Commun. Korean Math. Soc. **18** (2003), No. 4, pp. 767–779

A DISCONTINUOUS GALERKIN METHOD FOR A MODEL OF POPULATION DYNAMICS

MI-YOUNG KIM† AND Y. X. YIN

ABSTRACT. We consider a model of population dynamics whose mortality function is unbounded. We approximate the solution of the model using a discontinuous Galerkin finite element for the age variable and a backward Euler for the time variable. We present several numerical examples. It is experimentally shown that the scheme converges at the rate of $h^{3/2}$ in the case of piecewise linear polynomial space.

1. Introduction

When modelling a population dynamics, some significant variables should be considered. Thus, depending on the phenomenon that has to be modelled, the population is given a structure that is often responsible for special behaviors not occurring when the structure is absent. Age is one of the most natural and important parameter structuring a population. At the level of the single individual, many internal variables strictly depend on the age variable since different ages provide different reproductions, different survival capacities and different behaviors. In this paper, we consider a nonlinear age-dependent model of population dynamics and introduce a discontinuous Galerkin finite element method to approximate the solution of the model. The organization of the remainder of the paper is as follows. In Section 2, we introduce the basic parameters and derive some models of population dynamics. In Section 3, we demonstrate a discretization procedure for the nonlinear model. Finally, in Section 4, we show the numerical results.

Received February 4, 2003.
2000 Mathematics Subject Classification: 65M10, 65M20, 92A15, 65C20.
Key words and phrases: age-dependent population dynamics, discontinuous Galerkin method, integro-differential equation.
†The research of this author was supported by Inha University Research Grant (INHA-21965).

2. The model of the population dynamics

The simplest population model is the well-known Malthusian law. In the model we consider a single homogeneous population. That is, we assume that all individuals of the population are identical so that the only variable that we have to deal with is the number of the individuals as a function of time, $p(t)$. We also suppose that the population lives isolated in an invariant habitat with no limit to resources. Thus the population is subject to constant fertility and mortality rates that we call β and μ, respectively. Then the growth is governed by the following equation:

$$\frac{d}{dt}p(t) = \beta p(t) - \mu p(t).$$

Thus, we easily see that

$$p(t) = p(0)e^{\alpha t}.$$

Here $\alpha = \beta - \mu$ is usually called the Malthusian parameter of the population.

A linear model we want to introduce is a strict analogue of the Malthus model. We consider a single population living isolated in an invariant habitat. We assume that all of its individuals are perfectly equal but for their age. In particular, we assume that there are no sex differences. According to this setting, fertility and mortality are intrinsic parameters of the population growth and they depend on age. Thus the evolution of the population is described by its age density function at time t,

$$u(a,t), \qquad a \in [0, a_\dagger], \quad t \geq 0,$$

where $a_\dagger$ is the maximum age which an individual of the population may reach. We assume $a_\dagger < +\infty$. The integral $\int_{a_1}^{a_2} u(a,t)da$ then gives the number of individuals at time t, with age in the interval $[a_1, a_2]$ and

$$p(t) = \int_0^{a_\dagger} u(a,t)da$$

is the total population at time t. The age specific fertility $\beta(a)$ is the number of newborns in one time unit coming from a single individual whose age is in the infinitesimal age interval $[a, a + da]$. Thus $\int_{a_1}^{a_2} \beta(a)u(a,t)da$ gives the number of newborns in one time unit coming from individuals with age in $[a_1, a_2]$. The age specific mortality $\mu(a)$ is the death rate of people having age in $[a, a + da]$: Then the total death rate is given by

$$D(t) = \int_0^{a_\dagger} \mu(a)u(a,t)da,$$

which gives the total number of deaths occurring in one time unit. We also consider the total birth rate

$$B(t) = \int_0^{a_\dagger} \beta(a)u(a,t)da,$$

which gives the total number of newborns in one time unit. Another meaningful quantity is the survival probability, i.e., the probability for an individual to survive to age a, given by

