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IMPROVED UPPER BOUNDS OF PROBABILITY

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ABSTRACT. Let $A_1, A_2, \cdots, A_n$ be a sequence of events on a given probability space. Let m_n be the number of those $A_j's$ which occur. Upper bounds of $P(m_n \geq 1)$ are obtained by means of probability of consecutive terms which reduce the number of terms in binomial moments $S_{2,n}, S_{3,n}$ and $S_{4,n}$.

1. Introduction

Several problems of probability theory lead to the need of estimating the distribution of the number $m_n = m_n(A)$ of occurrences in a sequence $A_1, A_2, \cdots, A_n$ of events. When the estimation of this distribution is in terms of linear combinations of the binomial moments of $m_n(A)$, we speak of Bonferroni-type inequality. That is, let

$$S_{k,n} = E\left[\binom{m_n}{k}\right], \qquad 0 \leq k \leq n. \tag{1.1}$$

Then, with constants $c_{k,n}(r)$ and $d_{k,n}(r)$, $0 \leq k \leq n$, $r \leq 0$, the inequalities

$$\sum_{k=0}^{n} d_{k,n}(r) S_{k,n} \leq P(m_n(A) = r) \leq \sum_{k=0}^{n} c_{k,n}(r) S_{k,n} \tag{1.2}$$

are called Bonferroni-type inequality. Here the term constant means that $c_{k,n}(r)$ and $d_{k,n}(r)$ do not depend on the underlying probability space and nor on the choice of the events $A_1, A_2, \cdots, A_n$.

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By turning to indicator variables we immediately get that, for $1 \leq k \leq n$,

$$S_{k,n} = \sum P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}), \tag{1.3}$$

where the summation is over all subscripts satisfying $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Kounias [4] has proved that

$$P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) - \max_{j} \sum_{i \neq j} P(A_i \cap A_j) \tag{1.4}$$

which improves on the simple Bonferroni upper bound of $\sum P(A_i)$. Margolin and Maurer [7] generalize this result by using more than just $\sum P(A_i)$ from the classical estimates. Hunter [3], whose result is re-obtained in the paper of Worsley [9], presents an improved upper bound which is constructed by edges on a graph.

Lee [6] has proved that

$$P(m_n \geq 1) \leq S_{1,n} - \sum_{i<j \leq i+2} P(A_i \cap A_j) + \sum_{i=1}^{n-2} P(A_i \cap A_{i+1} \cap A_{i+2}). \tag{1.5}$$

Taking averages which over $i = 1, 2, \cdots, n$ of (1.5), we get the following Bonferroni-type inequality.

$$P(m_n \geq 1) \leq S_1 - \frac{(2n-3)}{\binom{n}{2}} S_2 + \frac{(n-2)}{\binom{n}{3}} S_3.$$

This inequality is known that it is the best possible upper bound in terms of S_1, S_2 and S_3 (see Kwerel [5]).

The classical lower bound of degree four is

$$S_{1,n} - S_{2,n} + S_{3,n} - S_{4,n} \leq P(m_n \geq 1)$$

and our idea is to reduce the number of terms in $S_{2,n}$, $S_{3,n}$ and $S_{4,n}$ in order to get an upper bound. For a related idea, see the graph-dependent models of Renyi [8] and Galambos [2].

In this direction, we obtain the inequalities of the theorems that follow.

2. The results

The upper bounds are improved by the following results.

THEOREM 1. *For positive integers* $n \geq 4$,

$$
\begin{aligned}
&P(m_n \geq 1) \\
&\leq S_{1,n} - \sum_{i<j\leq i+3} P(A_i \cap A_j) + \sum_{i=1}^{n-2} P(A_i \cap A_{i+1} \cap A_{i+2}) \\
&\quad + \sum_{i=1}^{n-3}[P(A_i \cap A_{i+1} \cap A_{i+3}) + P(A_i \cap A_{i+2} \cap A_{i+3})] \\
&\quad - \sum_{i=1}^{n-3} P(A_i \cap A_{i+1} \cap A_{i+2} \cap A_{i+3}).
\end{aligned}
\tag{2.1}
$$

Taking the averages of the above upper bound over $i = 1, 2, \cdots, n$, we get Theorem 2.

