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ON SOME SPECIAL CONDITIONS OF n-TH ORDER NON-OSCILLATORY NONLINEAR SYSTEMS

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ABSTRACT. Krylov-Bogoliubov-Mitropolskii method has been extended to obtain asymptotic solution of n-th order nonlinear differential system characterized by certain non-oscillatory processes. The damping force is considered in such a manner that one of the characteristic roots of the linear system becomes small and others are in integral multiple. The method is illustrated by an example. The solutions for different initial conditions show a good agreement with those obtained by numerical method.

1. Introduction

Krylov-Bogoliubov-Mitropolskii (KBM) method [1, 2] is well known in the theory of nonlinear oscillations. The method was originally developed by Krylov and Bogoliubov [1] for obtaining periodic solution of a second order system with small nonlinearities. Then the method was amplified and justified by Bogoliubov and Mitropolskii [2] and later extended to nonlinear over-damped systems by Murty *et al* [3, 4]; but the solutions obtained in [3, 4] were unable to give satisfactory results when the characteristic roots of the linear system are in integral multiple. On the other hand, Sattar [5] examined a critically damped nonlinear system. The critically damped solution obtained by Sattar gives correct results only for certain initial conditions. Shamsul [6] further investigated critically damped and over-damped nonlinear systems and was able to find desired solutions. Bojadziev [7] and Sattar [8] respectively investigated 3-dimensional damped and over-damped systems. Shamsul [9] studied a third-order over-damped system under some special conditions especially when the characteristic roots are in integral multiple. Then the method of [9] has been extended to an n-th order over-damped

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system in [10]. Shamsul [11] has also presented a unified method for n-th order damped and over-damped systems.

In case of an over-damped system, the nature of asymptotic solutions changes as the damping force changes. So, depending on damping forces, different type of approximate solutions of a nonlinear system have been found in some previous papers (see [8, 9, 10] for details). Yet all the previous over-damped solutions are useless for some particular damping forces, in where one of the characteristic roots becomes small or vanishes. In order to cover these situations, a new asymptotic solution (based on KBM method) has been presented.

2. The method

Let us consider a weakly nonlinear system governed by an n-th order differential equation

$$x^{(n)} + k_1 x^{(n-1)} + \cdots + k_n x = -\varepsilon f(x, \dot{x}, \cdots, x^{(n-1)}), \tag{1}$$

where $x^{(i)}, i = n, n-1, \cdots$, represents i-th derivatives, $k_1,\ k_2, \cdots$ are constants, ε is a small parameter and f is a nonlinear function. When $\varepsilon = 0$, the characteristic equation of (1) has n roots. In the case of non-oscillatory processes, the roots are real and non-positive, say $-\lambda_j$, $j = 1, 2, \cdots, n$ and the solution of the linear equation is

$$x(t, 0) = \sum_{j=1}^{n} a_{j,0} e^{-\lambda_j t}, \tag{2}$$

where $a_{j,0},\ j = 1, 2, \cdots, n$ are arbitrary constants.

When $\varepsilon \neq 0$, we propose an asymptotic solution of (1) in the form

$$x(t, \varepsilon) = \sum_{j=1}^{n} a_j(t) e^{-\lambda_j t} + \varepsilon u_1(a_1, a_2, \cdots, a_n, t) + \varepsilon^2 \cdots, \tag{3}$$

where each a_j satisfies a first order differential equation

$$\dot{a} = \varepsilon A_j(a_1, a_2, \cdots, a_n, t) + \varepsilon^2 \cdots. \tag{4}$$

Confining only to the first few terms,1, 2, $\cdots, m$, in the series expansions of (3) and (4), we evaluate the functions $u_1,\ u_2, \cdots$ and A_j, $B_j, \cdots,\ j = 1, 2, \cdots, n$ such that $a_j(t)$ appearing in (3) and (4) satisfy the given differential equation (1) with an accuracy of ε^{m+1}. In order to determine these unknown functions it was assumed by Murty *et al* [3] that the functions $u_1,\ u_2, \cdots$ do not contain the terms involving $e^{-\lambda_j t}$, $j = 1, 2, \cdots, n$, since these are included in the series expansion (3) at order ε^0. To obtain some special non-oscillatory solution of (1), we impose another restrictions that u_1, $u_2, \cdots$ do not contain terms involving $te^{-(i_1\lambda_1+i_2\lambda_2+\cdots+i_n\lambda_n)t}$ even if the smallest root, say $\lambda_1 \to 0$ or/and the rest of the roots are in integral multiple.