$$\Pi(a) = e^{-\int_0^a \mu(\sigma)d\sigma}, \quad a \in [0, a_\dagger].$$

Thus it must satisfy $\Pi(a_\dagger) = 0$. Moreover, the function

$$K(a) = \beta(a)\Pi(a), \quad a \in [0, a_\dagger]$$

is called the maternity function and synthesize the dynamics of the population; it is related to the parameter

$$R = \int_0^{a_\dagger} \beta(a)\Pi(a)da,$$

which is called the net reproduction rate and gives the number of the newborns that an individual is expected to produce during his reproductive life. In fact, we expect the population to show an increasing trend when $R > 1$, decreasing if $R < 1$, stable when $R = 1$. The linear model of population dynamics describes the evolution of the population regulated by prescribed linear biological parameters. Consider the function

$$N(a,t) = \int_0^a u(\sigma,t)d\sigma$$

which represents the number of individuals that, at time t, age$\leq a$. Then for $h > 0$ we have

(2.1)

$$N(a+h,t+h) = N(a,t) + \int_t^{t+h} B(s)ds - \int_0^h \int_0^{a+s} \mu(\sigma)u(\sigma,t+s)d\sigma ds.$$

Here $\int_t^{t+h} B(s)ds$ gives the input of all newborns in the time interval $[t, t+h]$, which have age$\leq h$ and consequently have to be included in the number $N(a+h, t+h)$. Moreover, since $\int_0^{a+s} \mu(\sigma)u(\sigma, t+s)d\sigma$ is the number of individuals who die at time $t+s$ having age $\leq a+s$, $\int_0^h \int_0^{a+s} \mu(\sigma)u(\sigma, t+s)d\sigma ds$ gives the loss from the initial group of $N(a,t)$ individuals and from the newborns, through the time interval $[t, t+h]$. As the second step, we differentiate (2.1) with respect to h, and set $h = 0$. We then have

$$u(a,t) + \int_0^a u_t(\sigma, t)d\sigma = B(t) - \int_0^a \mu(\sigma)u(\sigma,t)d\sigma. \tag{2.2}$$

Putting $a = 0$, we get $u(0,t) = B(t)$ and differentiating (2.2) with respect to a, we get

$$u_t(a,t) + u_a(a,t) + \mu(a)u(a,t) = 0.$$

Thus we obtain the following system

$$\begin{aligned} & u_t(a,t) + u_a(a,t) + \mu(a)u(a,t) = 0, \quad 0 < a < a_\dagger, \quad t > 0, \\ & u(0,t) = \int_0^{a_\dagger} \beta(\sigma)u(\sigma,t)d\sigma, \quad t > 0, \\ & u(a,0) = u_0(a), \quad 0 \leq a < a_\dagger. \end{aligned} \tag{2.3}$$

Here the third equation of (2.3) is the initial condition which we have added. The system is a basic linear model which describes the evolution of a single population under the phenomenological conditions specified at the beginning of this section. In order to be biologically meaningful, we assume the followings for the basic functions $\beta(\cdot)$, $\mu(\cdot)$ and for the initial condition u_0:

$$\begin{aligned} & \beta(\cdot),\ \mu(\cdot),\ u_0(a) \text{ are nonnegative,} \\ & \int_0^{a_\dagger} \mu(\sigma)d\sigma = +\infty. \end{aligned}$$

For the mathematical treatment, we also assume that

$$\beta(\cdot) \in L^\infty(0, a_\dagger), \ \ \mu(\cdot) \in L^1_{loc}(0, a_\dagger), \ \ u_0 \in L^1(0, a_\dagger).$$

