $\phi(0) = \phi(1) = 0$,

$$\langle u(x,t), \phi(x) \rangle + \int_0^t \langle u(x,s), \phi''(x) \rangle ds + \int_0^t \langle f(u(x,s)), \phi(x) \rangle ds$$
$$= \langle u_0, \phi \rangle + \int_0^t \int_0^1 \phi(x)\sigma(u(x,s))W(dx,ds)$$
$$+ \int_0^t \int_0^1 \phi(x)\eta(dx,ds) \quad \text{a.s. },$$

where f and σ are globally Lipschitz.

(4) $\int_0^t \int_{R_+} u d\eta = 0$.

We say that the reflected parabolic equation(RPE) has a solution (u, η) if the pair (u, η) satisfies (1), (2), (3) and (4).

Now we consider a sequence of orthogonal martingale measures, $\{M^n\}$ which converges in distribution to W in the Skorohod topology on $D_{\mathcal{S}'(R)}[0,T]$, where $\mathcal{S}'(R)$ is the dual of Schwartz space and $D_{\mathcal{S}'(R)}[0,T]$ is the Skorohod space. It is the space of all functions $f : [0,1] \to \mathcal{S}'(R)$ which are right continuous and have left limits at each $t \in [0,T]$. Consider the following equation:

(1.2)
$$\frac{\partial u^n(x,t)}{\partial t} - \frac{\partial^2 u^n(x,t)}{\partial x^2} + f(u^n(x,t)) = \sigma(u^n(x,t))M^n(x,t) + \eta^n(x,t)$$
$$u^n(\cdot,0) = u_0; u^n(0,t) = u^n(1,t) = 0.$$

See in detail [2] or [10] for the definition of orthogonal martingale measure. It is easily proved that under the Condition M2 in section 2, there is a solution, (u^n, η^n) of (1.2) which satisfies the corresponding (1), (2), (3) and (4). Since the proof is similar with the proof of the existence of solution to (1.1) (see Theorem 4.1 in [4]) and we focus on the limit behavior of solution sequence, we do not want to present that proof in this paper.

We are going to show that under certain conditions $(u^n, \eta^n) \Rightarrow (u, \eta)$ in the Skorohod topology on $D_{C[0,1]}[0,T] \times D_{\mathcal{S}'(R)}[0,1]$, for any $T > 0$.

2. Existence and uniqueness, and a comparison theorem

Let $R = (-\infty, \infty)$ and $R_+ = (0, \infty)$ as usual. The coefficients f and σ in equation (1.2) are measurable functions from $[0,1] \times \Omega \times R_+ \times R$ into R which satisfy the following conditions:

CONDITION C.
(C1) $f(x, \omega, t; 0) = 0$.
(C2) f and σ are Lipschitz such that for each $T > 0$, $M > 0$ there exists a constant $C(T)$ satisfying

$$|f(x,\omega,t;z) - f(x,\omega,t;r)| + |\sigma(x,\omega,t;z) - \sigma(x,\omega,t;r)| \le C(T)|z-r|,$$

for all $(x, \omega, t) \in (0,1) \times \Omega \times (0,T)$, $z, r \in [-M, M]$.
(C3) For each $T > 0$ there exists a constant $\bar{C}(T)$ satisfying

$$|f(x,\omega,t;z)| + |\sigma(x,\omega,t;z)| \le \bar{C}(T)(1+|z|),$$

for all $(x, \omega, t, z) \in (0,1) \times \Omega \times (0,T) \times R$.

We shall write $f(u(x,t))$(resp.$\sigma(u(x,t))$ instead of $f(x,\omega,t;u(x,t))$ (resp. $\sigma(x,\omega,t;u(x,t))$). Let M be a continuous orthogonal martingale measure and consider the parabolic equation:

(2.1)

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t)) = \sigma(u(x,t))M(dx,dt) \\ &u(\cdot,0) = u_0; u(0,t) = u(1,t) = 0. \end{aligned}$$