THEOREM 2. *For positive integers* $n \geq 4$,

$$
P(m_n \geq 1) \leq S_1 - \frac{3(n-2)}{\binom{n}{2}} S_2 + \frac{3n-8}{\binom{n}{3}} S_3 - \frac{n-3}{\binom{n}{4}} S_4. \tag{2.2}
$$

3. Proofs

PROOF OF THEOREM 1. We use the method of indicators. That is, let $I(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$ be 1 if $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ occurs or 0 otherwise.

Then $I(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = I(A_{i_1})I(A_{i_2}) \cdots I(A_{i_k})$ and $E[I(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})] = P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$. Furthermore, the indicator variable $I(m_n \geq 1)$ is 1 if $m_n \geq 1$ and 0 if $m_n = 0$. Note also that $\sum_{i=1}^{n} I(A_i) = m_n$ and $S_{1,n} = E[m_n]$.

We thus have to prove

$$
m_n - \sum_{i<j\leq i+3} I(A_i)I(A_j) + \sum_{i=1}^{n-2} I(A_i)I(A_{i+1})I(A_{i+2})
$$

$$(3.1)\quad \begin{aligned} &+\sum_{i=1}^{n-3}[I(A_i)I(A_{i+1})I(A_{i+3})+I(A_i)I(A_{i+2})I(A_{i+3})] \\ &-\sum_{i=1}^{n-3}I(A_i)I(A_{i+1})I(A_{i+2})I(A_{i+3}) \geq 1 \end{aligned}$$

if $m_n \geq 1$ and the left hand side of (3.1) is greater than zero or equal to zero if $m_n = 0$.

The latter case is evident, having zero on both sides. Also, if $m_n = 1$, both side of (3.1) equal 1 and if $m_n = 2$, left hand side of (3.1) is $2 - \binom{0}{1} \geq 1$.

Hence, for the sequel we may assume $m_n \geq 3$.

Next, we place the events $A_1, A_2, \cdots, A_n$ at every sample point into blocks which consist of events of the kind $A_{j+1} \cap \cdots \cap A_{j+k_j}$,which is a full block if neither A_j nor A_{j+k_j+1} occurs. Assume that in this way, at a given sample point, we have t blocks. We distinguish six cases.

case (i): For all j, $k_j \geq 3$; that is, every full block has at least three events. We can express

$$\sum_{i<j\leq i+3} I(A_i)I(A_j),$$

$$\sum_{i=1}^{n-2} I(A_i)I(A_{i+1})I(A_{i+2}) + \sum_{i=1}^{n-3}[I(A_i)I(A_{i+1})I(A_{i+3}) + I(A_i)I(A_{i+2})I(A_{i+3})]$$

and

$$\sum_{i=1}^{n-3} I(A_i)I(A_{i+1})I(A_{i+2})I(A_{i+3})$$

by means of blocks ; that is, if the t blocks have length $k_j, 1 \leq j \leq t$, then the above sums equal

$$(3.2)\quad \sum_{j=1}^{t} 3(k_j-2) + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 3(t-1) \end{pmatrix},\ \sum_{j=1}^{t}[3(k_j-3)+1] + \begin{pmatrix} 0 \\ 2 \\ \vdots \\ 2(t-1) \end{pmatrix}$$

and $\sum_{j=1}^{t}(k_j-3)$, respectively, where

$$\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 3(t-1) \end{pmatrix}$$

denotes the number $\sum_{j=1}^{t-1} L_d^j, L_d^j$ being 3 if $d = 2$ and 1 if $d = 3$ and 0 if $d \geq 4$ and

$$\begin{pmatrix} 0 \\ 2 \\ \vdots \\ 2(t-1) \end{pmatrix}$$

denotes the number $\sum_{j=1}^{t-1} L_d^j, L_d^j$ being 2 if $d = 2$ and 0 if $d \geq 3$ and d is the difference between last number of j-th block and first number of next one. Since $\sum_{j=1}^{t} k_j = m_n$, by (3.2), the left hand side of (3.1) becomes

$$t - \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ t-1 \end{pmatrix} \geq 1.$$