In [3], Gurtin and MacCamy introduced a nonlinear model which is more realistic than linear one, by assuming that the fertility and mortality depend on

$$p(t) = \int_0^{a_\dagger} u(a,t)da.$$

Thus $\beta(a)$ and $\mu(a)$ are replaced by $\beta(a,p(t)),\mu(a,p(t))$, respectively. The linear model is then modified into the following:

$$(2.4)\quad \begin{aligned} &u_t(a,t)+u_a(a,t)+\mu(a,p(t))u(a,t)=0, \quad 0<a<a_\dagger, \quad t>0,\\ &u(0,t)=\int_0^{a_\dagger}\beta(a,p(t))u(a,t)da, \quad t>0,\\ &u(a,0)=u_0(a), \quad 0\le a<a_\dagger,\\ &p(t)=\int_0^{a_\dagger}u(a,t)da, \quad t\ge 0. \end{aligned}$$

We refer to [5], for details and for the proof of the existence and uniqueness of the solution to (2.4) and also for (2.3).

3. Discretization procedure

In this section, we introduce the discretization procedure to approximate the solution to problem (2.4). We apply a backward Euler in time and a discontinuous Galerkin in the age variable.

We first consider the semi-discretization procedure leaving time variable to be continuous. Let $I=(0,a_\dagger]$ and let $L^2(I)$ and $H^1(I)$ be the standard L^2 and Sobolev spaces on I, respectively. Then, let $V=H^1(I)$ and multiply the first equation of (2.4) for a given t by $v\in V$ and integrate over I. We then obtain, by integration by parts, the following variational formulation: Find $u(\cdot,t)\in L^2(I)$ such that

(3.1)
$$\begin{aligned} &\int_0^{a_\dagger}u_t(a,t)v(a)da-\int_0^{a_\dagger}u(a,t)v'(a)da+\int_0^{a_\dagger}\mu(a,p(t))u(a,t)v(a)da\\ &=\int_0^{a_\dagger}\beta(a,p(t))u(a,t)dav(0)-u(a_\dagger,t)v(a_\dagger), \quad \forall v\in V. \end{aligned}$$

Now let $I_m=(a_{m-1},a_m]$ and assume that $I=\cup_{m=1}^M I_m$. For a given non-negative integer q, let

$$V_h=\{v:I\to R \mid v|_{I_m} \text{ is a polynomial of degree } \le q\}$$

be a finite dimensional subspace of V. We notice that the functions v in V_h may be discontinuous at the discrete age level a_m. To account for this we introduce the notation

$$v_+(a_m)=\lim_{s\to 0^+}v(a_m+s), \quad v_-(a^m)=\lim_{s\to 0^-}v(a_m+s),$$

and we also define the jump $[v]$ at the inter-node by

$$[v](a_m) = v_+(a_m) - v_-(a_m).$$

If $u^h(\cdot, t)$ and v belong to V_h, then we have, by integration by parts, that

$$\begin{aligned} &\int_0^{a_\dagger} u_a^h(a,t)v(a)da \\ &= -u^h(0,t)v(0) - \int_0^{a_\dagger} u^h(a,t)v'(a)da + u^h(a_\dagger,t)v(a_\dagger) \\ &= -u^h(0,t)v(0) - \sum_{m=1}^{M} \int_{I_m} u^h(a,t)v'(a)da + u^h(a_\dagger,t)v(a_\dagger). \end{aligned} \tag{3.2}$$

Since

$$\int_{I_m} u^h(a,t)v'(a)da = u^h(a,t)v(a)\Big|_{a=a_{m-1}^+}^{a=a_m^-} - \int_{I_m} u_a^h(a,t)v(a)da, \tag{3.3}$$

we obtain, from (3.2),

$$\begin{aligned} &\int_0^{a_\dagger} u_a^h(a,t)v(a)da \\ &= -u^h(0,t)v(0) - \sum_{m=1}^{M} \left(u^h(a,t)v(a)\Big|_{a=a_{m-1}^+}^{a=a_m^-} \right) \\ &+ \sum_{m=1}^{M} \int_{I_m} u_a^h(a,t)v(a)da + u^h(a_\dagger,t)v(a_\dagger). \end{aligned} \tag{3.4}$$