If M satisfies a certain condition as following Condition M1, adapting the method of Walsh [10] we can easily show that (2.1) has a unique cadlag adapted process which satisfies: for any $t \in R_+$, $\phi \in C^2[0,1]$ with $\phi(0) = \phi(1) = 0$,

$$\begin{aligned} &\langle u(x,t), \phi(x)\rangle + \int_0^t \langle u(x,s), A\phi(x)\rangle ds + \int_0^t \langle f(u(x,s)), \phi(x)\rangle ds \\ &= \langle u_0, \phi\rangle + \int_0^t \int_0^1 \phi(x)\sigma(u(x,s))M(dx,ds) \quad \text{a.s. ,} \end{aligned}$$

where $A = -\frac{\partial^2}{\partial x^2}$. Equivalently $u(x,t)$ satisfies the integral equation

$$\begin{aligned} u(x,t) = &\int_0^1 u_0(y) G_t(x,y)dy - \int_0^t \int_0^1 f(u(y,s))G_{t-s}(x-y)dyds \\ &+ \int_0^t \int_0^1 \sigma(u(y,s))G_{t-s}(x,y)M(dy,ds), \end{aligned}$$

where G is the Green's function associated to the operator $\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions. Several authors have shown that (2.1) has a unique continuous solution([6], [7]).

Let M be a martingale measure and $\pi(dx, dy)$ be its covariance measure.

CONDITION M1. We assume that there exists a nonnegative bounded function, $h(x)$ on $[0,1]$ such that

$$\pi(dx, ds) = h(x)dxds.$$

NOTATION. Let $d\mu = h(x)dx$ and $L_2([0,1], d\mu)$ be the Hilbert space of real valued $d\mu$-square integrable functions on $[0,1]$. For any function f let $f^+ = \max\{f(x), 0\}$, $f^- = -\min\{f(x), 0\}$ and $\|f(\cdot)\|_\infty = \sup_{x\in[0,1]} |f(x)|$.

The following theorems are modifications of Theorem 2.1 and Theorem 3.1 in [4].

THEOREM 2.1. *Let u_0 be as before and suppose that f, σ satisfy Condition C. Also assume that M satisfies the condition $M1$. Then equation (2.1) has a unique continuous and $\mathcal{F}_t$ adapted solution $\{u(x,t); 0 \le x \le 1, t \ge 0\}$.*

THEOREM 2.2. *Let M be a martingale measure satisfying Condition M1 and let two pairs of coefficients f, σ and g, σ be globally Lipschitz with $f \ge g$. We denote by u (resp. v) the solution of (2.1) corresponding to f (resp. g) with the same initial condition. Then, a.s., for all $(x,t) \in [0,1] \times R_+$,*

$$u(x,t) \le v(x,t).$$

PROOF. Let $\{e_k\}$ be an orthonormal basis of $L_2([0,1], d\mu)$ such that $\|e_k\|_\infty \le C$ for all $k = 1, 2, \ldots$. We let

$$M_t^k = \int_0^t \int_0^1 e_k(x) M(dx, ds).$$

Then $\{M^k\}$ is a family of mutually independent martingale processes. If we follow the arguments of the proof of Theorem 2.1 [4] under Condition M1, we can get the result applying the generalized Ito's formula in [9].□

3. The convergence of $\{u^n\}$

Let M^n, $n = 1, 2, \ldots$ be orthogonal martingale measures whose covariance measure, $\langle M_n(dx, ds)\rangle = \pi_n(dx, ds)$ satisfy the following Condition M2.

CONDITION M2. For each $n = 1, 2, \ldots$, there exists a $h_n(x) : [0, 1] \times R_+ \to R$ such that for each $T > 0$

$$\pi_n(dx, ds) = h_n(x)dxds, \quad \sup_n \sup_{x\in[0,1]} h_n(x) < \infty.$$

Let u_0^n (resp. u_0) be a positive continuous function on $[0, 1]$ such that $u_0^n(0) = u_0^n(1) = 0$ (resp. $u_0(0) = u_0(1) = 0$). We consider the following penalized SPDEs: for $n = 1, 2, \ldots$.