Hence, we get (3.1).

case (ii) : For all j, $k_j = 2$; that is, every full block has only two events. We have

$$\sum_{i<j\leq i+3} I(A_i)I(A_j) = \sum_{j=1}^{t} 1 + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 3(t-1) \end{pmatrix}, \tag{3.3}$$

$$\begin{aligned} &\sum_{i=1}^{n-2} I(A_i)I(A_{i+1})I(A_{i+2}) \\ &+ \sum_{i=1}^{n-3} [I(A_i)I(A_{i+1})I(A_{i+3}) + I(A_i)I(A_{i+2})I(A_{i+3})] \\ &= \sum_{j=1}^{t} 0 + \begin{pmatrix} 0 \\ 2 \\ \vdots \\ 2(t-1) \end{pmatrix} \end{aligned} \tag{3.4}$$

and

$$\sum_{i=1}^{n-3} I(A_i)I(A_{i+1})I(A_{i+2})I(A_{i+3}) = 0. \tag{3.5}$$

Since $\sum_{j=1}^{t} 2 = 2t = m_n$, in view of (3.3), (3.4) and (3.5), the left hand side of (3.1) is

$$t - \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ t-1 \end{pmatrix} \geq 1.$$

Once again, (3.1) obtains.

case (iii) : For all j, $k_j = 1$; that is, every full block has only one event. We now have

$$\sum_{i<j\leq i+3} I(A_i)I(A_j) = \sum_{j=1}^{t} 0 + \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ t-1 \end{pmatrix}, \tag{3.6}$$

$$\begin{aligned} &\sum_{i=1}^{n-2} I(A_i)I(A_{i+1})I(A_{i+2}) \\ &+ \sum_{i=1}^{n-3} [I(A_i)I(A_{i+1})I(A_{i+3}) + I(A_i)I(A_{i+2})I(A_{i+3})] = 0 \end{aligned} \tag{3.7}$$

and

$$\sum_{i=1}^{n-3} I(A_i)I(A_{i+1})I(A_{i+2})I(A_{i+3}) = 0. \tag{3.8}$$

Since $\sum_{j=1}^{t} 1 = t = m_n$ in view of (3.6), (3.7) and (3.8), the left hand side of (3.1) is

$$t - \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ t-1 \end{pmatrix} \geq 1.$$

Once again, (3.1) obtains.

case (iv) : There exist some i, j and r with $k_i = 1, k_j = 2$ and $k_r \geq 3$; that is, there are several blocks which have only one, two and at least three events at the same time.

Assume that we have t_1, t_2, t_3 blocks where they consist t_1 blocks with $k_r \geq 3$, t_2 blocks with $k_j = 2$, t_3 blocks with $k_i = 1$. We now have

(3.9)
$$\sum_{i<j\leq i+3} I(A_i)I(A_j) = \sum_{r=1}^{t_1} 3(k_r - 2) + \sum_{j=1}^{t_2} 1 + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 3t_1 + 3t_2 + t_3 - 1 \end{pmatrix},$$

(3.10)
$$\begin{aligned} &\sum_{i=1}^{n-2} I(A_i)I(A_{i+1})I(A_{i+2}) \\ &+ \sum_{i=1}^{n-3} [I(A_i)I(A_{i+1})I(A_{i+3}) + I(A_i)I(A_{i+2})I(A_{i+3})] \\ &= \sum_{r=1}^{t_1} [3(k_r - 3) + 1] + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 2(t_1 + t_2) \end{pmatrix} \quad \text{and} \end{aligned}$$