Differentiating $x(t,\epsilon)$ n-times with respect to t, substituting the derivatives $x^{(n)}$, $x^{(n-1)}, \cdots, \dot{x}$ and x in the original equation (1) and equating the coefficients of ε, we obtain

$$\prod_{j=1}^{n}\left(\frac{\partial}{\partial t}+\lambda_j\right)u_1+\sum_{j=1}^{n}e^{-\lambda_j t}\left(\prod_{k=1,k\neq j}^{n}\left(\frac{\partial}{\partial t}-\lambda_j+\lambda_k\right)\right)A_j \\ = -f^{(0)}(a_1,a_2,\cdots,a_n,t), \tag{5}$$

where $f^{(0)} = f(x_0, \dot{x}_0, \cdots, x^{(n-1)})$ and $x_0 = \sum_{j=1}^{n} a_j(t)e^{-\lambda_j t}$.

In general, the function $f^{(0)}$ can be expanded in Taylor's series as [3]

$$f^{(0)} = \sum_{i_1=0,i_2=0,\cdots,i_n=0}^{\infty,\infty,\cdots,\infty} F_{i_1,i_2,\cdots,i_n} e^{-(i_1\lambda_1+i_2\lambda_2+\cdots+i_n\lambda_n)t}. \tag{6}$$

Substituting $f^{(0)}$ from (6) into (5) and imposing the restriction that u_1 excludes terms $te^{-(i_1\lambda_1+i_2\lambda_2+\cdots+i_n\lambda_n)t}$, we shall able to find the unknown functions u_1 and A_j, $j = 1, 2, \cdots, n$, which complete the determination of the first order solution of (1). The method can be carried out to higher order approximations in a similar way.

3. Example

As an example of the above procedure, we consider a nonlinear mechanical system with internal friction and relaxation [7, 9, 11, 13]

$$\begin{aligned} &m\ddot{x} + \sigma = 0, \\ &\gamma\dot{\sigma} + \sigma = \beta\dot{x} + \alpha x + s x^3, \qquad s << 1. \end{aligned} \tag{7}$$

Here x is the deformation, m is the mass of the system and α, β, γ and s are constants. The terms with coefficients α and s represent respectively the linear and nonlinear elasticity, the term with coefficient β corresponds to the linear viscous damping and the term with coefficient γ reflects the linear relaxation. In some cases characterized by small internal friction one can neglect the effect of relaxation. However, there are situation in which the influence of relaxation is significant, for instance in plastic materials, and a study of such cases based on the assumption of lack of relaxation may severely limit their closeness to reality (see [12] for details). By imposing certain restrictions, Osiniskii [13] investigated a damped case of the system (7). He had transformed the system (7) to the following third order nonlinear differential equation

$$(8)\quad \dddot{x} + \gamma^{-1}\ddot{x} + \beta m^{-1}\gamma^{-1}\dot{x} + \alpha m^{-1}\gamma^{-1}x = -\varepsilon x^3, \qquad \varepsilon = sm^{-1}\gamma^{-1}.$$

Then Bojadziev [7] found a damped solution of (7) removing those restrictions imposed by Osiniskii [13]. Shamsul [11] has rediscovered Bojadziev's solution and used it as a unified solution for damped and over-damped systems. Shamsul [9] has also found approximate solutions of (8) [*i.e.*, the system (7)] by considering $\lambda_1 = 3\lambda_2$ or/and $\lambda_2 = 3\lambda_3$, or $\lambda_1 = \lambda_2 + 2\lambda_3$. It is noted that Sattar's [5] over-damped solution and Shamsul's [11] unified solutions were unable to give desired results in these cases. However, all the over-damped solutions obtained in [8, 11, 9] are useless when one of the characteristic roots of (8) becomes small or vanishes. The main theme of this paper is to find an asymptotic solution of (8) and as well as to generalize this technique for an n-th order system when the smallest root becomes small or vanishes. It would be mentioned that Sattar's [5] over-damped solution and Shamsul's [11] unified solutions are very similar to classical type solution found by Murty *et al* [4]. These solutions are only useful when the characteristic roots are not in integral multiple.