$$\begin{aligned} &\frac{\partial u_\epsilon^n(x,t)}{\partial t} - \frac{\partial^2 u_\epsilon^n(x,t)}{\partial x^2} + f(u_\epsilon^n(x,t)) \\ &= \sigma(u_\epsilon^n(x,t))M_n(dx,dt) + \frac{1}{\epsilon}(u_\epsilon^n(x,t))^- \\ &u_\epsilon^n(\cdot,0) = u_0^n; u_\epsilon^n(0,t) = u_\epsilon^n(1,t) = 0, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\frac{\partial u_\epsilon(x,t)}{\partial t} - \frac{\partial^2 u_\epsilon(x,t)}{\partial x^2} + f(u_\epsilon(x,t)) \\ &= \sigma(u_\epsilon(x,t))W(dx,dt) + \frac{1}{\epsilon}(u_\epsilon(x,t))^- \\ &u_\epsilon(\cdot,0) = u_0; u_\epsilon(0,t) = u_\epsilon(1,t) = 0, \end{aligned} \tag{3.2}$$

where $(u_\epsilon^n(x,t))^- = -\min\{u_\epsilon^n(x,t), 0\}$. For each $n = 0, 1, \ldots$ and $\epsilon > 0$, (3.1) (resp. (3.2)) admits a unique solution u_ϵ^n (resp. u_ϵ) which satisfies that $u_{\epsilon'}^n \le u_\epsilon^n$ a.s. for $\epsilon < \epsilon'$ by Theorem 2.2.

We consider a version of the Kolmogorov Lemma.

KOLMOGOROV LEMMA [4]. *Let $\mathcal{A}$ be a cube in R^n and $\{X_\alpha, \alpha \in \mathcal{A}\}$ be a real valued stochastic process. Suppose there exist constants $k > 1, K > 0$ and $\epsilon > 0$ such that*

$$E[|X_\alpha - X_\beta|^k] \le K|\alpha - \beta|^{n+\epsilon}.$$

Then

(1) *X has a continuous version,*

(2) *there exist constants a and γ, depending only on n, k and ϵ, and a random variable Y such that a.s., for all $\alpha, \beta \in R^2$,*

$$|X_\alpha - X_\beta| \leq Y|\alpha - \beta|^{\frac{\epsilon}{k}} (\log \frac{\gamma}{|\alpha - \beta|})^{\frac{2}{k}},$$

and

$$E[Y^k] \leq aK.$$

THEOREM 3.1. *If f, σ satisfy Condition C and $\{M_n\}$ satisfies Condition M2, then for each fixed $\epsilon > 0$, $\{u_\epsilon^n\}$ is relatively compact in $D_{C[0,1]}[0, T]$.*

PROOF. To show the relative compactness of $\{u_\epsilon^n\}$, we show that for any fixed $T > 0$,

$$\sup_n E(\sup_{s \leq T} \|u_\epsilon^n(\cdot, s)\|_\infty^p) < \infty, \text{ for some } p > 0.$$

Let

$$F_\epsilon(z) = f(z) - \frac{1}{\epsilon} z^-, \text{ for } z \in R,$$

then the solution of (3.1) satisfies the following:

$$\begin{aligned} u_\epsilon^n(x,t) = & \int_0^1 u_0^{n,\epsilon}(y) G_t(x,y) dy - \int_0^t \int_0^1 F_\epsilon(u_\epsilon^n(y,s)) G_{t-s}(x,y) dy ds \\ & + \int_0^t \int_0^1 \sigma(u_\epsilon^n(y,s)) G_{t-s}(x,y) M^n(dy, ds). \end{aligned}$$

For $(x,t) \in [0,1] \times R_+$, letting $p > 6$, $\frac{1}{p} + \frac{1}{q} = 1$, $r = \frac{2p}{p-2} < 3$, we get for some constant C

$$\begin{aligned} & E[|u_\epsilon^n(x,t)|^p] \\ \leq & C[|\int_0^1 u_0^n(y) G_t(x,y) dy|^p \\ & + (\int_0^t \int_0^1 G_{t-s}^q(x,y) dy ds)^{\frac{p}{q}} E[\int_0^t \int_0^1 |F_\epsilon(y,s,u_\epsilon^n)|^p dy ds] \\ & + (\int_0^t \int_0^1 G_{t-s}^r(x,y) \pi_n(dy,ds))^{\frac{p-2}{2}} E \int_0^t \int_0^1 |\sigma(x,s,u_\epsilon^n)|^p \pi_n(dy,ds)], \end{aligned}$$

and since $r < 3$, $\int_0^t \int_0^1 G_{t-s}^r(x,y)dyds < \infty$(see [10]). By the assumptions on f and σ, we deduce that for some constant $\bar{C}_\epsilon(t)$,

$$E(|u_\epsilon^n(x,t)|^p) \leq \bar{C}_\epsilon(t)(1 + E\int_0^t \sup_{y,r\leq s} |u^n(y,r)|^p ds).$$