(3.11)
$$\sum_{i=1}^{n-3} I(A_i)I(A_{i+1})I(A_{i+2})I(A_{i+3}) = \sum_{r=1}^{t_1} (k_r - 3), \text{respectively.}$$

Since $\sum_{r=1}^{t_1} k_r + \sum_{j=1}^{t_2} 2 + \sum_{i=1}^{t_3} 1 = m_n$, in view of (3.9), (3.10) and (3.11), the left hand side of (3.1) is

$$t - \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ t-1 \end{pmatrix} \geq 1.$$

Once again, (3.1) obtains.

case (v) : There exist some i and j with $k_i = 2$, $k_j \geq 3$; that is, every full block has two and at least three events. Assume that we has t_1, t_2

blocks where they consist t_1 blocks with $k_j \geq 3$, t_2 blocks with $k_i = 2$. We now have
(3.12)

$$\sum_{i<j\leq i+3} I(A_i)I(A_j) = \sum_{j=1}^{t_1} 3(k_j-2) + \sum_{i=1}^{t_2} 1 + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 3(t_1+t_2-1) \end{pmatrix},$$

$$\begin{aligned} &\sum_{i=1}^{n-2} I(A_i)I(A_{i+1})I(A_{i+2}) \\ &\quad + \sum_{i=1}^{n-3}[I(A_i)I(A_{i+1})I(A_{i+3}) + I(A_i)I(A_{i+2})I(A_{i+3})] \\ &= \sum_{j=1}^{t_1}[3(k_j-3)+1] + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 2(t_1+t_2-1) \end{pmatrix} \end{aligned} \tag{3.13}$$

and

$$\sum_{i=1}^{n-3} I(A_i)I(A_{i+1})I(A_{i+2})I(A_{i+3}) = \sum_{j=1}^{t_1}(k_j-3). \tag{3.14}$$

Since $\sum_{j=1}^{t_1} k_j + \sum_{i=1}^{t_2} 2 = m_n$, in view of (3.12), (3.13) and (3.14), the left hand side of (3.1) is

$$t - \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ t-1 \end{pmatrix} \geq 1.$$

Once again, (3.1) obtains.

case (vi) : There exist some i and j with $k_i = 1, k_j \geq 3$ or $k_i = 1, k_j = 2$. In the same manner as in (v), we get (3.1).

This completes the proof. □

PROOF OF THEOREM 2. Let $A_1, A_2, \cdots, A_n$ be a sequence of events on a given probability space, and let $x = m_n$ be the number of those $A_j's$ which occur.

By the binomial moments of (1.1), the right hand side of (2.2) becomes

$$\binom{x}{1} - \frac{3(n-2)}{\binom{n}{2}}\binom{x}{2} + \frac{3n-8}{\binom{n}{3}}\binom{x}{3} - \frac{n-3}{\binom{n}{4}}\binom{x}{4}. \tag{3.15}$$

We thus have to prove that

$$f(x) = \binom{x}{1} - \frac{3(n-2)}{\binom{n}{2}}\binom{x}{2} + \frac{3n-8}{\binom{n}{3}}\binom{x}{3} - \frac{n-3}{\binom{n}{4}}\binom{x}{4} \geq 1 \tag{3.16}$$

if $x \geq 1$ and (3.15) is greater than zero or equal to zero if $x = 0$.

The latter case is evident, having zero on both sides. Also, if $x = 1$, both side of (3.16) equal 1 and if $x = 2$, left hand side of (3.16) is $2 - \frac{6(n-2)}{n(n-1)} \geq 1$ for $n \geq 2$ and if $x = 3$, left hand side of (3.16) is $3 - \frac{18(n-2)}{n(n-1)} + \frac{6(3n-8)}{n(n-1)(n-2)} \geq 1$ for $n \geq 3$. Hence, for the sequel we may assume $x \geq 4$.