It is obvious that equation (8) is a particular case of (1) in where $n = 3$, $\lambda_1 + \lambda_2 + \lambda_3 = k_1 = \gamma^{-1}$, $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = k_2 = \beta m^{-1}\gamma^{-1}$, $\lambda_1\lambda_2\lambda_3 = k_3 = \alpha m^{-1}\gamma^{-1}$ and $f = x^3$. Therefore,

(9)

$$\begin{aligned} f^{(0)} = {} & a_1^3 e^{-3\lambda_1 t} + 3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1^2 a_3 e^{-(2\lambda_1+\lambda_3)t} \\ + {} & 3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} + 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3a_1 a_3^2 e^{-(\lambda_1+2\lambda_3)t} \\ + {} & a_2^3 e^{-3\lambda_2 t} + 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 3a_2 a_3^2 e^{-(\lambda_2+2\lambda_3)t} + a_3^3 e^{-3\lambda_3 t}. \end{aligned}$$

We assume that u_1 excludes the terms involving $te^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3)t}$ for all values of $\lambda_1 \geq 0$. In accordance to Shamsul's [9] assumptions, u_1 also excludes terms involving $te^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3)t}$ when rest of the roots are in integral multiple. So, the equation of u_1 does not contain $e^{-3\lambda_1 t}$, $e^{-(2\lambda_1+\lambda_2)t}$ and $e^{-(2\lambda_1+\lambda_3)t}$ of $f^{(0)}$. Moreover, u_1 respectively excludes the term $e^{-(\lambda_1+2\lambda_3)t}$ when $\lambda_3 = 2\lambda_2$, and $e^{-3\lambda_3 t}$ when $\lambda_3 = 3\lambda_2$. Substituting the values of $f^{(0)}$ form (9) into (5), we can separate it (when

$\lambda_3 = 2\lambda_2$) into four independent equations as follows

$$\begin{aligned}
&\prod_{j=1}^{3}\left(\frac{\partial}{\partial t}+\lambda_j\right)u_1 \\
(10)\quad &= -\,(6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3a_1a_3^2e^{-(\lambda_1+2\lambda_3)t} \\
&+\, a_2^3e^{-3\lambda_2 t} + 3a_2^2a_3e^{-(2\lambda_2+\lambda_3)t} + 3a_2a_3^2e^{-(\lambda_2+2\lambda_3)t} + a_3^3e^{-3\lambda_3 t}),
\end{aligned}$$

$$(11)\qquad \left(\frac{\partial}{\partial t}-\lambda_1+\lambda_2\right)\left(\frac{\partial}{\partial t}-\lambda_1+\lambda_3\right)A_1 = -a_1^3e^{-2\lambda_1 t},$$

$$(12)\qquad \left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)A_2 = -3a_1^2a_2e^{-2\lambda_1 t},$$

and

$$\begin{aligned}
(13)\qquad &\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_3\right)A_3 \\
&= -\,3a_1^2a_3e^{-2\lambda_1 t} - 3a_1a_2^2e^{(\lambda_3-\lambda_1-2\lambda_2)t}.
\end{aligned}$$

The solution of (10) is

$$\begin{aligned}
(14)\qquad u_1 &= c_2a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t} + c_3a_1a_3^2e^{-(\lambda_1+2\lambda_3)t} + c_4a_2^3e^{-3\lambda_2 t} \\
&+\, c_5a_2^2a_3e^{-(2\lambda_2+\lambda_3)t} + c_6a_2a_3^2e^{-(\lambda_2+2\lambda_3)t} + c_7a_3^3e^{-3\lambda_3 t},
\end{aligned}$$

where

$$\begin{aligned}
&c_2 = 6[(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)]^{-1}, \\
&c_3 = 3[2\lambda_3(\lambda_1+\lambda_3)(\lambda_1-\lambda_2+2\lambda_3)]^{-1}, \\
(15)\qquad &c_4 = [2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)]^{-1}, \\
&c_5 = 3[2\lambda_2(\lambda_2+\lambda_3)(-\lambda_1+2\lambda_2+\lambda_3)]^{-1}, \\
&c_6 = 3[2\lambda_3(\lambda_2+\lambda_3)(-\lambda_1+\lambda_2+2\lambda_3)]^{-1}, \\
&c_7 = [2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)]^{-1}.
\end{aligned}$$