Also, by the computations made in Corollary 3.4 [10], we can get the following; for $p > 6$, $s,t \leq T$

(3.5)

$$\begin{aligned} &E[|u_\epsilon^n(x,t) - u_\epsilon^n(y,s)|^p] \\ &\leq \bar{C}_\epsilon(T)|(x,t)-(y,s)|^{\frac{p}{4}-3}(1 + E\int_0^{t\vee s} (\sup_{z,\theta\leq r} |u_\epsilon^n(z,\theta)|^p dr). \end{aligned}$$

Choosing $p > 20$, by the Kolmogorov lemma, we can find a random variable $Y_{\epsilon,p}^n$ such that

$$|u_\epsilon^n(x,t) - u_\epsilon^n(y,s)|^p \leq Y_{\epsilon,p}^n|(x,t)-(y,s)|^{\frac{p}{4}-5}\big(\log\big(\frac{\gamma}{|(x,t)-(y,s)|}\big)\big)^2$$

and

$$E[(Y_{\epsilon,p}^n)^p] \leq aK(1 + E\int_0^t \sup_{z,y\leq r} |u_\epsilon^n(z,y)|^p dr). \tag{3.6}$$

Choosing $s = 0, y = 0$ in (3.5), we deduce by the Lemma that for any $T > 0$ there exists $C_{\epsilon,T}$(independently of n) such that

$$E(\sup_{x,s\leq T} |u_\epsilon^n(x,s)|^p) \leq C_{\epsilon,T}(1 + E(\int_0^T \sup_{y,r\leq s} |u_\epsilon^n(y,r)|^p ds)).$$

Applying Gronwall's lemma and letting $M_T(\epsilon) = C_{\epsilon,T} \cdot e^{T\cdot C_{\epsilon,T}}$, we have

$$\sup_n E[\sup_{x,s\leq T} |u_\epsilon^n(x,s)|^p] < M_T(\epsilon).$$

To show the other criteria of relative compactness of $\{u_\epsilon^n\}$ (for any fixed ϵ), for $0 \leq s \leq t \leq T$, $|t-s| < \delta$,

$$\begin{aligned} &|u_\epsilon^n(x,t) - u_\epsilon^n(x,s)|^p \\ &\leq |\int_s^t \int_0^1 F_\epsilon(u_\epsilon^n(y,s))G_{t-s}(x,y)dyds \\ &\quad + \int_s^t \int_0^1 \sigma(u_\epsilon^n(y,s))G_{t-s}(x,y)W(dy,ds)dxds|^p \\ &\leq C_\epsilon\{\int_s^t \int_0^1 G_{t-s}^q(x,y)dyds)^{\frac{p}{q}} \int_s^t \int_0^1 |F_\epsilon(u_\epsilon^n(y,s))|^p dyds \\ &\quad + (\int_0^t \int_0^1 G_{t-s}^r(x,y)dyds)^{\frac{p-2}{2}} \int_s^t \int_0^1 |\sigma(u_\epsilon^n(y,s)|^p dyds\}, \end{aligned}$$

for some constant C, where $\frac{1}{p}+\frac{1}{q}=1$, $r=\frac{2p}{p-2}<3$, and hence all integrals in the above right hand side are finite. Then as before, there exists a constant $\bar{C}_\epsilon$ such that

$$E[\|u_\epsilon^n(\cdot,t)-u_\epsilon^n(\cdot,s)\|_\infty^p|\mathcal{F}_s^n]\le E[\bar{C}_\epsilon(1+\sup_{0\le s\le T}\|u_\epsilon^n(\cdot,s)\|_\infty^p)\delta|\mathcal{F}_s^n].$$

