Commun. Korean Math. Soc. **18** (2003), No. 4, pp. 737–743

EXISTENCE RESULTS FOR VECTOR NONLINEAR INEQUALITIES

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ABSTRACT. The purpose of this paper is to consider some existence results for vector nonlinear inequalities without any monotonicity assumption. As consequences of our main result, we give some existence results for vector equilibrium problem, vector variational-like inequality problem and vector variational inequality problems as special cases.

1. Introduction and preliminaries

A vector equilibrium problem has been studied in [1]-[6], [8, 9, 14, 16], which includes vector variational inequalities, vector optimization problems, vector saddle point problems and Nash equilibrium problem for vector-valued functions as special cases; see, for example [11]-[13], [15, 18] and references therein.

In the analysis of solutions to vector equilibrium problem, it is commonly assumed that the defined mapping is monotone or generalized monotone [5, 8, 10, 13].

Recently, Ansari *et al.* [3] established an existence result for scalar nonlinear inequalities for general mappings without the monotonicity assumption. And they derived some existence results for the scalar case of equilibrium problem, variational inequality and variational-like inequality without the monotonicity assumption.

The purpose of this paper is to consider some existence results for vector nonlinear inequalities without the monotonicity assumption. As

Received October 28, 2002.
2000 Mathematics Subject Classification: 49J40.
Key words and phrases: vector nonlinear inequality, vector variational inequality, vector variational-like inequality, vector equilibrium problem, Tarafdar's fixed point theorem.
This work was supported by Korea Research Foundation Grant (KRF-2002-DP0013).

consequences of our main result, we give some existence results for vector equilibrium problems, vector variational-like inequality problems and vector variational inequality problems as special cases.

Let X be a Hausdorff topological vector space and Y a topological vector space. Let K be a nonempty convex subset of X, $L(X,Y)$ the space of all linear continuous operators from X to Y and $\{C(x) : x \in K\}$ a family of closed convex cones in Y. Assume that f is in $L(X,Y)$ and $\varphi : K \times K \to Y$ is a mapping.

Consider the existence of solutions to the following form of vector nonlinear inequality **(VNI)**:

Find $x_0 \in K$ such that

$$\varphi(x_0, y) - \langle f, y - x_0 \rangle \notin -intC(x_0) \text{ for all } y \in K,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $L(X,Y)$ and X, and $intC(x)$ the interior of $C(x)$.

The following theorem will play a crucial role in proving the existence result of solutions for vector nonlinear inequality.

THEOREM 1 [17]. *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $T : K \to 2^K$ be a set-valued mapping such that*

(a) *for each $x \in K$, $T(x)$ is a nonempty convex subset of K;*
(b) *for each $y \in K$, $T^{-1}(y) = \{x \in K : y \in T(x)\}$ contains an open set O_y which may be empty;*
(c) $\bigcup_{y \in K} O_y = K$;
(d) *there exists a nonempty set D_0 contained in a compact convex subset D_1 of K such that $D = \bigcap_{x \in D_0} {O_x}^c$ is either empty or compact, where ${O_x}^c$ is the complement of O_x in K.*

Then there exists $x \in K$ such that $x \in T(x)$.

2. Main results

We now prove the main result of this paper.

THEOREM 2. *Let X be a Hausdorff topological vector space and Y a topological vector space. Let K be a nonempty convex subset of X, $L(X,Y)$ the space of all linear continuous operators from X to Y and $\{C(x) : x \in K\}$ a family of closed convex cones in Y. Let a set-valued* *mapping* $W : K \to 2^Y$ *be defined by* $W(x) = Y \setminus \{-intC(x)\}$ *such that* $Gr(W) := \{(x,y) \in X \times Y : x \in X, y \in W(x)\}$ *is closed in* $X \times Y$. *Assume that* f *is in* $L(X,Y)$, $\varphi : K \times K \to Y$ *is a mapping such that* $x \mapsto \varphi(x,\cdot)$ *is continuous,* $x \mapsto \varphi(\cdot,x)$ *is convex and* $\varphi(x,x) = 0$ *for all* $x \in K$ *and the following coercive condition is satisfied:*

There exists a compact convex subset D_1 *of* K *such that, for each* $x \in K \setminus D_1$, *there exists a* $y \in D_1$ *with*

$$\varphi(x,y) - \langle f, y - x \rangle \in -intC(x).$$

Then **(VNI)** *has a solution in* K.