We observe that the inflow on the boundary a_{m-1} of I_m is the outflow on the boundary a_{m-1} of I_{m-1}. Thus, with $u_-^h(0,t) = u^h(0,t)$, semi-discretization of the problem (3.1) is given as follows:

$$\begin{aligned} &\sum_{m=1}^{M} \int_{I_m} u_t^h(a,t)v(a)da + \sum_{m=1}^{M} [u^h](a_{m-1},t)v_+(a_{m-1}) \\ &+ \sum_{m=1}^{M} \int_{I_m} u_a^h(a,t)v(a)da \\ &= -\sum_{m=1}^{M} \int_{I_m} \mu(a,p(t))u^h(a,t)v(a)da, \quad \forall v \in V_h, \end{aligned} \tag{3.5}$$

where $u^h_-(0,t) = \int_0^{a_\dagger} \beta(a,p(t))u^h(a,t)da$.

Since v varies independently on each subinterval I_m, we may alternatively formulate (3.5) as follows: For $m = 1,\cdots,M$, given $U_-(a_{m-1},t)$, find $U(a,t) \equiv u^h|_{I_m} = u^h\chi_{I_m} \in V_h^m$ such that

(3.6)

$$\int_{I_m} U_t(a,t)v(a)da + U_+(a_{m-1},t)v_+(a_{m-1}) - U_-(a_{m-1},t)v_+(a_{m-1}) + \int_{I_m} U_a(a,t)v(a)da = -\int_{I_m} \mu(a,p(t))U(a,t)v(a)da, \quad v \in V_h^m,$$

where

$$U_-(0,t) = \int_0^{a_\dagger} \beta(a,p(t))u^h(a,t)da, \tag{3.7}$$

$$V_h^m = \{p : I_m \to R \,|\, p \text{ is a polynomial of degree } \le q\}.$$

Here we note that the integral equation (3.7) and $p(t)$ are computed using the two-point Gaussian quadrature.

For the numerical experiment, we now consider the case $q = 1$ and apply a backward Euler method in time variable.

We note that $U(a,t)$ is of the form

$$U(a,t) = \sum_{i=0}^{1} \xi_{m,i}(t)\varphi_i^m(a), \quad \text{on } I_m,$$

where

$$\varphi_0^m(a) = -\frac{1}{h_a}(a-a_{m-1})+1,\ \varphi_1^m(a) = \frac{1}{h_a}(a-a_{m-1}),\ h_a = a_m - a_{m-1}.$$

Thus we finally obtain the following fully discretized problem: For given $n \ge 1$ and $1 \le m \le M$, find $\xi_{m,i}^n$, $i = 0,1$ such that

(3.8)

$$\frac{1}{h_t}\sum_{i=0}^{1}\xi_{m,i}^n \int_{I_m} \varphi_i^m(a)v(a)da - \frac{1}{h_t}\sum_{i=0}^{1}\xi_{m,i}^{n-1}\int_{I_m}\varphi_i^m(a)v(a)da + \xi_{m,0}^n v_+(a_{m-1}) - \xi_{m-1,1}^n v_+(a_{m-1}) + \sum_{i=0}^{1}\xi_{m,i}^n\int_{I_m}(\varphi_i^m)'(a)v(a)da$$

$$= -\sum_{i=0}^{1}\xi_{m,i}^n\int_{I_m}\mu(a,p(t^n))\varphi_i^m(a)v(a)da,$$

$$\xi_{0,1}^n = \sum_{m=1}^{M}\sum_{i=0}^{1}\xi_{m,i}^n\int_{I_m}\beta(a,p(t^n))\varphi_i^m(a)da,$$

where $\xi^n_{m,i} = \xi_{m,i}(t^n)$ and $h_t = t^n - t^{n-1}$ is the mesh size of time. Here we note that $\xi^0_{m,i}$'s are provided by the approximation of the initial condition such as