Let

$$\gamma_n^\epsilon(\delta)=\bar{C}_\epsilon(1+\sup_{0\le s\le T}\|u_\epsilon^n(\cdot,s)\|_\infty^p)\delta.$$

Then

$$\lim_{\delta\to 0}\sup_n E[\gamma_\epsilon^n(\delta)]=0.$$

Hence for each (fixed) $\epsilon>0$, $\{u_\epsilon^n\}$ is relatively compact in $D_{C[0,1]}[0,T]$ by Theorem 3.8.6 [5]. □

Let u_ϵ be a limit of $\{u_\epsilon^n\}$. Then u_ϵ is the unique solution of the penalized SPDE, (3.2). (The uniqueness of solution of (3.2) is well known in [4, Theorem 3.1])

THEOREM 3.2. *If f,σ satisfy Condition C and $\{M_n\}$ satisfies Condition M2, and u be the solution of (1.1). Suppose $M_n\Rightarrow W$ in the Skorohod topology on $D_{S'(R)}[0,T]$, then $u_\epsilon^n\Rightarrow u$ in the Skorohod topology on $D_{C[0,1]}[0,T]$ as $n\to\infty$ and then $\epsilon\to 0$.*

PROOF. Note that u_ϵ^n solves the SPDE, (3.1) in the following sense : for any $t\in R_+,\phi\in C_c^\infty([0,1]\times R_+)$

$$\begin{aligned}&\langle u_\epsilon^n(x,t),\phi_t\rangle+\int_0^t\langle A\phi_s,u_\epsilon^n(x,s)\rangle ds+\int_0^t\langle f(u_\epsilon^n(x,s)),\phi_s\rangle ds\\ &=\langle u_\epsilon^n(0),\phi_0\rangle+\int_0^t\int_0^1\sigma(u_\epsilon^n(x,s))\phi(x,s)M^n(dx,ds)\\ &+\frac{1}{\epsilon}\int_0^t\langle(u_\epsilon^n(x,s))^-,\phi_s\rangle ds\text{ a.s.,}\end{aligned}\tag{3.8}$$

where $\phi_s(\cdot)\equiv\phi(\cdot,s)$. All the terms on the left-hand side of (3.8) converge in distribution as $n\to\infty$. Also applying Theorem 2.1 [2],

$$\begin{aligned}&\int_0^t\int_0^1\sigma(u_\epsilon^n(x,s))\phi_s(x)M^n(dx,ds)\\ \Rightarrow&\int_0^t\int_0^1\sigma(u^\epsilon(x,s))\phi_s(x)W(dx,ds)\end{aligned}$$

as $n \to \infty$. We denote $\eta_\epsilon^n(dx,dt) = \frac{1}{\epsilon}(u_\epsilon^n(x,t))^- dxdt$ on $[0,1] \times R_+$. By the above theorem, we can see that there exists a positive distribution η_ϵ, $\eta_\epsilon(dx,dt) = \frac{1}{\epsilon}(u_\epsilon(x,t))^- dxdt$ such that

$$\frac{1}{\epsilon}\int_0^t \phi_s(x)\eta_\epsilon^n(dx,dt) \Rightarrow \frac{1}{\epsilon}\int_0^t \phi_s(x)\eta_\epsilon(dx,dt),$$

as $n \to \infty$ since η_ϵ^n is a positive distribution.

Since the unique existence of limit of u_ϵ as $\epsilon \to 0$ is already known in [4], set $u = \lim_{\epsilon \to 0} u^\epsilon = \limsup_{\epsilon \to 0} u^\epsilon$ a.s.. Then all the terms of (3.8) except the last one converge in distribution as $\epsilon \to 0$. Hence we deduce that η_ϵ converges to a positive distribution, say η on $[0,1] \times R_+$ and $\frac{1}{\epsilon}\int_0^t \langle u_\epsilon^n(s)^-, \phi_s\rangle ds \Rightarrow \int_0^t \phi_s(x)\eta_s(dx,dt)$ as $\epsilon \to 0$. Therefore for all $t \geq 0$ and $\phi \in C_c^\infty([0,1] \times R_+)$, $u(t)$ satisfies the following;