PROOF. We define $A : K \to 2^K$ by, for each $y \in K$,

$$A(y) := \{x \in K : \varphi(x,y) - \langle f, y - x \rangle \notin -intC(x)\}.$$

Then the solution set of (VNI) is $S = \bigcap_{y \in K} A(y)$. Now we show that, for each $y \in K$, $A(y)$ is closed. In fact, let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in $A(y)$ such that $x_\lambda \to x \in K$. Since $x_\lambda \in A(y)$, we have

$$\varphi(x_\lambda, y) - \langle f, y - x_\lambda \rangle \in W(x_\lambda).$$

For all $f \in L(X,Y)$, $\langle f, y - x_\lambda \rangle \to \langle f, y - x \rangle$ and, since $x \mapsto \varphi(x,\cdot)$ is continuous and W has a closed graph, we get

$$\varphi(x,y) - \langle f, y - x \rangle \notin -intC(x).$$

Hence $x \in A(y)$, $A(y)$ is closed.

We will prove that the solution set S is nonempty. Assume that $S = \emptyset$. We define a set-valued mapping $B : K \to 2^K$ by, for each $x \in K$,

$$\begin{aligned} B(x) : &= \{y \in K : x \notin A(y)\} \\ &= \{y \in K : \varphi(x,y) - \langle f, y - x \rangle \in -intC(x)\}. \end{aligned}$$

Then $B(x)$ is a nonempty convex. Indeed, let $y_1, y_2 \in B(x)$ and $t \in (0,1)$, then

$$\varphi(x,y_1) - \langle f, y_1 - x \rangle \in -int(x) \text{ and } \varphi(x,y_2) - \langle f, y_2 - x \rangle \in -int(x).$$

So, we have

$$t\{\varphi(x,y_1) - \langle f, y_1 - x \rangle\} + (1-t)\{\varphi(x,y_2) - \langle f, y_2 - x \rangle\} \in -intC(x).$$

Since $x \mapsto \varphi(\cdot, x)$ is convex, we have

$$\begin{aligned} &\varphi(x, ty_1 + (1-t)y_2) - \langle f, ty_1 + (1-t)y_2 - x\rangle \\ &\leq t\varphi(x, y_1) + (1-t)\varphi(x, y_2) - t\langle f, y_1 - x\rangle - (1-t)\langle f, y_2 - x\rangle \\ &= t\left\{\varphi(x, y_1) - \langle f, y_1 - x\rangle\right\} + (1-t)\left\{\varphi(x, y_2) - \langle f, y_2 - x\rangle\right\}, \end{aligned}$$

which means that

$$\begin{aligned} &t\{\varphi(x, y_1) - \langle f, y_1 - x\rangle\} + (1-t)\{\varphi(x, y_2) - \langle f, y_2 - x\rangle\} \\ &- \{\varphi(x, ty_1 + (1-t)y_2) - \langle f, ty_1 + (1-t)y_2 - x\rangle\} \in C(x). \end{aligned}$$

Thus,

$$\varphi(x, ty_1 + (1-t)y_2) - \langle f, ty_1 + (1-t)y_2 - x\rangle \in -intC(x).$$

Hence $ty_1 + (1-t)y_2 \in B(x)$ and $B(x)$ is convex. Now, for each $y \in K$, the set

$$\begin{aligned} B^{-1}(y) &= \{x \in K : y \in B(x)\} \\ &= \left\{x \in K : \varphi(x, y) - \langle f, y - x\rangle \in -intC(x)\right\} \\ &= \left\{x \in K : \varphi(x, y) - \langle f, y - x\rangle \notin -intC(x)\right\}^C \\ &= [A(y)]^C \\ &= O_y \end{aligned}$$

is open in K.