Then the solution of (11)-(13) is

$$\begin{aligned}
(16)\qquad &A_1 = -l_1a_1^3e^{-2\lambda_1 t}, \quad A_2 = m_1a_1^2a_2e^{-2\lambda_1 t}, \\
&A_3 = -n_1a_1^2a_3e^{-2\lambda_1 t} - n_2a_1a_2^2e^{(\lambda_3-\lambda_1-2\lambda_2)},
\end{aligned}$$

where

(17)

$$l_1 = [(\lambda_2-3\lambda_1)(\lambda_3-3\lambda_1)]^{-1},\ m_1 = 3[(\lambda_1+\lambda_2)(\lambda_3-\lambda_2-2\lambda_1)]^{-1},$$
$$n_1 = 3[(\lambda_1+\lambda_2)(\lambda_3-\lambda_2+2\lambda_1)]^{-1},\ n_2 = 3[2\lambda_3(\lambda_1+\lambda_2)]^{-1}.$$

When $\lambda_3 = 3\lambda_2$, the functions A_1 and A_2 will be remain unchanged, while u_1 and A_3 will be changed in accordance Shamsul's [9] assumption. In this case, the functional relations of u_1 and A_3 are respectively

$$(18)\quad \begin{aligned} u_1 &= c_1 a_1 a_2^2 e^{-(\lambda+2\lambda_2)t} + c_2 a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + c_3 a_1 a_3^2 e^{-(\lambda_1+2\lambda_3)t} \\ &\quad + c_5 a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + c_6 a_2 a_3^2 e^{-(\lambda_2+2\lambda_3)t} + c_7 a_3^3 e^{-3\lambda_3 t}, \end{aligned}$$

and

$$(19)\qquad A_3 = -n_1 a_1^2 a_3 e^{-2\lambda_1 t} - n_3 a_2^3 e^{(\lambda_3-3\lambda_2)}.$$

The new coefficients of u_1 and A_3 are

$$(20)\quad c_1 = 3[2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)]^{-1}, \qquad n_3 = [2\lambda_2(3\lambda_2-\lambda_1)]^{-1},$$

and all other coefficients are given by (15) and (17).

Substituting the values of A_1, A_2 and A_3 into (4) and then integrating with respect to t, we obtain

$$(21)\quad \begin{aligned} a_1 &= \frac{a_{1,0}}{\sqrt{1+\varepsilon\lambda_1^{-1} l_1 a_{1,0}^2 (1-e^{-2\lambda_1 t})}}, \\ a_2 &= a_{2,0}\sqrt[r]{1+\varepsilon\lambda_1^{-1} l_1 a_{1,0}^2 (1-e^{-2\lambda_1 t})}, \ r = m_1/l_1, \\ a_3 &\cong a_{3,0} + \varepsilon[n_1 a_{1,0}^2 a_{3,0}(e^{-2\lambda_1 t}-1)/(2\lambda_1) \\ &\quad + n_2 a_{1,0} a_{2,0}^2 (e^{-\lambda_1 t}-1)/\lambda_1], \ \lambda_3 = 2\lambda_2, \\ a_3 &\cong a_{3,0} + \varepsilon[n_1 a_{1,0}^2 a_{3,0}(e^{-2\lambda_1 t}-1)/(2\lambda_1) - n_3 a_{2,0}^3 t], \ \lambda_3 = 3\lambda_2. \end{aligned}$$

Therefore, the first order solution of (7) is

$$(22)\qquad x(t,\varepsilon) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + \varepsilon u_1,$$

where a_1, a_2 and a_3 are given by (21) and u_1 is given by (14) or (18). This solution is valid for all values of $\lambda_1 > 0$. However, as the limit $\lambda_1 \to 0$, (22) is also useful where a_1, a_2 and a_3 will be computed by

$$(23)\quad \begin{aligned} a_1 &= \frac{a_{1,0}}{\sqrt{1+2\varepsilon l_1 a_{1,0}^2 t}}, \\ a_2 &= a_{2,0}\sqrt[r]{1+2\varepsilon l_1 a_{1,0}^2 t}, \ r = m_1/l_1, \\ a_3 &\cong a_{3,0} - \varepsilon(n_1 a_{1,0}^2 a_{3,0} t + n_2 a_{1,0} a_{2,0}^2 t), \qquad \lambda_3 = 2\lambda_2, \\ a_3 &\cong a_{3,0} - \varepsilon(n_1 a_{1,0}^2 a_{3,0} t + n_3 a_{2,0}^3 t), \qquad \lambda_3 = 3\lambda_2. \end{aligned}$$

It is noted that the equation of a_3, *i.e.*, the third equation of (4) has not an exact solution. It has been solved by assuming that a_1, a_2 and a_3 are constant in the right hand side of (4) (see [3, 6, 9] for details).