(3.9)

$$\begin{aligned}
&\langle u(x,t), \phi(x,t)\rangle + \int_0^t \langle A\phi(x,s), u(x,s)\rangle ds + \int_0^t \langle f(u(x,s)), \phi(x,s)\rangle ds \\
&= \langle u_0, \phi_0\rangle + \int_0^t \int_0^1 \sigma(u(x,s))\phi(x,s)W(dx,ds) \\
&\quad + \int_0^t \int_0^1 \phi(x,s)\eta(dx,ds)
\end{aligned}$$

□

THEOREM 3.3. *Let $u^n, n = 1,2,\ldots$ be the solution of* (1.2). *Then for any $T > 0$,*

$$u^n \Rightarrow u \quad \text{in } D_{C[0,1]}[0,T],$$

where u is the solution of (3.9).

PROOF. First, we want to show for any n and $0 \leq t \leq T$

$$\sup_n E[\|u_\epsilon^n(\cdot,t) - u^n(\cdot,t)\|_\infty] \to 0,$$

as $\epsilon \to 0$. Let $\bar{v}_\epsilon^n$ be the solution of

$$\begin{aligned}
&\frac{\partial \bar{v}_\epsilon^n(x,t)}{\partial t} + A\bar{v}_\epsilon^n(x,t) = \sigma(u_\epsilon^n(x,t))M_n(dx,dt) \\
&\bar{v}_\epsilon^n(\cdot,0) = u_0; \bar{v}_\epsilon^n(0,t) = \bar{v}_\epsilon^n(1,t) = 0,
\end{aligned} \tag{3.10}$$

and $\bar{v}^n$ be the solution of the following:

$$\frac{\partial \bar{v}^n(x,t)}{\partial t} + A\bar{v}^n(x,t) = \sigma(u^n(x,t))M_n(dx,dt)$$

$$\bar{v}^n(\cdot,0) = u_0; \bar{v}^n(0,t) = \bar{v}^n(1,t) = 0. \tag{3.11}$$

Let $z_\epsilon^n = u_\epsilon^n - \bar{v}_\epsilon^n$, z_ϵ^n be the solution of

$$\frac{\partial z_\epsilon^n(x,t)}{\partial t} + Az_\epsilon^n + f(z_\epsilon^n + \bar{v}_\epsilon^n) = \frac{1}{\epsilon}(z_\epsilon^n(x,t) + \bar{v}_\epsilon^n(x,t))^-$$

$$z_\epsilon^n(\cdot,0) = 0; z_\epsilon^n(0,t) = z_\epsilon^n(1,t) = 0, \tag{3.12}$$

and $\bar{z}_\epsilon^n$ be the solution of the following:

$$\frac{\partial \bar{z}_\epsilon^n(x,t)}{\partial t} + A\bar{z}_\epsilon^n(x,t) + f_\epsilon(\bar{z}_t^{n,\epsilon} + \bar{v}_t^n) = 0 \tag{3.13}$$

$$\bar{z}_\epsilon^n(\cdot,0) = 0; \bar{z}_\epsilon^n(0,t) = \bar{z}_\epsilon^n(1,t) = 0,$$

where $f_\epsilon(r) = f(r) - \frac{1}{\epsilon}r^-$.

Since $\|\bar{v}_\epsilon^n(\cdot,t) - \bar{v}^n(\cdot,t)\|_\infty \geq \|\bar{z}_\epsilon^n(\cdot,t) - z_\epsilon^n(\cdot,t)\|_\infty$ by the same argument of (13) in [8], we have

(3.14)

$$\begin{aligned}
&\|u_\epsilon^n(\cdot,t) - u^n(\cdot,t)\|_\infty \\
&\leq \|\bar{v}_\epsilon^n(\cdot,t) - \bar{v}^n(\cdot,t)\|_\infty + \|\bar{z}_\epsilon^n(\cdot,t) - \bar{z}^n(\cdot,t)\|_\infty + \|\bar{z}_\epsilon^n(\cdot,t) - z_\epsilon^n(\cdot,t)\|_\infty \\
&\leq 2\|\bar{v}_\epsilon^n(\cdot,t) - \bar{v}^n(\cdot,t)\|_\infty + \|\bar{z}_\epsilon^n(\cdot,t) - \bar{z}^n(\cdot,t)\|_\infty
\end{aligned}$$