Next, we show that $\bigcup_{y \in K} O_y = \bigcup_{y \in K} B^{-1}(y) = K$. Let $x \in K$. Since $B(x) \neq \emptyset$, we can choose y such that $y \in B(x)$. Hence $x \in B^{-1}(y) = O_y$.

Finally, by the coercive condition, for each $x \in K \setminus D_1$, there exists a $y \in D_1$ with

$$\varphi(x, y) - \langle f, y - x\rangle \in -intC(x),$$

that is, $x \notin A(y)$. This implies that

$$D = \bigcap_{y \in D_1} {O_y}^C = \bigcap_{y \in D_1} A(y) \subset D_1.$$

Since, for each $y \in K$, $A(y)$ is closed and D_1 is a compact set, it follows that D is compact. Hence the set-valued mapping $B : K \to 2^K$ satisfies all the conditions in Theorem 1. So there exists a point $x_0 \in K$ such that $x_0 \in B(x_0)$, that is,

$$\varphi(x_0, x_0) - \langle f, x_0 - x_0\rangle \in -intC(x_0),$$

which is a contradiction. Hence the solution set S is nonempty. Therefore, **(VNI)** has a solution in K. □

REMARK 1. If K is compact, then the coercive condition in Theorem 2 is automatically satisfied since we can set $D_1 = K$.

As consequences of Theorem 2, we have the following:

COROLLARY 3. *Considering $f \equiv 0$ in Theorem 2, we can show the existence result of solution to the following vector equilibrium problem:*
Find $x_0 \in K$ such that

$$\varphi(x_0, y) \notin -intC(x_0) \text{ for all } y \in K.$$

REMARK 2. Let X be a Hausdorff topological vector space and Y a locally convex vector space. Let K be a closed convex subset of X. Given a bifunction $F : K \times K \to Y$, the vector equilibrium problem is to find $x_0 \in K$ such that

$$F(x_0, y) \not< 0 \text{ for all } y \in K,$$

which means that $0 - F(x_0, y) \notin intC$. Bianchi *et al.* [5] and Hadjisavvas *et al.* [8] obtained the existence theorem of solution to vector equilibrium problem for pseudomonotone and quasimonotone mappings, respectively.

COROLLARY 4. *Considering $\varphi(x_0, y) = \langle T(x_0), \theta(y, x_0)\rangle$ and $f \equiv 0$ in Theorem 2, we can show the existence result of solution to the following vector variational-like inequality problem:*
Find $x_0 \in K$ such that

$$\langle T(x_0), \theta(y, x_0)\rangle \notin -intC(x_0) \text{ for all } y \in K,$$

where $T : K \to L(X, Y)$ and $\theta : K \times K \to X$ are mappings.

REMARK 3. Siddiqi *et al.* [15] considered the existence theorem of solution to vector variational-like inequality problem without the monotonicity condition by using Fan's section theorem [7].

COROLLARY 5. *Considering $\varphi(x_0, y) = \langle T(x_0), y - x_0\rangle$ and $f \equiv 0$ in Theorem 2, we can show the existence result of solution to the following vector variational inequality problem:*
Find $x_0 \in K$ such that

$$\langle T(x_0), y - x_0\rangle \notin -intC(x_0) \text{ for all } y \in K,$$

where $T : K \to L(X, Y)$ is a mapping.

REMARK 4. Taking $Y = \mathbb{R}$ and $C(x) = \mathbb{R}^+$ in Theorem 2, we can derive Theorem 2 in [3]. Further, from Corollary 3 to Corollary 5, we can derive some scalar equilibrium problems, scalar variational-like inequality problems and scalar variational inequality problems, respectively, obtained by many authors.

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