4. Results and discussion

On some special conditions, asymptotic solutions of a third-order nonlinear system have been found based on KBM method (in Sec. 3). In order to test the accuracy of approximate solutions obtained by a perturbation method, we sometimes compare the approximate solutions to the numerical solutions (considered to be exact). With regard to such a comparison concerning the presented KBM method of this paper, we refer to the work of Murty *et al* [3, 4]. In our paper, for a particular set of initial conditions, we have compared the asymptotic solution (22) to those obtained by *Runge-Kutta* (fourth-order) method when $\lambda_3 = 2\lambda_2$ and $\lambda_3 = 3\lambda_2$ for two different values of λ_1.

First of all, $x(t, \varepsilon)$ has been calculated by approximate solution (22) in which a_1, a_2 and a_3 are evaluated by (21) with initial conditions $[x(0) = 1, \dot{x}(0) = -0.25, \ddot{x}(0) = -0.75]$ or $a_{1,0} = 0.250530$, $a_{2,0} = 1.304914$, $a_{3,0} = -0.568194$ for $m = 1$, $\alpha = \frac{2}{31}$, $\beta = \frac{23}{31}$, $\gamma = \frac{10}{31}$, $s = \frac{1}{31}$ or, $\lambda_1 = 0.1$, $\lambda_2 = 1$, $\lambda_3 = 2$ and $\varepsilon = 0.1$. Then corresponding numerical solution of (7) or (8) has been computed by *Runge-Kutta* method and percentage errors are calculated. All the results are presented in Table 1. From Table 1, it is clear that the percentage errors of the solution (22) is much smaller than 1% (it is to be noted that for $\varepsilon = 0.1$, error(s) of the first order perturbation solution should occurs 1%). It is interesting to note that as the difference of two roots λ_2 and λ_3 increases, the error(s) decreases. To clarify this matter, $x(t, \varepsilon)$ has again been calculated by solution (22) in which a_1, a_2 and a_3 are evaluated by (21) with same initial conditions $[x(0) = 1, \dot{x}(0) = -0.25, \ddot{x}(0) = -0.75]$ or , $a_{1,0} = 0.456575$, $a_{2,0} = 0.721313$, $a_{3,0} = -0.139374$ for $m = 1$, $\alpha = \frac{3}{41}$, $\beta = \frac{34}{41}$, $\gamma = \frac{10}{41}$, $s = \frac{1}{41}$ or, $\lambda_1 = 0.1$, $\lambda_2 = 1$, $\lambda_3 = 3$ and $\varepsilon = 0.1$. Corresponding numerical solution has been computed and the percentage errors have been calculated. The results are given in Table 2. Comparing the percentage errors of two tables, we conclude that errors in Table 2, have occurred smaller than those in Table 1.

Table 1

t	$a_1 e^{-\lambda_1 t}$	x	x_{nu}	$E(\%)$
0.0	0.250530	1.000000	1.000000	0.0000
1.0	0.225613	0.632149	0.631975	0.0275
2.0	0.203356	0.374093	0.373748	0.0923
3.0	0.183427	0.249815	0.249382	0.1736
5.0	0.149483	0.158844	0.158408	0.2752
7.0	0.122009	0.123296	0.122925	0.3018
10.0	0.090132	0.090197	0.089921	0.3069
15.0	0.054551	0.054551	0.054384	0.3071
20.0	0.033060	0.033060	0.032959	0.3064

Table 2

t	$a_1 e^{-\lambda_1 t}$	x	x_{nu}	$E(\%)$
0.0	0.456575	1.000000	1.000000	0.0000
1.0	0.409057	0.670181	0.670665	-0.0722
2.0	0.367196	0.468976	0.469533	-0.1186
3.0	0.330125	0.368529	0.369048	-0.1406
5.0	0.267750	0.273101	0.273479	-0.1382
7.0	0.217857	0.218595	0.218872	-0.1266
10.0	0.160486	0.160523	0.160712	-0.1176
15.0	0.096923	0.096923	0.097033	-0.1134
20.0	0.058695	0.058695	0.058761	-0.1123