$$u_0(a) = \sum_{i=0}^{1} \xi^0_{m,i}\varphi^m_i(a), \quad \text{on } I_m.$$

Noting that, for $i, j = 1, 2$,

$$\int_{I_m} \varphi^m_i(a)\varphi^m_j(a)da = \frac{1}{6}h_a, \quad i \neq j,$$

$$\int_{I_m} \varphi^m_i(a)\varphi^m_j(a)da = \frac{1}{3}h_a, \quad i = j,$$

$$\int_{I_m} (\varphi^m_0)'(a)\varphi^m_0(a)da = \int_{I_m} (\varphi^m_0)'(a)\varphi^m_1(a)da = -\frac{1}{2},$$

$$\int_{I_m} (\varphi^m_1)'(a)\varphi^m_1(a)da = \int_{I_m} (\varphi^m_1)'(a)\varphi^m_0(a)da = \frac{1}{2},$$

we now apply $v(a) = \varphi^m_k(a)$, $k = 0, 1$, to the first equation of (3.8) and we obtain the following two equations for each $m = 1, \cdots, M$:

$$\begin{aligned}
&\left(\frac{h_a}{3h_t} + \frac{1}{2} - A_m\right)\xi^n_{m,0} + \left(\frac{h_a}{6h_t} + \frac{1}{2} - B_m\right)\xi^n_{m,1} \\
&= \frac{h_a}{3h_t}\xi^{n-1}_{m,0} + \frac{h_a}{6h_t}\xi^{n-1}_{m,1} + \xi^n_{m-1,1}, \\
&\left(\frac{h_a}{6h_t} - \frac{1}{2} - B_m\right)\xi^n_{m,0} + \left(\frac{h_a}{3h_t} + \frac{1}{2} - C_m\right)\xi^n_{m,1} \\
&= \frac{h_a}{6h_t}\xi^{n-1}_{m,0} + \frac{h_a}{3h_t}\xi^{n-1}_{m,1},
\end{aligned}$$

where

$$\begin{aligned}
A_m &= -\int_{I_m} \mu(a, p(t^n))(\varphi^m_0(a))^2 da, \\
B_m &= -\int_{I_m} \mu(a, p(t^n))\varphi^m_0(a)\varphi^m_1(a) da, \\
C_m &= -\int_{I_m} \mu(a, p(t^n))(\varphi^m_1(a))^2 da.
\end{aligned}$$

We notice that there are two unknowns and two equations on each I_m. Consequently, we have $2M$ equations and $2M$ unknowns in total and the equations are coupled due to the birth process $(2.4)_2$. We now move the first equation to the last. Then, we see that the nonzero structure of the coefficient matrix is of the form:

$$\begin{pmatrix} \mathrm{x} & \mathrm{x} & & & & & & \\ & \mathrm{x} & \mathrm{x} & \mathrm{x} & & & & \\ & & \mathrm{x} & \mathrm{x} & & & & \\ & & & \mathrm{x} & \mathrm{x} & \mathrm{x} & & \\ & & & & \mathrm{x} & \mathrm{x} & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \dots & \mathrm{x} \end{pmatrix}$$

After simple row operations, the matrix reduces to a upper triangular matrix and the system is then solved directly by back substitution.

4. Numerical results

In this section we present some numerical results. In the test we computed the order of convergence of the algorithms by the usual formula:

$$r(h) = \frac{\log \frac{E(h)}{E(\frac{h}{2})}}{\log 2},$$

where $E(h)$ is the approximation error defined by

$$E(h) = \|U(\cdot, t^n) - u(\cdot, t^n)\|_2,$$

for population density, where $\|\cdot\|_2$ denotes L^2 norm and the notation $U(\cdot, t^n)$, $u(\cdot, t^n)$ denote approximate solution and exact solution, respectively.

In the following example, we tested the scheme with nonlinear mortality μ.