Observing (3.10)-(3.13), we see that

$$\bar{u}_\epsilon^n = \bar{z}_\epsilon^n + \bar{v}^n$$

solves

$$\begin{aligned}
&\frac{\partial \bar{u}_\epsilon^n(x,t)}{\partial t} + A\bar{u}_\epsilon^n(x,t) + f(\bar{u}_\epsilon^n(x,t)) \\
&= \sigma(u^n(x,t))M_n(dx,dt) + \frac{1}{\epsilon}(\bar{u}_\epsilon^n(x,t))^-.
\end{aligned} \tag{3.15}$$

It was shown in [8] that the solution of (3.15) converges as ϵ tends to 0 to a (continuous) function $\bar{u}^n$ on $[0,1] \times [0,T]$. So $\bar{z}_\epsilon^n$ converges uniformly to a continuous function $\bar{z}^n$. Note that $z_\epsilon^n = u_\epsilon^n - \bar{v}_\epsilon^n$ and $\|\bar{z}_\epsilon^n - z_\epsilon^n\|_\infty \leq$

$\|\bar{v}^n - \bar{v}_\epsilon^n\|_\infty$. By Lemma 6.2 of [4], we have $E[\|\bar{v}^n - \bar{v}_\epsilon^n\|_\infty] \to 0$. Hence (3.14) $\to 0$ a.s. as $\epsilon \to 0$.

The next step was also used to prove a limit,(4.6) in [3]. Since u_ϵ is continuous, whenever $t_n \in [0,1], n = 1, 2 \ldots, \lim_{n\to\infty} t_n = t$

(3.16)

$$\begin{aligned} &\|u_\epsilon^n(\cdot, t_n) - u^n(\cdot, t)\|_\infty \\ &\le \|u_\epsilon^n(\cdot, t_n) - u_\epsilon(\cdot, t)\|_\infty + \|u_\epsilon(\cdot, t) - u_\epsilon^n(\cdot, t)\|_\infty + \|u_\epsilon^n(\cdot, t) - u^n(\cdot, t)\|_\infty \\ &\to 0, \text{ a.s.} \end{aligned}$$

as $n \to \infty$ and $\epsilon \to 0$. Since for each ϵ, $u_\epsilon^n \Rightarrow u_\epsilon$ as $n \to \infty$ and $u_\epsilon \Rightarrow u$ as $\epsilon \to 0$ according to Theorem 4.1 [4], by Proposition 3.6.5. [5] and Theorem 1.4.2 [1] (3.16) implies that

$$u^n \Rightarrow u \quad \text{in the Skorohod topology on } D_{C[0,1]}[0,T].$$

□

References

[1] P. Billingsley, *Convergence of probability measures*, Wiley, 1968.

[2] N. Cho, *Weak convergence of stochastic integrals driven by martingale measure*, Stochastic Process Appl **59** (1995), 55–79.

[3] ______, *On the intersection local time of super Brownian motion*, Stochastics and Stochastics reports **67** (1999), 191–205.

[4] C. Donald-Martin and E. Pardoux, *White noise driven SPDEs with reflection*, Probab. Theory Relat. Fields **95** (1993), 1–24.

[5] E. Ethier and T. Kurtz, *Markov processes: Characterization and convergence*, Wiley, 1986.

[6] N. V. Krylov, *SPDE in L_q spaces*, Electronic J. of Probab. **5** (2000), 1–29.

[7] C. Mueller, *Long time existence for the heat equation with a noise*, Probab. Theory Relat. Fields **90** (1991), 505–518.

[8] D. Nualart and E. Pardoux, *White noise driven quasilinear SPDEs with reflection*, Probab. Theory Relat. Fields **93** (1992), 77–89.

[9] E. Pardoux, *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics **3** (1979), 127–167.

[10] J. Walsh, *An introduction to stochastic partial differential equations*, Springer Lect. notes in mathematics **1180** (1986), 265–439.

Department of general science and
Department of computer science and information
Hansung University
Seoul 136-792, Korea
E-mail: ncho@hansung.ac.kr
jmkwon@hansung.ac.kr