The error(s) of approximate solution (22), are not changed abruptly as λ_1 changes. When $m = 1$, $\alpha = 0$, $\beta = \frac{3}{4}$, $\gamma = \frac{1}{4}$, $s = \frac{1}{40}$ or $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been calculated by (22) in which a_1, a_2 and a_3 are evaluated by (23) with initial conditions $[x(0) = 1, \dot{x}(0) = -0.25, \ddot{x}(0) = -0.75]$ or $a_{1,0} = 0.393548$, $a_{2,0} = 0.797038$, $a_{3,0} = -0.150077$. Corresponding numerical solution has been computed and percentage errors are calculated. The results are given in Table 3. Comparing Table 2 and Table 3, we see that the percentage errors are not greatly changed though λ_1 has changed form 0.1 to 0. However, as the difference of two roots λ_2 and λ_3 changes, the errors change significantly. When $m = 1$, $\alpha = 0$, $\beta = \frac{2}{3}$, $\gamma = \frac{1}{3}$, $s = \frac{1}{30}$ or $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has again been calculated by (22) in which a_1, a_2 and a_3 are evaluated by (23) with initial conditions $[x(0) = 1, \dot{x}(0) = -0.25, \ddot{x}(0) = -0.75]$ or $a_{1,0} = 0.214378$, $a_{2,0} = 1.364474$, $a_{3,0} = -0.593203$. Corresponding numerical solution has been computed and percentage errors are calculated. The results are given in Table 4.

Comparing Table 4 and Table 3, we see that the percentage errors are changed significantly, but the errors in Table 4 are similar to those in Table 1, while the errors in Table 3 are similar to those in Table 2.

The method has another merit that only the first term of the approximate solution (22), *i.e.*, $a_1e^{-\lambda_1 t}$ gives perturbation results of (7) when t is large, especially $t = O(\varepsilon^{-1})$. Since λ_1 is the smallest root, $a_1e^{-\lambda_1 t}$ dies out very slowly (for $\lambda_1 > 0$) while the other components $a_je^{-\lambda_j t}$, $j = 2, 3, \cdots$ die out quickly. Thus $x(t, \varepsilon) \cong a_1e^{-\lambda_1 t}$ when $t = O(\varepsilon^{-1})$. In Tables 1-4, $a_1e^{-\lambda_1 t}$ has been given and it has been compared to $x(t, \varepsilon)$.

The difference between (22) and those obtained in [9] is that $x(t, \varepsilon)$ [presented by (22)] dies out vary slowly, since one of the roots is small or vanishes. On the contrary, the solutions presented in [9] die out quickly as all the roots were considered significant. One can not use the previous solutions when one of the roots is small or vanishes. In these cases one or more coefficients of variational equations of a_j, $j = 1, 2, 3$ of all previous solutions or u_1 contains secular type terms te^{-t} or the coefficients of u_1 becomes large; but these are not desired in an asymptotic solution.

Table 3

t	$a_1e^{-\lambda_1 t}$	x	x_{nu}	$E(\%)$
0.0	0.393548	1.000000	1.000000	0.0000
5.0	0.383766	0.389779	0.390570	-0.2025
10.0	0.374680	0.374725	0.375288	-0.1500
15.0	0.366209	0.366210	0.366578	-0.1004
20.0	0.358288	0.358288	0.358493	-0.0572
30.0	0.343876	0.343876	0.343825	0.0051
40.0	0.331074	0.331074	0.330838	0.0713
50.0	0.319602	0.319602	0.319232	0.1159

Table 4

t	$a_1e^{-\lambda_1 t}$	x	x_{nu}	$E(\%)$
0.0	0.214378	1.000000	1.000000	0.0000
5.0	0.211957	0.221773	0.221399	0.1689
10.0	0.209615	0.209686	0.209305	0.1820
15.0	0.207350	0.207351	0.206960	0.1889
20.0	0.205157	0.205157	0.204757	0.1954
30.0	0.200971	0.200971	0.200557	0.2064
40.0	0.197031	0.197031	0.196607	0.2157
50.0	0.193315	0.193315	0.192883	0.2240

5. Conclusion

An asymptotic solution has been obtained for n-th order nonlinear differential system characterized by non-oscillatory processes. The method is a generalization of KBM method [1, 2] and can be used to obtain desired solution for certain damping forces. Thus it is no longer to treat individual cases separately. The method is important especially when one of the characteristic roots of the linear equation is small and other are in integral multiple. The solution is even useful when one of the roots vanishes.

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