EXAMPLE 4.1. We solve problem (2.4) with the following data: $a_\dagger = 1$, $\beta(a,p) = 4$, $\mu(a,p) = \frac{1}{1-a} + p$ and $u_0(a) = \omega(a)$, given below.

We then find the exact solution of separable type $u(a,t) = w(a)p(t)$, where $p(t)$ is the solution of

$$p' = (\alpha^* - p)p, \; p(0) = 1,$$

so that the total population p is given as

$$p(t) = \frac{\alpha^*}{(\alpha^* - 1)\exp(-\alpha^* t) + 1},$$

and

$$w(a) = 4(1-a)\exp(-\alpha^* a).$$

Here α^* is given by the relation

$$\alpha^* = \int_0^1 (\beta - \frac{1}{1-a}) w(a) da$$

and is computed as

$$\alpha^* \approx 2.5569290855.$$

	t=1.0		t=1.5	
h_a	$E(h)$	$r(h)$	$E(h)$	$r(h)$
1/10	0.349760	0.958031	0.216245	1.1209717
1/20	0.180042	0.987458	0.099426	1.0805020
1/40	0.090807	0.996271	0.047015	1.0457960
1/80	0.045521	0.998827	0.022773	1.0243409
1/160	0.022779	0.999556	0.011196	1.0124237
1/320	0.011393	0.999873	0.005550	1.0062522

Table 1. Convergence estimates for Example 4.1.

We then note that the compatibility condition at $(0,0)$ is satisfied, which guarantees the continuity of the solution $u(a,t)$. In the approximation, we lag the coefficients to compute the nonlinear terms. Table 1 shows the convergence estimates of the scheme applied to the problem.

In the following example, we tested the scheme with linear and bounded mortality μ.

EXAMPLE 4.2. We solve problem (2.4) with the following data: $a_\dagger = 1$, $\beta(a) = 20a(1-a)$, $\mu(a) = 10\exp(-100(1-a))$ and $u_0(a) = \omega(a)$, given below.

We then find the exact solution of separable type $u(a,t) = \omega_0\omega(a)p(t)$, where $p(t)$ is the solution of

$$p' = \alpha^* p, \quad p(0) = 1,$$

so that the total population p is given as

$$p(t) = \exp(\alpha^* t),$$

and

$$\omega(a) = \exp(-\int_0^a \mu(\xi)d\xi - \alpha^* a).$$

Here α^* is given by the relation

$$1 = \int_0^1 \beta(a)\omega a da$$

and is computed as

$$\alpha^* \approx 2.78576939.$$

ω_0 is also given by the relation

$$1 = \int_0^1 \omega_0 \omega(a)da$$

and is computed as

$$\omega_0 \approx 2.9669447356.$$

As in Example 1, the compatibility condition at $(0,0)$ is satisfied, which guarantees the continuity of the solution $u(a,t)$.

We used Gaussian quadrature in the computation of the integral. Tables 2-3 show the estimated order of convergence with mesh sizes h_a and h_t satisfying $h_t/h_a = \frac{1}{2}$ and $h_t/(h_a)^{\frac{3}{2}} = \frac{1}{2}$, respectively.

In the following example, we tested the scheme with unbounded mortality μ.

EXAMPLE 4.3. We solve problem (2.4) with the following data: $a_\dagger = 1$, $\beta(a) = e$, $\mu(a) = \frac{1}{1-a}$ and $u_0(a) = \omega(a)$, given below.

We then find the exact solution of separable type $u(a,t) = w(a)p(t)$, where

$$p(t) = \exp(t)$$

and

$$\omega(a) = (1-a)\exp(-a).$$

Tables 4-5 show the estimated order of convergence with mesh sizes h_a and h_t satisfying $h_t/h_a = \frac{1}{2}$ and $h_t/h_a^{\frac{3}{2}} = \frac{1}{2}$, respectively.