Let $g(x) = f(x) - 1$. We must prove that $g(x) \geq 0$ for any integer values x, $4 \leq x \leq n$.

Then

$$\begin{aligned} g(x) &= \binom{x}{1} - \frac{3(n-2)}{\binom{n}{2}}\binom{x}{2} + \frac{3n-8}{\binom{n}{3}}\binom{x}{3} - \frac{n-3}{\binom{n}{4}}\binom{x}{4} - 1 \\ &= -(x-1)(x-(n-2))(x-(n-1))(x-n). \end{aligned}$$

Now, for any positive integers the polynomial $g(x)$ obtains its minimum value 0 at $x = 1, n-2, n-1, n$.

Hence, for any integers $x \geq 4, g(x) \geq 0$.

This completes the proof. □

4. Numerical examples

EXAMPLE 4-1. Let X_j be the time to failure of the j-th component of a piece of equipment. Assume that each X_j is a unit exponential variate; that is, for each j,

$$P(X_j < x) = 1 - e^{-x}, (x > 0).$$

Consider a group of five components, X_1, X_2, X_3, X_4, X_5. We assume that we just know the following information.

(a) X_i is dependent on X_{i+1} ,X_{i+2} and X_{i+3} ; that is, X_1 and X_2 are dependent, so are X_1 and X_3, X_1 and X_4, X_2 and X_3, X_2 and X_4, X_2 and X_5, X_3 and X_4, X_3 and X_5, finally, X_4 and X_5.

(b) X_i,X_{i+1} and X_{i+2} are dependent on each other and X_i,X_{i+1} and X_{i+3} are dependent on each other and X_i,X_{i+2} and X_{i+3} are dependent on each other. ; that is, X_1,X_2 and X_3 are dependent, so are X_1,X_2 and X_4, X_1,X_2 and X_4, X_1,X_3 and X_4, X_2,X_3 and X_4, X_2, X_3 and X_5, X_2,X_4 and X_5, finally X_3,X_4 and X_5.

(c) X_i, X_{i+1}, X_{i+2} and X_{i+3} are dependent on each other: that is, X_1, X_2, X_3 and X_4 is dependent, so are X_2, X_3, X_4 and X_5.

No other information is available on the interdependence of the components. We also specify the multivariate distributions of the X_j.

For simplicity, let the multivariate distributions for all dependent components specified in (a), (b) and (c) be the same. Let

$$\begin{aligned}
&P(X_1 < x, X_2 < y) \\
&= P(X_1 < x, X_3 < y) = P(X_1 < x, X_4 < y) = P(X_2 < x, X_3 < y) \\
&= P(X_2 < x, X_4 < y) = P(X_2 < x, X_5 < y) = P(X_3 < x, X_4 < y) \\
&= P(X_3 < x, X_5 < y) = P(X_4 < x, X_5 < y) \\
&= (1 - e^{-x})(1 - e^{-y})(1 - \frac{1}{2}e^{-x-y}),
\end{aligned}$$

$$\begin{aligned}
&P(X_1 < x, X_2 < y, X_3 < z) \\
&= P(X_2 < x, X_3 < y, X_4 < z) = P(X_3 < x, X_4 < y, X_5 < z) \\
&= P(X_1 < x, X_2 < y, X_4 < z) = P(X_2 < x, X_3 < y, X_5 < z) \\
&= P(X_1 < x, X_3 < y, X_4 < z) = P(X_2 < x, X_4 < y, X_5 < z) \\
&= (1 - e^{-x})(1 - e^{-y})(1 - e^{-z})(1 - \frac{1}{3}e^{-x-y-z}),
\end{aligned}$$

$$\begin{aligned}
&P(X_1 < x, X_2 < y, X_3 < z, X_4 < u) \\
&= P(X_2 < x, X_3 < y, X_4 < z, X_5 < u) \\
&= (1 - e^{-x})(1 - e^{-y})(1 - e^{-z})(1 - e^{-u})(1 - \frac{1}{4}e^{-x-y-z-u}).
\end{aligned}$$

No further assumption is made.

We would like to estimate $P(W_5 \geq x)$ where $W_5 = \min(X_1, X_2, X_3, X_4, X_5)$. We choose the events $A_j = (X_j < x)$ and then $(m_5 = 0) =$

$(W_5 \geq x)$. For a numerical calculation, let us choose $x = 0.1$. We then estimate $P(W_5 \geq 0.1)$. We have

$$S_{1,5} = \sum_{i=1}^{5} P(A_i) = 5(1 - e^{-0.1}) = 0.4758,$$

$$\sum_{i<j\leq i+3} P(A_i \cap A_j) = 9[(1 - e^{-0.1})^2(1 - \frac{1}{2}e^{-0.2})] = 0.0481,$$

$$\sum_{i=1}^{3} P(A_i \cap A_{i+1} \cap A_{i+2}) + \sum_{i=1}^{2}[P(A_i \cap A_{i+1} \cap A_{i+3}) + P(A_i \cap A_{i+2} \cap A_{i+3})] = 7[(1 - e^{-0.1})^3(1 - \frac{1}{3}e^{-0.3})] = 0.0045$$

and

$$\sum_{i=1}^{2} P(A_i \cap A_{i+1} \cap A_{i+2} \cap A_{i+3}) = 2[(1 - e^{-0.1})^4(1 - \frac{1}{4}e^{-0.4})] = 0.0001.$$

Theorem 1 now gives $P(m_n \geq 1) \leq 0.4321$.

EXAMPLE 4-2. Consider a numerical example 2 in the paper of Bukszar and Prekopa [1]. Let A_1, A_2, A_3, A_4, A_5 be events with the following probabilities; $P_1 = P_2 = P_3 = P_4 = P_5 = 0.38$ $P_{1,2} = 0.15$, $P_{1,3} = 0.13$, $P_{1,4} = 0.14$, $P_{1,5} = 0.12$, $P_{2,3} = 0.20$, $P_{2,4} = 0.21$, $P_{2,5} = 0.18$, $P_{3,4} = 0.19$, $P_{3,5} = 0.16$, $P_{4,5} = 0.17$, $P_{1,2,3} = P_{1,2,4} = P_{1,2,5} = P_{1,3,4} = P_{1,3,5} = P_{1,4,5} = P_{2,3,4} = P_{2,3,5} = P_{2,4,5} = P_{3,4,5} = 0.07$.

The cherry tree upper bound by Bukszar and Prekopa [1] is following

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) \leq S_1 - \sum_{i,j \in \xi} P(A_i \cap A_j). \tag{4.1}$$

This yields $P(A_1 \cup A_2 \cup \cdots \cup A_5) \leq 0.87$.

Now we have

$$S_1 = \sum_{i=1}^{5} P(A_i) = 1.9, \qquad \sum_{i<j\leq i+3} P(A_i \cap A_j) = 1.53,$$

$$\sum_{i=1}^{3} P(A_i \cap A_{i+1} \cap A_{i+2}) + \sum_{i=1}^{2}[P(A_i \cap A_{i+1} \cap A_{i+3}) + P(A_i \cap A_{i+2} \cap A_{i+3})] = 0.49,$$

and assume that $\sum_{i=1}^{2} P(A_i \cap A_{i+1} \cap A_{i+2} \cap A_{i+3}) = 0.07$.
Then theorem 1 gives $P(m_n \geq 1) \leq 0.79$.

Upper bound for $P(\cup_{i=1}^{5} A_i)$

inequality	example 4-1	example 4-2
(1.4)	0.4544	1.16
(1.5)	0.4403	0.96
(4.1)		0.87
(2.1)	0.4321	0.79

In the above table, we see that (2.1) is the best upper bound.

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