In all Examples 1-3, we have computed the order of convergence with mesh sizes h_t and h_a satisfying the CFL condition. Our numerical experiments show that the rate of convergence is $O(h^{3/2})$, which is the expected result when the discontinuous Galerkin finite element method

	t=1.0		t=1.5	
h_a	$E(h)$	$r(h)$	$E(h)$	$r(h)$
1/40	0.544225	0.981742	3.078201	0.980551
1/80	0.275578	0.990576	1.559989	0.990027
1/160	0.138692	0.995243	0.785405	0.994970
1/320	0.069575	0.997617	0.394074	0.997476
1/640	0.034845	0.998799	0.197382	0.998728

Table 2. Convergence estimates for Example 4.2 with $h_t/h_a = \frac{1}{2}$.

	t=1.0		t=1.5	
h_a	$E(h)$	$r(h)$	$E(h)$	$r(h)$
1/16	0.625167	1.498192	3.693625	1.497120
1/32	0.221307	1.457412	1.308503	1.450494
1/64	0.080588	1.844685	0.478776	1.843776

Table 3. Convergence estimates for Example 4.2 with $h_t/h_a^{\frac{3}{2}} = \frac{1}{2}$.

	t=1.0		t=1.5	
h_a	$E(h)$	$r(h)$	$E(h)$	$r(h)$
1/8	0.037641	0.955579	0.088485	0.922937
1/16	0.019409	0.985136	0.046670	0.965967
1/32	0.009805	0.994272	0.023892	0.983907
1/64	0.004922	0.997657	0.012080	0.992376
1/128	0.002465	0.999414	0.006072	0.996203

Table 4. Convergence estimates for Example 4.3 with $h_t/h_a = \frac{1}{2}$.

	t=1.0		t=1.5	
h_a	$E(h)$	$r(h)$	$E(h)$	$r(h)$
1/16	0.006143	1.536240	0.013687	1.500920
1/32	0.002118	1.552631	0.004836	1.523636
1/64	0.000722	1.553340	0.001682	1.528609
1/128	0.000246	1.533123	0.000583	1.529140

Table 5. Convergence estimates for Example 4.3 with $h_t/h_a^{\frac{3}{2}} = \frac{1}{2}$.

with piecewise linear polynomial space is applied: Here we notice that the rate of convergence does not deteriorate even if the coefficient function is unbounded. Theoretical estimate for the rate of convergence is provided in [8].

We also see that, if we take the mesh of $h_t = \frac{1}{2}h_a$ (which means relatively large time step size), then the convergence is slow. On the other hand, if we take the mesh of $h_t = \frac{1}{2}h_a^{3/2}$, (which means relatively small time step size), then the convergence is fast. It would be better to use the Crank-Nicolson method in time combining with a discontinuous Galerkin method in age to have fast convergence while keeping relatively large time step size.

References

[1] M. B. Allen and E. L. Isaacson, *Numerical Analysis For Applied Science*, John wiley & sons, 1998.

[2] R. L. Burden and J. D. Faires, *Numerical Analysis*, Brooks/Cole Publishing Company, 1997.

[3] M. Gurtin and R. C. MacCamy, *Non-linear age-dependent population dynamics*, Archs Ration. Mech. Analysis **54** (1974), 281–300.

[4] R. Haberman, *Elementary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Third Edition, Prentice-Hall, 1998.

[5] M. Iannelli, *Mathematical Theory of Age-Structured Population Dynamics*, Giardini Editori E Stampatori, Pisa, 1994.

[6] C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, The Cambridge University Press, 1987.

[7] M.-Y. Kim and Y. Kwon, *A collocation method for the Gurtin-MacCamy equation with finite life-span*, SIAM J. Numer. Anal. **39** (2002), no. 6, 1914–1937.

[8] M.-Y. Kim, *Discontinuous Galerkin method for the Lotka-McKendrick equation with finite life-span*, Preprint.

Department of Mathematics
Inha University
Incheon 402-751, Korea
E-mail: mykim@math.inha.ac.kr
yxyin@math.inha.ac.kr